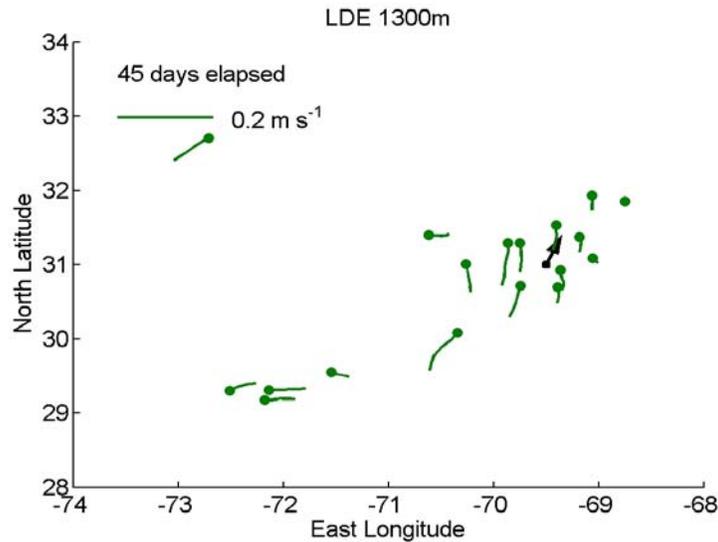


# Lagrangian and Eulerian Representations of Fluid Flow: Kinematics and the Equations of Motion



James F. Price

Woods Hole Oceanographic Institution  
Woods Hole, MA, 02543

July 31, 2006

---

**Summary:** This essay introduces the two methods that are widely used to observe and analyze fluid flows, either by observing the trajectories of specific fluid parcels, which yields what is commonly termed a Lagrangian representation, or by observing the fluid velocity at fixed positions, which yields an Eulerian representation. Lagrangian methods are often the most efficient way to sample a fluid flow and the physical conservation laws are inherently Lagrangian since they apply to moving fluid volumes rather than to the fluid that happens to be present at some fixed point in space. Nevertheless, the Lagrangian equations of motion applied to a three-dimensional continuum are quite difficult in most applications, and thus almost all of the theory (forward calculation) in fluid mechanics is developed within the Eulerian system. Lagrangian and Eulerian concepts and methods are thus used side-by-side in many investigations, and the premise of this essay is that an understanding of both systems and the relationships between them can help form the framework for a study of fluid mechanics.

The transformation of the conservation laws from a Lagrangian to an Eulerian system can be envisaged in three steps. (1) The first is dubbed the Fundamental Principle of Kinematics; the fluid velocity at a given time and fixed position (the Eulerian velocity) is equal to the velocity of the fluid parcel (the Lagrangian velocity) that is present at that position at that instant. Thus while we often speak of Lagrangian velocity or Eulerian velocity, it is important to keep in mind that these are merely (but significantly) different ways to represent a given fluid flow. (2) A similar relation holds for time derivatives of fluid properties: the time rate of change observed on a specific fluid parcel,  $D()/Dt = \partial()/\partial t$  in the Lagrangian system, has a counterpart in the Eulerian system,  $D()/Dt = \partial()/\partial t + \mathbf{V} \cdot \nabla()$ , called the material derivative. The material derivative at a given position is equal to the Lagrangian time rate of change of the parcel present at that position. (3) The physical conservation laws apply to extensive quantities, i.e., the mass or the momentum of a specific fluid volume. The time derivative of an integral over a moving fluid volume (a Lagrangian quantity) can be transformed into the equivalent Eulerian conservation law for the corresponding intensive quantity, i.e., mass density or momentum density, by means of the Reynolds Transport Theorem (Section 3.3).

Once an Eulerian velocity field has been observed or calculated, it is then more or less straightforward to compute parcel trajectories, a Lagrangian property, which are often of great practical interest. An interesting complication arises when time-averaging of the Eulerian velocity is either required or results from the observation method. In that event, the FPK does not apply. If the high frequency motion that is filtered out is wavelike, then the difference between the Lagrangian and Eulerian velocities may be understood as Stokes drift, a correlation between parcel displacement and the spatial gradient of the Eulerian velocity.

In an Eulerian system the local effect of transport by the fluid flow is represented by the advective rate of change,  $\mathbf{V} \cdot \nabla()$ , the product of an unknown velocity and the first partial derivative of an unknown field variable. This nonlinearity leads to much of the interesting and most of the challenging phenomena of fluid flows. We can begin to put some useful bounds upon what advection alone can do. For variables that can be written in conservation form, e.g., mass and momentum, advection alone can not be a net source or sink when integrated over a closed or infinite domain. Advection represents the transport of fluid properties at a definite rate and direction, that of the fluid velocity, so that parcel trajectories are the characteristics of the advection equation. Advection by a nonuniform velocity may cause linear and shear deformation (rate) of a fluid parcel, and it may also cause a fluid parcel to rotate. This fluid rotation rate, often called vorticity follows a particularly simple and useful conservation law.

