## DERIVATIVE OF ln $y$ AND $\sin^{-1} y$

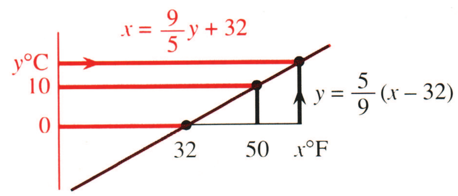
There is a remarkable special case of the chain rule. It occurs when $f(y)$ and $g(x)$ are "***inverse functions***." That idea is expressed by a very short and powerful equation: $f(g(x)) = x$. Here is what that means.

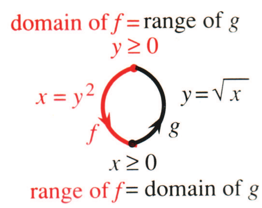
***Inverse functions***: Start with any input, say $x = 5$. *Compute* $y = g(x)$, say $y = 3$. Then compute $f(y)$, and *the answer must be* 5. What one function does, the inverse function undoes. If $g(5) = 3$ then $f(3) = 5$. ***The inverse function $f$ takes the output $y$ back to the input $x$.***

**EXAMPLE 1** $g(x) = x - 2$ and $f(y) = y + 2$ are inverse functions. Starting with $x = 5$, the function $g$ subtracts 2. That produces $y = 3$. Then the function $f$ adds 2. *That brings back* $x = 5$. To say it directly: ***The inverse of*** $y = x - 2$ ***is*** $x = y + 2$.

**EXAMPLE 2** $y = g(x) = \frac{5}{9}(x - 32)$ and $x = f(y) = \frac{9}{5}y + 32$ are inverse functions (for
temperature). Here $x$ is degrees Fahrenheit and $y$ is degrees Celsius. From $x = 32$ (freezing in Fahrenheit) you find $y = 0$ (freezing in Celsius). The inverse function takes $y = 0$ back to $x = 32$.

Notice that $\frac{5}{9}(x - 32)$ subtracts 32 *first*. The inverse $\frac{9}{5}y + 32$ adds 32 *last*. In the same way $g$ multiplies last by $\frac{5}{9}$ while $f$ multiplies first by $\frac{9}{5}$.




°F to °C to °F. Always $g^{-1}(g(x)) = x$ and $g(g^{-1} = (y)) = y$. If $f = g^{-1}$ then $g = f^{-1}$.

***The inverse function is written*** $f = g^{-1}$ ***and pronounced*** "$g$ inverse." ***It is not*** $1/g(x)$.

If the demand $y$ is a function of the price $x$, then the price is a function of the demand. Those are inverse functions. ***Their derivatives obey a fundamental rule***: $dy/dx$ ***times*** $dx/dy$ ***equals*** 1. In Example 2, $dy/dx$ is $5/9$ and $dx/dy$ is $9/5$.

There is another important point. When $f$ and $g$ are applied in the *opposite order*, they still come back to the start. First $f$ adds 2, then $g$ subtracts 2. The chain $g(f(y)) = (y + 2) - 2$ brings back $y$. ***If $f$ is the inverse of $g$ then $g$ is the inverse of*** $f$. The relation is completely symmetric, and so is the definition:

***Inverse function***: ***If*** $y = g(x)$ ***then*** $x = g^{-1}(y)$. ***If*** $x = g^{-1}(y)$ ***then*** $y = g(x)$.

The loop in the figure goes from $x$ to $y$ to $x$. The composition $g^{-1}(g(x))$ is the "identity function." Instead of a new point $z$ it returns to the original $x$. This will make the chain rule particularly easy—leading to $(dy/dx)(dx/dy) = 1$.

**EXAMPLE 3** $y = g(x) = \sqrt{x}$ and $x = f(y) = y^2$ are inverse functions.

Starting from $x = 9$ we find $y = 3$. The inverse gives $3^2 = 9$. The square of $\sqrt{x}$ is $f(g(x)) = x$. In the opposite direction, the square root of $y^2$ is $g(f(y)) = y$.

***Caution*** That example does not allow $x$ to be negative. The domain of $g$—the set of numbers with square roots—is restricted to $x \geqslant 0$. This matches the range of $g^{-1}$. The outputs $y^2$ are nonnegative. With *domain of* $g =$ *range of* $g^{-1}$, the equation $x = (\sqrt{x})^2$ is possible and true. The nonnegative $x$ goes into $g$ and comes out of $g^{-1}$.

