

PROFESSOR: OK, this lecture, this day, is differential equations day. I just feel even though these are not on the BC exams, that we've got everything we need to actually see calculus in use. We've got the derivatives of the key functions and ready for a differential equation. And there it is.

When I look at that equation-- so it's a differential equation because it has the derivatives of y as well as y itself in the equation. And when I look at it, I see it's a second order equation because there's a second derivative. It's a linear equation because second derivative, first derivative, and y itself are separate, no multiplying of y times y prime. In fact, the only multiplications are by these numbers m , r and k , and those are constant numbers coming from the application. So I have a constant coefficient, linear, second order, differential equation, and I'd like to solve it. And we can do it because it uses the very functions that we know how to find derivatives of.

Let me take two or three special cases where those functions appear purely. So one special case is if I knock out the second derivative term, and let me just choose-- rewrite it in the way it looks easiest and best. It's just the derivative of y as some multiple of y . It's now first order, and we know the function that solves that equation.

When the derivative equals the function itself, that's the exponential. If I want to have an extra factor a here, then I need e to the at . And usually, in fact, we expect that with a first order equation, in the solution there'll be some constant that we can set later to match the starting condition. And that constant shows up here. It's just if e to the at solves that equation, as it does, because when I take the derivative, down comes on a , so does c times e to the at . That's because the equation is linear. So that's a nice one. Pure exponential.

OK, ready for this one. So this one, I don't have this middle term. I just have the second derivative and the function. And let me again change to letters I like, putting the ky on the other side with a minus sign. So this ω^2 will be k/m when I reorganize that equation.

My point is we can solve this equation, the second derivative equaling minus the function. We've met that. That's this equation that the sine and the cosine both solve. There we get two solutions for this second order equation. And just as with the a in this problem, so with this number here, I'll just jiggle the sine and the cosine a little bit so that we get ω to come

down twice when I take two derivatives. So the solution here will be-- one solution will be the cosine of ωt because two derivatives of the cosine is minus the cosine-- that's what this asks for-- with the factor ω coming out twice.

And another solution will be sine of ωt for the same reason. And again, now with a second order equation, I'm expecting a couple of constants to be able to choose later. And here they are: $c \cos \omega t$ and $d \sin \omega t$. That's the general solution to that equation. So we know that.

And these are the two important ones. There's another less important one and an extremely simple one. Suppose all that went away, and I just had as a third very special case $d^2 y / dt^2 = 0$. We sure know the solution to that. What functions have second derivative equals 0? Well, a constant function does, certainly. A constant, even its first derivative is 0 much less its second. And then t does. Its first derivative is 1 and then the second derivative is 0. So there show up the powers of t . Well, the first two powers, t^0 and t^1 , show up in that very special case. Sine and cosine show up here. e^t shows up here. And now let me tell you the good part of this lecture.

The solution to this equation and in fact to equations of third, fourth, all orders, are products of these ones that we know: exponentials times sines and cosines times powers of t . That's all we need to solve constant coefficient differential equations. So I plan now to go ahead and solve the-- and move toward this equation. I should have said that's a fundamental equation of engineering. m stands for some mass.

Oh yeah, let me draw a picture, and you'll see why I chose to choose t rather than x , because things are happening in time. And what is happening? Let me show you. What's happening in time is typically this would be a problem with us some kind of a spring hanging down, and on that spring is a mass m .

OK, so what happens if I-- I've pulled that mass down, so I've stretched the spring, and then I let go. Then what does the spring do? Well, the spring will pull the mass back upwards, and it will pull it back up to the point where it squeezes, compresses the spring. The spring will be-- like instead of being stretched out, it'll be the opposite. It'll be compressed in. And then when compressed in, it'll push the mass. Being compressed, the spring will push. It'll push the mass down again and up again, and I get oscillation.

Oscillation is what I'm seeing here. And what are examples of oscillation in real life? The

spring or a clock, especially a grandfather clock that's swinging back and forth, back and forth, so I'll just put a clock. Music, a violin string is oscillating. That's where the beautiful sound comes from. Our heart is in and out, in and out, a regular oscillation. I could add molecules. They oscillate extremely quickly. So this equation that I'm aiming for comes up in biology, in chemistry-- for molecules, it's a very important equation-- in physics and mechanics and engineering for springs. It comes up in economics. It's everywhere. And by choosing constant coefficients, I have the basic model and the simplest model.

OK, so I'll talk in this language of springs, but it's all these oscillations that lead to equations like that. Actually, this model often would have r equals 0. So let me take that case r equals 0 again. So where does the equation come from? Can I do two cents worth of physics and then go back to the math, solving the equation.

The physics is just remembering Newton's Law: f equals ma . So there is the m , the mass. The a is the acceleration. That's the second derivative, right? You don't mind if I write that as second derivative. Acceleration, the mass is constant. We're not going at the speed of light here so we can assume that mass is not being converted to energy. It's mass.

And then what's the force? Well, for this spring force, for this spring, what did we say? So y is going this way. It's the disposition, the displacement of the spring. When it's down like that, when y is positive, the spring is pulling back. The force from the spring is pulling back. And the force from the spring is proportional to y . And the proportionality constant is my number k .