**Cover page graphic:** SOFAR float trajectories (green worms) and horizontal velocity measured by a current meter (black vector) during the Local Dynamics Experiment conducted in the Sargasso Sea. The float trajectories are five-day segments, and the current vector is scaled similarly. The northeast to southwest oscillation seen here appears to be a (short) barotropic Rossby wave. (See the full essay for an animation of these data.)

# Contents

<b>1</b>	<b>The challenge of fluid mechanics is mainly the kinematics of fluid flow.</b>	<b>5</b>
1.1	Physical properties of materials; what distinguishes fluids from solids? . . . . .	5
1.1.1	The response to pressure — in linear deformation liquids are not very different from solids . . . . .	6
1.1.2	The response to shear stress — solids deform and fluids flow . . . . .	9
1.2	A first look at the kinematics of fluid flow . . . . .	13
1.3	Two ways to observe fluid flow and the Fundamental Principle of Kinematics . . . . .	14
1.4	The goal and the plan of this essay; Lagrangian to Eulerian and back again . . . . .	17
<b>2</b>	<b>The Lagrangian (or material) coordinate system.</b>	<b>19</b>
2.1	The joy of Lagrangian measurement . . . . .	21
2.2	Transforming a Lagrangian velocity into an Eulerian velocity . . . . .	23
2.3	The Lagrangian equations of motion in one dimension . . . . .	24
2.3.1	Mass conservation; mass is neither lost or created by fluid flow . . . . .	24
2.3.2	Momentum conservation; $F = Ma$ in a one dimensional fluid flow . . . . .	28
2.3.3	The one-dimensional Lagrangian equations reduce to an exact wave equation . . . . .	30
2.4	The agony of the three-dimensional Lagrangian equations . . . . .	31
<b>3</b>	<b>The Eulerian (or field) coordinate system.</b>	<b>33</b>
3.1	Transforming an Eulerian velocity field to Lagrangian trajectories . . . . .	34
3.2	Transforming time derivatives from Lagrangian to Eulerian systems; the material derivative . . . . .	35
3.3	Transforming integrals and their time derivatives; the Reynolds Transport Theorem . . . . .	38
3.4	The Eulerian equations of motion . . . . .	41
3.4.1	Mass conservation represented in field coordinates . . . . .	41
3.4.2	The flux form of the Eulerian equations; the effect of fluid flow on properties at a fixed position . . . . .	44
3.4.3	Momentum conservation represented in field coordinates . . . . .	46
3.4.4	Fluid mechanics requires a stress tensor (which is not as difficult as it first seems) . . . . .	47
3.4.5	Energy conservation; the First Law of Thermodynamics applied to a fluid . . . . .	53
3.5	A few remarks on the Eulerian equations . . . . .	54
<b>4</b>	<b>Depictions of fluid flows represented in field coordinates.</b>	<b>55</b>
4.1	Trajectories (or pathlines) are important Lagrangian properties . . . . .	55
4.2	Streaklines are a snapshot of parcels having a common origin . . . . .	58
4.3	Streamlines are parallel to an instantaneous flow field . . . . .	58
<b>5</b>	<b>Eulerian to Lagrangian transformation by approximate methods.</b>	<b>60</b>

5.1	Tracking parcels around a steady vortex given limited Eulerian data . . . . .	60
5.1.1	The zeroth order approximation, or PVD . . . . .	60
5.1.2	A first order approximation, and the velocity gradient tensor . . . . .	61
5.2	Tracking parcels in gravity waves . . . . .	63
5.2.1	The zeroth order approximation, closed loops . . . . .	64
5.2.2	The first order approximation yields the wave momentum and Stokes drift . . . . .	64
<b>6</b>	<b>Aspects of advection, the Eulerian representation of fluid flow.</b>	<b>67</b>
6.1	The modes of a two-dimensional thermal advection equation . . . . .	68
6.2	The method of characteristics implements parcel tracking as a solution method . . . . .	70
6.3	A systematic look at deformation due to advection; the Cauchy-Stokes Theorem . . . . .	74
6.3.1	The rotation rate tensor . . . . .	77
6.3.2	The deformation rate tensor . . . . .	79
6.3.3	The Cauchy-Stokes Theorem collects it all together . . . . .	81
<b>7</b>	<b>Lagrangian observation and diagnosis of an oceanic flow.</b>	<b>82</b>
<b>8</b>	<b>Concluding remarks; where next?</b>	<b>86</b>
<b>9</b>	<b>Appendix: A Review of Composite Functions</b>	<b>87</b>
9.1	Definition . . . . .	88
9.2	Rules for differentiation and change of variables in integrals . . . . .	89

MIT OpenCourseWare  
<http://ocw.mit.edu>

Resource: Online Publication.Fluid Dynamics  
James Price

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.