To summarize: ***The domain of a function matches the range of its inverse***. The inputs to $g^{-1}$ are the outputs from $g$. The inputs to $g$ are the outputs from $g^{-1}$.

***If*** $g(x) = y$ ***then solving that equation for*** $x$ ***gives*** $x = g^{-1}(y)$:

$$\text{if } y = 3x - 6 \quad \text{then } x = \tfrac{1}{3}(y + 6) \quad (\text{this is } g^{-1}(y))$$
$$\text{if } y = x^3 + 1 \quad \text{then } x = \sqrt[3]{y - 1} \quad (\text{this is } g^{-1}(y))$$

In practice that is how $g^{-1}$ is computed: *Solve* $g(x) = y$. This is the reason inverses are important. Every time we solve an equation we are computing a value of $g^{-1}$.

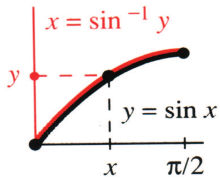
Not all equations have one solution. ***Not all functions have inverses***. For each $y$, the equation $g(x) = y$ is only allowed to produce one $x$. That solution is $x = g^{-1}(y)$. If there is a second solution, then $g^{-1}$ will not be a function—because a function cannot produce two $x$'s from the same $y$.

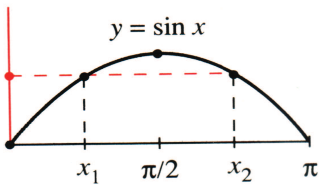
**EXAMPLE 4** There is more than one solution to $\sin x = \frac{1}{2}$. Many angles have the same sine. On the interval $0 \leqslant x \leqslant \pi$, the inverse of $y = \sin x$ is not a function. The figure shows how two $x$'s give the same $y$.

Prevent $x$ from passing $\pi/2$ and the sine has an inverse. Write $x = \sin^{-1} y$.

*The function* $g$ *has no inverse if two points* $x_1$ *and* $x_2$ *give* $g(x_1) = g(x_2)$. Its inverse would have to bring the same $y$ back to $x_1$ and $x_2$. No function can do that; $g^{-1}(y)$ cannot equal both $x_l$ and $x_2$. There must be only one $x$ for each $y$.

***To be invertible over an interval,*** $g$ ***must steadily increase or steadily decrease***.




Inverse exists (one $x$ for each $y$). No inverse function (two $x$'s for one $y$).

## THE DERIVATIVE OF $g^{-1}$

It is time for calculus. Forgive me for this very humble example.

**EXAMPLE 5** (ordinary multiplication) The inverse of $y = g(x) = 3x$ is $x = f(y) = \frac{1}{3}y$.

This shows with special clarity the rule for derivatives: ***The slopes*** $dy/dx = 3$ ***and*** $dx/dy = \frac{1}{3}$ ***multiply to give*** $1$. This rule holds for all inverse functions, even if their slopes are not constant. It is a crucial application of the chain rule to the derivative of $f(g(x)) = x$.

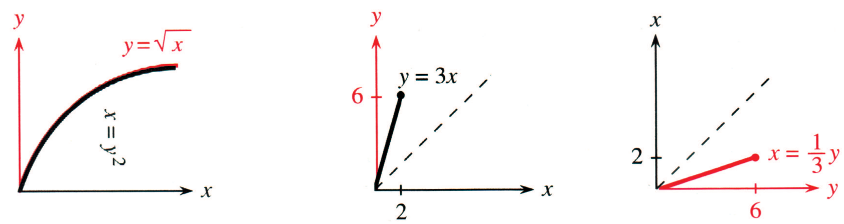
> (***Derivative of inverse function***) From $f(g(x)) = x$ the chain rule gives $f'(g(x))g'(x) = 1$. Writing $y = g(x)$ and $x = f(y)$, this rule looks better:
>
> $$\frac{dx}{dy}\frac{dy}{dx} = 1 \qquad \textit{or} \qquad \frac{dx}{dy} = \frac{1}{dy/dx}. \tag{1}$$
>
> The slope of $x = g^{-1}(y)$ times the slope of $y = g(x)$ equals one.

