Applications of Risk-neutral Pricing

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15.450 Recitation 5

Outline

- We are going to study two examples that illustrate the concepts we have learned in the lectures.
- The first example will go over pricing of a futures contract and analyze a European option on the futures contract.
- The second example concerns the valuation of a quanto derivative.

Futures on a Stock

- Assume the basic Black-Scholes economy where the stock price follows
$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$
and the risk-free rate is r.
- Now suppose there is a futures market for the stock. A trader buying a futures contract enters into an obligation to buy the stock at a pre-specified price (futures price) at a pre-specified time (maturity, call it T_F here). Let F_t be the futures price at time t.

Futures Price

- Since the futures price is such that the futures contract is worth zero to both parties today, it satisfies

$$E_t^Q\left[\exp\left(-r\left(T_F-t\right)\right)\left(S_{T_F}-F_t\right)\right]=0$$

- From this characterization of F_t under the risk-neutral measure, we would like to solve for F_t.
- Here, the risk-free rate is deterministic and F_t is known at time t, hence the above equation can be rewritten as

$$\exp\left(-r\left(T_F-t\right)\right)F_t=E_t^Q\left[\exp\left(-r\left(T_F-t\right)\right)S_{T_F}\right]$$

- But notice that the right hand side is simply the stock price at time t, S_t, by the definition of the risk-neutral measure.
- Therefore

$$\begin{aligned}\exp\left(-r\left(T_F-t\right)\right)F_t &= S_t\\ F_t &= \exp\left(r\left(T_F-t\right)\right)S_t\end{aligned}$$

- Using this result, we can also calculate dF_t using Ito's Lemma:

$$\begin{aligned}dF_t &= -rF_t dt + F_t\frac{dS_t}{S_t}\\ &= F_t\left(\left(\mu-r\right)dt+\sigma dZ_t\right)\end{aligned}$$

- Also note that under the risk-neutral measure,

$$\begin{aligned}dF_t &= F_t\left(\left(\mu-r\right)dt+\sigma\left(dZ_t^Q-\frac{\mu-r}{\sigma}dt\right)\right)\\ &= \sigma F_t dZ_t^Q\end{aligned}$$

Option on the Futures Contract

- Suppose we have a European call option on the futures contract, with strike price K and maturity $T < T_F$ (the only novelty here is that the underlying security is a Futures contract, not a stock). Denote the option price at time t by $C(F_t, t)$. How do we compute this?
- Just as we did for the Black-Scholes formula! Again observe that the only difference is that the option is written on the futures price and it behaves according to

$$\frac{dF_t}{F_t} = (\mu - r)\,dt + \sigma\,dZ_t$$

  This is very similar to the dynamics of the stock price in the Black-Scholes formula.

The Black-Scholes Formula, Once Again

- This calculation yields

$$C(F_t, t) = \exp(-r(T-t))\left[F_t\Phi(d_1) - K\Phi(d2)\right]$$

where

$$d_1 = \frac{\log\left(\frac{F_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

PDE Method

- There is another way of characterizing the option price $C(F_t, t)$ using the PDEs.
- Under the risk-neutral measure,

$$
\begin{aligned}
C(F_t, t) &= E_t^Q \left[\exp(-r(T-t)) \max(F_T - K, 0)\right] \\
e^{-rt} C(F_t, t) &= E_t^Q \left[e^{-rT} \max(F_T - K, 0)\right]
\end{aligned}
$$

therefore, $e^{-rt} C(F_t, t)$ is a martingale under the risk-neutral measure. In fact, for any security whose value at time t is X_t, $e^{-rt} X_t$ is a martingale under the risk-neutral measure. Why is this true?

