Black-Scholes Formula

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15.450 Recitation 2

Expectation of a Lognormal Variable

- Suppose $X \sim N\left(\mu, \sigma^2\right)$. We want to know how to compute $E\left[e^X\right]$. This calculation is often needed (e.g., page 30 of Lecture Notes 1) because we usually assume that log return is distributed normally.

$$
\begin{aligned}
E\left[e^X\right] &= \int_{-\infty}^{\infty} e^x \phi(x)\, dx \\
&= \int_{-\infty}^{\infty} \exp(x) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 + x\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 + \left(-2\mu - 2\sigma^2\right)x + \mu^2\right)\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 + -2\left(\mu + \sigma^2\right)x + \left(\mu + \sigma^2\right)^2\right)\right) \exp \\
&= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x - \left(\mu + \sigma^2\right)\right)^2\right) dx \\
&= \exp\left(\mu + \frac{1}{2}\sigma^2\right)
\end{aligned}
$$

Change of Measure

- Given a probability measure (think probability distribution), a random variable that is positive and integrates to one defines a change of measure. In other words, suppose we have a probability measure P and a random variable ξ such that $E^P[\xi] = 1$. Then we can define a new probability measure Q through ξ by

$$Q(A) = \int_A \xi \, dP$$

- We can think of ξ as a redistribution of probability weights from P to Q. Hence it's called "change of measure" and denoted $\frac{dQ}{dP}$.

Normality-Preserving Change of Measure

- Now, there is a special class of random variables called exponential martingales that, as change of measures, preserve normality. In more concrete terms, suppose probability measure P is given by the normal distribution $N\left(\mu^P, \sigma^2\right)$. Then, if $\frac{dQ}{dP}$ is an exponential martingale, then the new probability measure Q is also normally distributed, with a different mean but with the same variance, $N\left(\mu^Q, \sigma^2\right)$.
- Such exponential martingales take on the form

$$\xi = \exp\left(-\eta \varepsilon^P - \frac{1}{2}\eta^2\right)$$

for arbitrary numbers η (later in Lecture Notes 2, we'll see that η can be stochastic processes as well).
- Furthermore, we know the exact relationship between μ^P and μ^Q: $\mu^P - \mu^Q = \eta\sigma$ (the previous notes had a typo and had σ^2 instead of σ).

Black-Scholes Formula

- Suppose under Q (the risk-neutral measure), the stock return is given by

$$\frac{S_{t+1}}{S_t} = \exp\left(r - \frac{1}{2}\sigma^2 + \sigma\varepsilon^Q\right)$$

where $\varepsilon^Q \sim N(0,1)$ under the Q-measure.

- Let's derive the Black-Scholes formula in this simple setting. Suppose $S_0 = 1$ and we have a call option that matures at $T = 1$ with a strike price K. The price of this call option is

$$\begin{aligned} C &= e^{-r} E^Q\left[\max(S_1 - K, 0)\right] \\ &= e^{-r} \int_{S_1=K}^{\infty} (S_1 - K)\, dQ \\ &= e^{-r} \int_{S_1=K}^{\infty} S_1 dQ - e^{-r} \int_{S_1=K}^{\infty} K dQ \end{aligned}$$

Continued

- Call the first term C_1 and the second term C_2.
- Let's calculate them separately.

$$\begin{aligned}
C_1 &= e^{-r}\int_{S_1=K}^{\infty} S_1 dQ \\
&= e^{-r}\int_{\frac{\ln K - r + \frac{\sigma^2}{2}}{\sigma}}^{\infty} \exp\left(r - \frac{\sigma^2}{2} + \sigma\varepsilon^Q\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\varepsilon^Q\right)^2\right) d\varepsilon \\
&= \int_{\frac{\ln K - r + \frac{\sigma^2}{2}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\left(\varepsilon^Q\right)^2 - 2\sigma\varepsilon^Q + \sigma^2\right)\right) d\varepsilon \\
&= \int_{\frac{\ln K - r + \frac{\sigma^2}{2}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\varepsilon^Q - \sigma\right)^2\right) d\varepsilon \\
&= \Phi\left(\sigma - \frac{\ln K - r + \frac{\sigma^2}{2}}{\sigma}\right) \\
&= \Phi\left(\frac{-\ln K + r + \frac{\sigma^2}{2}}{\sigma}\right)
\end{aligned}$$

Continued

- Now for C_2

$$C_2 = e^{-r}\int_{S_1=K}^{\infty} K dQ$$
$$= e^{-r}\int_{\frac{\ln K - r + \frac{\sigma^2}{2}}{\sigma}}^{\infty} K \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\varepsilon^Q\right)^2\right) d\varepsilon$$
$$= e^{-r} K \int_{\frac{\ln K - r + \frac{\sigma^2}{2}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\varepsilon^Q\right)^2\right) d\varepsilon$$
$$= e^{-r} K \Phi\left(\frac{-\ln K + r - \frac{\sigma^2}{2}}{\sigma}\right)$$

- So the option price is given by the Black-Scholes formula

$$C = C_1 - C_2$$
$$= \Phi\left(\frac{-\ln K + r + \frac{\sigma^2}{2}}{\sigma}\right) - e^{-r} K \Phi\left(\frac{-\ln K + r - \frac{\sigma^2}{2}}{\sigma}\right)$$

Numerical Integration

- Definite integrals can rarely be computed analytically. In those cases, we need to resort to numerical methods. Here, we present the simplest method using the Riemann sum approximation.
- As an example, let's say we want to compute

$$\int_1^\infty \frac{1}{x^2} dx$$

- We have to worry about two things: summation on the right tail and fineness of approximating rectangles.
- Refer to the MATLAB® code.

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