This is the chain rule with a special feature. Since $f(g(x)) = x$, *the derivative of both sides is* $1$. If we know $g'$ we now know $f'$. That rule will be tested on a familiar example. In the next section it leads to totally new derivatives.

**EXAMPLE 6** The inverse of $y = x^3$ is $x = y^{1/3}$. We can find $dx/dy$ two ways:

$$\text{directly}: \frac{dx}{dy} = \frac{1}{3}y^{-2/3} \qquad \text{indirectly}: \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}.$$

The equation $(dx/dy)(dy/dx) = 1$ is not ordinary algebra, but it is true. Those derivatives are limits of fractions. The fractions are $(\Delta x/\Delta y)(\Delta y/\Delta x) = 1$ and we let $\Delta x \to 0$.




Graphs of inverse functions: $x = \frac{1}{3}y$ is the mirror image of $y = 3x$.

Before going to new functions, I want to draw graphs. The figure shows $y = \sqrt{x}$ and $y = 3x$. What is special is that *the same graphs also show the inverse functions*. The inverse of $y = \sqrt{x}$ is $x = y^2$. The pair $x = 4, y = 2$ is the same for both. That is the whole point of inverse functions—if $2 = g(4)$ then $4 = g^{-1}(2)$. Notice that the graphs go steadily up.

The only problem is, the graph of $x = g^{-1}(y)$ is on its side. To change the slope from 3 to $\frac{1}{3}$, you would have to turn the figure. After that turn there is another problem—the axes don't point to the right and up. You also have to look in a mirror! (The typesetter refused to print the letters backward. He thinks it's crazy but it's not.) To keep the book in position, and the typesetter in position, we need a better idea.

The graph of $x = \frac{1}{3}y$ comes from ***turning the picture across the*** $45°$ ***line***. The $y$ axis becomes horizontal and $x$ goes upward. The point $(2,6)$ on the line $y = 3x$ goes into the point $(6,2)$ on the line $x = \frac{1}{3}y$. The eyes see a reflection across the $45°$ line. The mathematics sees the same pairs $x$ and $y$. The special properties of $g$ and $g^{-1}$ allow us to know two functions—and draw two graphs—at the same time.[1] ***The graph of*** $x = g^{-1}(y)$ ***is the mirror image of the graph of*** $y = g(x)$.

## EXPONENTIALS AND LOGARITHMS

The all-important example is $y = e^x$. Its inverse is the natural logarithm $x = \ln y$ :

$$f^{-1}\big(f(x)\big) = \ln(e^x) = x \qquad f\big(f^{-1}(y)\big) = e^{\ln y} = y$$

We know that the derivative of $e^x$ is $e^x$. So equation (1) will tell us the derivative of $x = \ln y$. This comes from the chain rule $(dx/dy)(dy/dx) = 1$.

$$\boxed{\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{e^x} = \frac{1}{y}}$$

The slope of $\ln y$ is therefore $1/y$. If you want to use different letters, there is nothing to stop you:

The function $f(x) = \ln x$ has slope $\frac{df}{dx} = \frac{1}{x}$.

We already knew the functions $x^n/n$ with slope $x^{n-1}$, but $n = 0$ and slope $x^{-1}$ was not allowed. Now we know that the natural logarithm fills this hole perfectly.

## THE INVERSE OF A CHAIN $h(g(x))$

The functions $g(x) = x - 2$ and $h(y) = 3y$ were easy to invert. For $g^{-1}$ we added 2, and for $h^{-1}$ we divided by 3. Now the question is: If we create the composite function $z = h(g(x))$, or $z = 3(x-2)$, what is its inverse?

Virtually all known functions are created in this way, from chains of simpler functions. *The problem is to invert a chain using the inverse of each piece*. The answer is one of the fundamental rules of mathematics:

[1] I have seen graphs with $y = g(x)$ and also $y = g^{-1}(x)$. For me that is wrong: it has to be $x = g^{-1}(y)$. If $y = \sin x$ then $x = \sin^{-1} y$.

The inverse of $z = h(g(x))$ is a chain of inverses ***in the opposite order***:

$$x = g^{-1}(h^{-1}(z)). \tag{2}$$

$h^{-1}$ is applied first because $h$ was applied last: $g^{-1}(h^{-1}(h(g(x)))) = x$.