Continued

- Now we're back to Kolmogorov Backward Equation. We need to compute the drift of $e^{-rt}C(F_t,t)$ and set it to zero to derive the PDE.
- Using Ito's Lemma, the drift of $e^{-rt}C(F_t,t)$ is

$$-re^{-rt}C(F_t,t)+e^{-rt}\frac{\partial C(F_t,t)}{\partial t}+\frac{1}{2}e^{-rt}\frac{\partial^2 C(F_t,t)}{\partial F_t^2}\sigma^2 F_t^2$$

so the PDE we're after is

$$0=-rC(F,t)+\frac{\partial C(F,t)}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 C(F,t)}{\partial F^2}F^2$$

Dynamic Replication

- Finally, how do we replicate the option on the futures contract? Suppose we can trade the riskless bond and the futures contract.
- The answer is, of course, we match the delta:

$$\theta_t = \frac{\partial C(F_t, t)}{\partial F_t}$$

- Since we already solved for $C(F_t, t)$ explicitly, we can say more:

$$\theta_t = \exp(-r(T-t))\Phi(d_1)$$

where

$$d_1 = \frac{\log\left(\frac{F_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

Quanto

- A quanto is a type of a derivative where the underlying is denominated in one currency but the settlement is in another currency. This is useful for traders who wish to have exposure to the foreign asset but do not want to face the exchange rate risk.
- To be concrete, suppose that the level of the UK FTSE 100 index (in British pounds) evolves according to

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dZ_t^S$$

  and the dollar value of a single British Pound, call it X_t, follows

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X dZ_t^X$$

  and $dZ_t^S dZ_t^X = \rho dt$.
- Suppose the riskfree rate in US and UK are r_{US} and r_{UK}, respectively.

Quanto Derivative

- Now consider a derivative that pays S_T dollars at time T (note dollars, not pounds). We want to find the price of this derivative security.
- According to the risk-neutral pricing, its value, call it V_0, is given by

$$V_0 = E_0^Q\left[\exp\left(-r_{US}T\right)S_T\right]$$

  where Q is the risk-neutral measure of the US investor.
- Now the question boils down to finding the dynamics of S_t under the measure Q.

Continued

- Write

$$\frac{dS_t}{S_t} = \mu_S^Q dt + \sigma_S dZ_t^{Q,S}$$
$$\frac{dX_t}{X_t} = \mu_X^Q dt + \sigma_X dZ_t^{Q,X}$$

  and $dZ_t^{Q,S} dZ_t^{Q,X} = \rho dt$. It remains to find the drifts μ_S^Q and μ_X^Q.
- The important observation here is that the US investor can invest in British risk-free bond or British stock market, but their instantaneous returns under the risk-neutral measure Q have to be r_{US}. Therefore

$$r_{US} dt = E_t^Q \left[\frac{d(X_t S_t)}{X_t S_t} \right] = \left(\mu_S^Q + \mu_X^Q + \rho \sigma_S \sigma_X \right) dt$$

  and

$$r_{US} dt = E_t^Q \left[\frac{d\left(X_t B_t^{UK}\right)}{X_t B_t^{UK}} \right] = \left(\mu_X^Q + r_{UK} \right) dt$$

Continued

- Therefore
$$\mu_X^Q = r_{US} - r_{UK}$$
and
$$\mu_S^Q = r_{UK} - \rho \sigma_S \sigma_X$$
- We can solve for S_T explicitly:
$$S_T = S_0 \exp\left(\left(\mu_S^Q - \frac{1}{2}\sigma_S^2\right) T + \sigma_S Z_T^{Q,S}\right)$$
- Finally,
$$\begin{aligned} V_0 &= E_0^Q\left[\exp\left(-r_{US} T\right) S_T\right] \\ &= S_0 \exp\left(\left(-r_{US} + \mu_S^Q - \frac{1}{2}\sigma_S^2\right) T + \frac{1}{2}\sigma_S^2 T\right) \\ &= S_0 \exp\left(\left(-r_{US} + \mu_S^Q\right) T\right) \\ &= S_0 \exp\left(\left(r_{UK} - r_{US} - \rho \sigma_S \sigma_X\right) T\right) \end{aligned}$$

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