

Fundamental Theorem of Asset Pricing

Brandon Lee

15.450 Recitation 1

No Arbitrage

- Roughly speaking, an arbitrage is a possibility of profit at zero cost.
- Often implicit is an assumption that such an arbitrage opportunity is scalable (can repeat it over and over).
- In this situation, presumably someone with a deep pocket and an ability to recognize the arbitrage opportunity (called arbitrageur) will exploit the arbitrage and in the course will eliminate the arbitrage opportunity in the market.
- From the point of view of an average investor, all arbitrage opportunities are already taken up by the arbitrageurs and therefore there is “no arbitrage”.
- We’re not saying there cannot be an arbitrage — arbitrageurs, limits to arbitrage, or systemic mispricing.
- However, “no arbitrage” is a very sensible and realistic description of the world we live in. We take “no arbitrage” as an axiom for the discussion in this class.

Fundamental Theorem of Asset Pricing

- The condition of no arbitrage is intuitive and primitive. Yet, it's reasonable to imagine that asset prices are determined by the total supply and demand of assets by the entire group of investors, each of whom has his or her own preferences (this approach won't get us far, it's just too difficult to analyze this economy).
- The breakthrough is the Fundamental Theorem of Asset Pricing. It allows us to study asset prices without having to worry about individual preferences or cross-sectional distribution of investors.
- The theorem states that the absence of arbitrage is equivalent to the existence of a state price density. One implies the other — the implications in both directions are important.

State Price Density

- Another name for the state price density is stochastic discount factor.
- Recall straightforward bond pricing from 15.401/15.415.

$$B = \frac{1}{1+r} C + \frac{1}{(1+r)^2} C + \dots + \frac{1}{(1+r)^T} (C + P)$$

- The terms $\frac{1}{(1+r)^t}$ are (time) discount factors. They do not take into account risk. The stochastic discount factor (or state price density) is a generalization of the simple discount factor that incorporates both risk and time value of money.
- Roughly speaking, state price density is large in “bad states” where a dollar is more valuable and small in “good states” where a dollar is less valuable.
- A state price density can price any asset (where an asset represents a stream of stochastic cashflows) (see page 11 of Lectures Notes 1).

Related Concepts

- Existence of a state price density is equivalent to existence of two other objects: state prices and risk neutral probabilities.
- Therefore, by the Fundamental Theorem of Asset Pricing, no arbitrage is equivalent to existence of any one of the three related objects: state price density, state prices, or risk neutral probabilities.
- Why have so many different concepts? Each provides a different perspective and insight, but the three concepts, as a group, are really the same thing!
- In what follows, to fix the idea and simplify notation, assume the following: we have date 0 and date 1, there are N possible states of nature at date 1, and state i occurs with probability p_i .

State Price Density

- Consider an asset that pays off x_i dollars in state i at date 1. The question is what is the value (price) of this asset at date 0.
- Using the SPD π , we calculate it as follows:

$$\begin{aligned} P &= E[\pi x] \\ &\equiv \sum_{i=1}^N p_i \pi_i x_i \end{aligned}$$

- In words, the price is the expected value of payoffs, discounted by the state price density.

State Prices

- The state-contingent security in state i is an asset that pays off \$1 in state i and 0 in all other states. The price of this state-contingent security is the state price in state i , denoted ϕ_i .
- State prices refer to the vector of state prices (ϕ_1, \dots, ϕ_N) .
- Using the state prices, the price of our asset is

$$P = \sum_{i=1}^N \phi_i x_i$$

- Intuition: John Cochrane's Happy Meal analogy - The price of a happy meal is the price of a hamburger plus the price of a soft drink plus the price of fries (and plus the price of a toy).

Risk Neutral Probabilities

- The actual probabilities are given by (p_1, \dots, p_N) . Are there a different set of probabilities and a risk-free rate such that the price of the asset is given by the expected payoff (under the alternative probabilities) at date 1, only discounted by the risk-free rate?
- Call such probabilities (q_1, \dots, q_N) and the risk-free rate r . Then

$$\begin{aligned} P &= \frac{1}{1+r} E^Q [x] \\ &\equiv \frac{1}{1+r} \sum_{i=1}^N q_i x_i \end{aligned}$$

- Thinking of an asset price as the expected payoff is somewhat naive because it doesn't take into account risk, but it's not completely wrong! Under the alternative probabilities Q , this is actually correct (hence the name risk neutral probabilities).

The Three Concepts Are Equivalent

- Existence of state prices \Leftrightarrow existence of state price density

$$\begin{aligned} P &= \sum_{i=1}^N \phi_i x_i \\ &= \sum_{i=1}^N p_i \pi_i x_i \\ &= E[\pi x] \end{aligned}$$

where $\pi_i = \frac{\phi_i}{p_i}$. We can also reverse the direction of the proof.

- Existence of state prices \Leftrightarrow existence of risk neutral probabilities

$$\begin{aligned} P &= \sum_{i=1}^N \phi_i x_i \\ &= (\phi_1 + \dots + \phi_N) \sum_{i=1}^N \frac{\phi_i}{\phi_1 + \dots + \phi_N} x_i \\ &= \frac{1}{1+r} \sum_{i=1}^N q_i x_i \end{aligned}$$

where we define $q_i = \frac{\phi_i}{\phi_1 + \dots + \phi_N}$ and make use of the fact that $\frac{1}{1+r} = \phi_1 + \dots + \phi_N$. Alternatively, given (q_1, \dots, q_N) and r , we can solve for (ϕ_1, \dots, ϕ_N) .

Summary

- A very intuitive, primitive, and simple restriction of no arbitrage has powerful consequences and guarantees existence of powerful pricing tools (Fundamental Theorem of Asset Pricing).
- Each of the three objects, state price density, state prices, and risk neutral probabilities, can price any asset (again, an asset is merely a stream of stochastic payoffs).
- The value of an asset depends on three elements: probabilities, attitude towards risk, and time value of money.
- The state prices incorporate all three elements. The state price density contain all the information about attitude towards risk and time value of money. The risk neutral probabilities contain all the information about (actual) probabilities and attitude towards risk.

MIT OpenCourseWare
<http://ocw.mit.edu>

15.450 Analytics of Finance

Fall 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.