That last equation looks like a mess, but it holds the key. In the middle you see $h^{-1}$ and $h$. That part of the chain does nothing! The inverse functions cancel, to leave $g^{-1}(g(x))$. *But that is* $x$. The whole chain collapses, when $g^{-1}$ and $h^{-1}$ are in the correct order—which is opposite to the order of $h(g(x))$.

**EXAMPLE 7** $z = h(g(x)) = 3(x-2)$ and $x = g^{-1}(h^{-1}(z)) = \frac{1}{3}z + 2$.

First $h^{-1}$ divides by 3. Then $g^{-1}$ adds 2. The inverse of $h \circ g$ is $g^{-1} \circ h^{-1}$. *It can be found directly by solving* $z = 3(x-2)$. A chain of inverses is like writing in prose—we do it without knowing it.

**EXAMPLE 8** Invert $z = \sqrt{x-2}$ by writing $z^2 = x-2$ and then $x = z^2 + 2$.

The inverse adds 2 and takes the square—*but not in that order*. That would give $(z+2)^2$, which is wrong. The correct order is $z^2 + 2$.

**EXAMPLE 9** Inverse matrices $(AB)^{-1} = B^{-1}A^{-1}$ (this linear algebra is optional).

Suppose a vector $x$ is multiplied by a square matrix $B$: $y = g(x) = Bx$. The inverse function multiplies by the ***inverse matrix***: $x = g^{-1}(y) = B^{-1}y$. It is like multiplication by $B = 3$ and $B^{-1} = 1/3$, except that $x$ and $y$ are vectors.

Now suppose a second function multiplies by another matrix $A$: $z = h(g(x)) = ABx$. The problem is to recover $x$ from $z$. The first step is to invert $A$, because that came last: $Bx = A^{-1}z$. Then the second step multiplies by $B^{-1}$ and brings back $x = B^{-1}A^{-1}z$. ***The product*** $B^{-1}A^{-1}$ ***inverts the product*** $AB$. The rule for matrix inverses is like the rule for function inverses—in fact it is a special case.

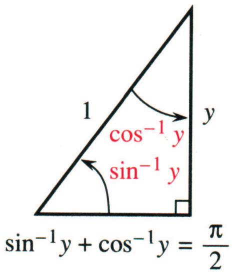
Mathematics is built on basic functions like the sine, and on basic ideas like the inverse. Therefore ***it is totally natural to invert the sine function***. The graph of $x = \sin^{-1} y$ is a mirror image of $y = \sin x$. This is a case where we pay close attention to the domains, since the sine goes up and down infinitely often. We only want *one piece* of that curve.

For the bold line the domain is restricted. *The angle* $x$ *lies between* $-\pi/2$ *and* $+\pi/2$. On that interval the sine is increasing, so ***each*** $y$ ***comes from exactly one angle*** $x$. If the whole sine curve is allowed, infinitely many angles would have $\sin x = 0$. The sine function could not have an inverse. By restricting to an interval where $\sin x$ is increasing, we make the function invertible.

The inverse function brings $y$ back to $x$. It is $x = \sin^{-1} y$ (the ***inverse sine***):

$$x = \sin^{-1} y \text{ when } y = \sin x \text{ and } |x| \leqslant \pi/2. \tag{3}$$

The inverse starts with a number $y$ between $-1$ and 1. It produces an angle $x = \sin^{-1} y$—***the angle whose sine is*** $y$. The angle $x$ is between $-\pi/2$ and $\pi/2$, with the




Graphs of $\sin x$ and $\sin^{-1} y$. Their slopes are $\cos x$ and $1/\sqrt{1-y^2}$.

requiblack sine. Historically $x$ was called the "arc sine" of $y$, and *arcsin* is used in computing. The mathematical notation is $\sin^{-1}$. *This has nothing to do with* $1/\sin x$.

