Stochastic Calculus II

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15.450 Recitation 4

Kolmogorov Backward Equation

- Consider a stochastic process X_t. X_t is a martingale if for $s > t$,

$$E_t[X_s] = X_t$$

  In other words, conditional expectation of future value is simply the current value (example: fair gamble). The notion of a martingale makes sense for both discrete and continuous time processes.
- Now let's look at this concept when X_t is an Ito process. Suppose

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dZ_t$$

- If X_t were a martingale, then over a small time interval dt,

$$\begin{aligned} E_t[X_{t+dt}] &= X_t \\ E_t[X_{t+dt} - X_t] &= 0 \\ E_t[dX_t] &= 0 \end{aligned}$$

  This is another way of thinking about martingales: expected changes are zero.

Continued

- But now recall the dynamics of dX_t:

$$
\begin{aligned}
E_t[dX_t] &= E_t[\mu(t, X_t)\,dt + \sigma(t, X_t)\,dZ_t] \\
&= \mu(t, X_t)\,dt
\end{aligned}
$$

- So X_t is a martingale if and only if $\mu(t, X_t) = 0$. This is intuitive: the drift term is responsible for expected change whereas the diffusion term is responsible for variance of change.
- The Kolmogorov Backward Equation simply formalizes this idea.

Continued

- Suppose we have an underlying process X_t where

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dZ_t$$

- Consider $Y_t = f(t, X_t)$. Suppose we know for some reason that Y_t is a martingale. We have seen two such examples: 1) conditional expectation, 2) security price discounted at risk-free rate under the risk-neutral measure. Then by the above argument, we know that the drift term of $dY_t = d(f(t, X_t))$ should be zero.
- By Ito's Lemma,

$$df(t, X_t) = \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu(t, X_t) + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} (\sigma(t, X_t))^2 \right) dt + \frac{\partial f(t, X_t)}{\partial X_t} \sigma(t, X_t)\, dZ_t$$

- So if $Y_t = f(t, X_t)$ is a martingale, then

$$\frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu(t, X_t) + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} (\sigma(t, X_t))^2 = 0$$

Option Replication

- How do you dynamically replicate an European option in the Black-Scholes setting? Answer: Match the "delta".
- Suppose you have the option price $f(t, S_t)$ on the one hand and the value of a replicating portfolio consisting of the stock and the bond, W_t. At time t, this replicating portfolio holds θ_t number of stocks and the rest is invested in the risk-free bond. This means

$$dW_t = \theta_t dS_t + r(W_t - \theta_t S_t)\,dt$$

- To have perfect replication, we need

$$d(f(t, S_t)) = dW_t$$

  In particular we need the coefficient in front of dS_t to match:

$$\frac{\partial f(t, S_t)}{\partial S_t} = \theta_t$$

- This argument is exactly as in yesterday's lecture where we tried to replicate an option in an environment with stochastic volatility using a stock and another option.

State Price Density and Risk-Neutral Measure

- We can always move from one to the other very easily in both discrete time and continuous time settings.
- In discrete time, if the state price density is given by

$$\frac{\pi_t}{\pi_{t-1}} = \exp\left(-r_{t-1} - \eta_{t-1}\varepsilon_t - \frac{1}{2}\eta_{t-1}^2\right)$$

  then the change of measure from P to Q is given by

$$\left(\frac{dQ}{dP}\right)_t = \exp\left(-\eta_{t-1}\varepsilon_t - \frac{1}{2}\eta_{t-1}^2\right)$$

- In continuous time, if the state price density is given by

$$\pi_t = \exp\left(-\int_0^t r_s ds - \int_0^t \eta_s dZs - \frac{1}{2}\int_0^t \eta_s^2 ds\right)$$

  then the change of measure from P to Q is given by

$$\left(\frac{dQ}{dP}\right)_t = \exp\left(-\int_0^t \eta_s dZs - \frac{1}{2}\int_0^t \eta_s^2 ds\right)$$

- We can always decompose the state price density into time discount and change of measure from P to Q.

Change of Measure in Continuous Time

- Similar as in the discrete time case. If

$$\left(\frac{dQ}{dP}\right)_t = \exp\left(-\int_0^t \eta_s dZs - \frac{1}{2}\int_0^t \eta_s^2 ds\right)$$

then the Brownian motion under P, Z_t^P acquires a drift but its diffusion coefficient does not change. Therefore

$$dZ_t^P = dZ_t^Q - \eta_t dt$$

- Suppose W_t^P is another Brownian motion under P and it has correlation ρ with Z_t^P. How does W_t^P look under the Q-measure?
- Do the decomposition

$$W_t^P = \rho Z_t^P + \sqrt{1-\rho^2} V_t^P$$

where V_t^P is another Brownian motion under P that's independent of Z_t^P. Conclude

$$dW_t^P = dW_t^Q - \rho \eta_t dt$$

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