Differential Equations of Growth

$\boldsymbol{\frac{dy}{dt}} = \boldsymbol{cy}$ Complete solution $y(t) = Ae^{ct}$ for any A

Starting from $y(0)$ $y(t) = y(0)e^{ct}$ $A = y(0)$

Now include a constant source term s This gives a new equation

$\boldsymbol{\frac{dy}{dt}} = \boldsymbol{cy} + s$ $s > 0$ is saving, $s < 0$ is spending, cy is interest

Complete solution $y(t) = -\frac{s}{c} + \boldsymbol{Ae^{ct}}$ (any A gives a solution)

$y = -\frac{s}{c}$ is a constant solution with $cy + s = 0$ and $\frac{dy}{dt} = 0$ and $A = 0$

For that solution, the spending s exactly balances the income cy

Choose $\boldsymbol{A}$ to start from $y(0)$ at $t = 0$ $y(t) = -\frac{s}{c} + \left(\boldsymbol{y(0) + \frac{s}{c}}\right) e^{ct}$

Now add a nonlinear term sP^2 coming from competition

$P(t) =$ world population at time t (for example) follows a new equation

$\boldsymbol{\frac{dP}{dt}} = \boldsymbol{cP - sP^2}$ $c =$ birth rate minus death rate

"LOGISTIC EQN" P^2 since each person competes with each person

To bring back a linear equation set $y = \frac{1}{P}$

Then $\frac{dy}{dt} = -\frac{dP/dt}{P^2} = \frac{(-cP + sP^2)}{P^2} = -\frac{c}{P} + s = -cy + s$

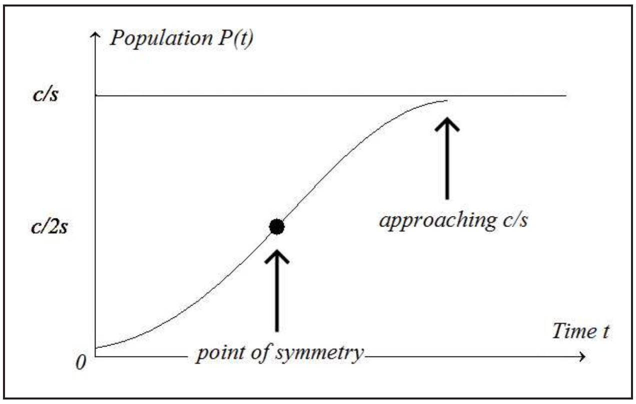
$y = 1/P$ produced our linear equation (no y^2) with $-c$ not $+c$

$y(t) = \frac{s}{c} + Ae^{-ct} = \frac{s}{c} + \left(y(0) - \frac{s}{c}\right) e^{-ct} =$ old solution with change to $-c$

At $t = 0$ we correctly get $y(0)$ CORRECT START

As $t \to \infty$ and $e^{-ct} \to 0$ we get $y(\infty) = \frac{s}{c}$ and $P(\infty) = \frac{c}{s}$

The population $P(t)$ increases along an $\boldsymbol{S}$**-curve** approaching $\frac{c}{s}$



$P = \frac{c}{2s}$ has $P'' = 0$ Inflection point Bending changes from up to down

CHECK $\frac{d^2P}{dt^2} = \frac{d}{dt}\left(cP - sP^2\right) = (c - 2sP)\frac{dP}{dt} = 0$ at $P = \frac{c}{2s}$

World population approaches the limit $\frac{c}{s} \approx 12$ billion (FOR THIS MODEL!)

Population now ≈ 7 billion Try Google for "World population"

Practice Questions

$\frac{\boldsymbol{dy}}{\boldsymbol{dt}} = \boldsymbol{c}\boldsymbol{y} - s$ has $s =$ spending rate not savings rate (with minus sign)

1. The constant solution is $y =$ _____ when $\frac{dy}{dt} = 0$

In that case interest income balances spending: $cy = s$

2. The complete solution is $y(t) = \frac{s}{c} + Ae^{ct}$. Why is $A = y(0) - \frac{s}{c}$?

3. If you start with $y(0) > \frac{s}{c}$ why does wealth approach ∞ ?

   If you start with $y(0) < \frac{s}{c}$ why does wealth approach $-\infty$?

4. The complete solution to $\frac{\boldsymbol{dy}}{\boldsymbol{dt}} = \boldsymbol{s}$ is $y(t) = st + A$

What solution $y(t)$ starts from $y(0)$ at $t = 0$?

5. If $\frac{\boldsymbol{dP}}{\boldsymbol{dt}} = -\boldsymbol{sP^2}$ and $y = \frac{1}{P}$ explain why $\frac{dy}{dt} = s$

Pure competition. Show that $P(t) \to 0$ as $t \to \infty$

6. If $\frac{\boldsymbol{dP}}{\boldsymbol{dt}} = \boldsymbol{cP} - \boldsymbol{sP^4}$ find a linear equation for $y = \frac{1}{P^3}$

Resource: Highlights of Calculus
Gilbert Strang