The figure shows the 30° angle $x = \pi/6$. Its sine is $y = \frac{1}{2}$. ***The inverse sine of*** $\frac{1}{2}$ ***is*** $\pi/6$. Again: The symbol $\sin^{-1}(1)$ stands for the angle whose sine is 1 (this angle is $x = \pi/2$). We are seeing $g^{-1}(g(x)) = x$:

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \qquad \sin(\sin^{-1} y) = y \text{ for } -1 \leqslant y \leqslant 1.$$

**EXAMPLE 10** (important) If $\sin x = y$ find a formula for $\cos x$.

Solution We are given the sine, we want the cosine. The key to this problem must be $\cos^2 x = 1 - \sin^2 x$. When the sine is $y$, the cosine is the square root of $1 - y^2$:

$$\cos x = \cos(\sin^{-1} y) = \sqrt{1-y^2}. \tag{4}$$

This formula is crucial for computing derivatives. We use it immediately.

## Problems

### Read-through questions

**Solve equations 1–6 for $x$, to find the inverse function $x = g^{-1}(y)$. When more than one $x$ gives the same $y$, write "no inverse."**

1. $y = 3x - 6$
2. $y = Ax + B$
3. $y = x^2 - 1$
4. $y = x/(x-1)$ [solve $xy - y = x$]
5. $y = 1 + x^{-1}$
6. $y = |x|$
7. Suppose $f$ is increasing and $f(2) = 3$ and $f(3) = 5$. What can you say about $f^{-1}(4)$?

8. Suppose $f(2) = 3$ and $f(3) = 5$ and $f(5) = 5$. What can you say about $f^{-1}$?

9. Suppose $f(2) = 3$ and $f(3) = 5$ and $f(5) = 0$. How do you know that there is no function $f^{-1}$?

10. ***Vertical line test***: If no vertical line touches its graph twice then $f(x)$ is a ***function*** (one $y$ for each $x$). ***Horizontal line test***: If no horizontal line touches its graph twice then $f(x)$ is ***invertible*** because _____ .

11. If $f(x)$ and $g(x)$ are increasing, which two of these might not be increasing?
    $f(x) + g(x)$ $\quad$ $f(x)g(x)$ $\quad$ $f(g(x))$ $\quad$ $f^{-1}(x)$ $\quad$ $1/f(x)$

12. If $y = 1/x$ then $x = 1/y$. If $y = 1 - x$ then $x = 1 - y$. The graphs are their own mirror images in the 45° line. Construct two more functions with this property $f = f^{-1}$ or $f(f(x)) = x$.

13. If $dy/dx = 1/y$ then $dx/dy =$ _____ and $x =$ _____ .

14. If $dx/dy = 1/y$ then $dy/dx =$ _____ (these functions are $y = e^x$ and $x = \ln y$, soon to be honoblack properly).

15. The slopes of $f(x) = \frac{1}{3}x^3$ and $g(x) = -1/x$ are $x^2$ and $1/x^2$. Why isn't $f = g^{-1}$? What is $g^{-1}$? Show that $g'(g^{-1})' = 1$.

**Find $dx/dy$ at the given point**.

16. $y = \sin x$ at $x = \pi/6$

17. $y = \sin 2x$ at $x = \pi/4$

18. $y = \sin x^2$ at $x = 3$

19. $y = x - \sin x$ at $x = 0$

20. If $y$ is a decreasing function of $x$, then $x$ is a _____ function of $y$. Prove by graphs and by the chain rule.

21. If $f(x) > x$ for all $x$, show that $f^{-1}(y) < y$.

22. (a) Show by example that $d^2x/dy^2$ is not $1/(d^2y/dx^2)$.
    (b) If $y$ is in meters and $x$ is in seconds, then $d^2y/dx^2$ is in _____ and $d^2x/dy^2$ is in _____ .

23. Suppose the richest $x$ percent of people in the world have $10\sqrt{x}$ percent of the wealth. Then $y$ percent of the wealth is held by _____ percent of the people.

24. We know that $\sin \pi = 0$. Why isn't $\pi = \sin^{-1} 0$?

25. **True or false**, with reason:

(a) $(\sin^{-1} y)^2 + (\cos^{-1} y)^2 = 1$

(b) $\sin^{-1} y = \cos^{-1} y$ has no solution

(c) $\sin^{-1} y$ is an increasing function

(d) $\sin^{-1} y$ is an odd function

(e) $\sin^{-1} y$ and $-\cos^{-1} y$ have the same slope—so they are the same.

(f) $\sin(\cos x) = \cos(\sin x)$

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