That fact that I just said is-- and I guess it's pulling opposite to a positive y . When y is positive, and this spring is way down, the force is pulling it up, pulling in the negative direction, upwards. So I need that minus sign. So that k is the stiffness of the spring. This is Hooke's Law, that the force coming from a spring is proportional to the stretch. And the constant in there is Hooke's constant, the spring constant k .

So now that if I put the minus ky over there as plus ky , you see my equation again. But this is the one that we were able to solve right here. Do you see that this is the case where r is 0 in this first model? r is 0 because r involves resistance: r for resistance, air resistance, damping. And right now, I don't have that. I just have a little spring that'll oscillate forever. It'll oscillate forever following sine and cosine. That's exactly what the spring will do. It will just go on forever, and the c and the d , their constants, will depend on how it started.

Did it start from rest? If it started from rest, there would be no sine term. It's easy to find c and

d later. The real problem is to solve the equation, and we've done it for this equation.

OK, and let's just remember this. Let me repeat that when I put that onto the other side and divide by k , then I see the-- oh, divided by m , sorry-- then I see the ω squared. So ω squared is k/m . Right. Oh, it is k/m , right.

OK, that's the simple case, pure sines and cosines. Now I'm letting in some resistance. So now I'm coming back to my equation. Let me write it again. $m y''$ double prime-- second derivative-- plus $2r y'$ prime-- first derivative-- plus ky -- 0-th derivative equals 0. I want to solve that equation now for any numbers m and r and k .

OK, well, the nice thing is that the exponential function takes us right to the answer, the best plan here. So this is the most important equation you would see in a differential equations course. And then the course kind of goes past, and then you easily forget this is the most important and the simplest. Why is it simple? Because if the key idea-- this is the key idea: try y equals e to the-- an exponential e to the something times t . Let me call that something λ . You might have preferred c . I'm happy with c or any other number there.

Yeah, OK, I'll call that λ , just because it gives it a little Greek importance. All right, so I try this. I substitute that into the equation, and I'm going to choose λ to make things work. OK, but here's the key. The key idea is so easy. Now, put it into the equation. So what happens when I put-- when this is y , take two derivatives. Well, let me start with taking no derivatives.

So I have down here the $k e^{\lambda t}$, and over here is a 0. And now let me back up to the first derivative. So that's $2r$ times the derivative of this guy y' . I'm just substituting this into the equation. So what's the derivative? We know that the derivative of this brings down the λ . We already did it. $\lambda e^{\lambda t}$, right? That's the derivative.

And what about this one? This one is going to be an $m y''$ double prime. What happens with two derivatives? Bring down λ twice, two times, so I have $\lambda^2 e^{\lambda t}$. And then in a minute, you know what I'm going to do. I'm going to cancel that common factor $e^{\lambda t}$, which is never 0 so I can safely divide it out, and then write the equation we get.

OK, let me write that out more with more space. $m \lambda^2$ plus $2r \lambda$ -- taking that-- plus k is 0. This is the equation. It's just an ordinary quadratic equation. It's a high school

algebra equation. Lambda appears squared because we had two derivatives. We had a second derivative, and so I need the quadratic formula to know-- I expect two answers, two lambdas. And that's normal for a second order equation, and I'll get two solutions: e to the lambda $1t$ and e to the lambda $2t$. Two different exponentials will both solve the problem.

All right, what's the lambda? Well, can I just recall the quadratic formula? Well, it's a little messy, but it's not too bad here, just to show that I remember it. And the 2 there is kind of handy with the quadratic formula because then I just get minus an r plus or minus the square root of r squared. And it's not minus $4km$, but because of the 2 there, it's just minus km . And then I divide by m .

OK, well, big deal. I get two roots. Let me use numbers. So what you see there is like solving the differential equation. Not too bad.

Now let me put in numbers to show what's typical, and, of course, as those numbers change, we'll see different lambdas. And actually, as the numbers change, that will take us between the exponential stuff and the oscillating stuff.

All right, let me take one where I think it start-- I think this will be-- so this is example one. I'll choose m equal 1 y double prime plus-- let me take r to be 3, so then I have $6y$ prime. And let me choose k to be 8 y equals 0. All right, now we've got numbers.

So the numbers are m equals 1, r equals 3, k equals 8. And I claim we can write the solution to that equation, or the two solutions, because there will be two lambdas. OK, so when I try e to the lambda t , plug it in, I'll get lambda twice, and then I'll get 6 lambda once, and then I'll get 8 without a lambda coming down, all multiplied by e to the lambda t , which I'm canceling, equals 0.

So I solve that equation either directly by recognizing that it factors into lambda plus 2 times lambda plus 4 equals 0 or by plugging in r and k and m in the quadratic formula. Either way, I'm learning that lambda is minus 2 or minus 4. Those are the two solutions. The two decay rates, you could call them, because they're up in the exponent.

So what's the solution? y of t , the solution to that equation, the general solution with a constant c and a constant d is an e to the minus $2t$ and an e to the minus $4t$. The two lambdas are in the exponent, and we've solved it. So that's the point. We have the ability to solve differential equations based on the three most important derivatives we know: exponential, sines-cosines,

powers of t .

OK, ready for example two? Example two, I'm just going to change that 8 to a 10, so you're going to see a 10 show up here. All right, but that will make a difference. It won't just be some new numbers. There'll be a definite difference here.

OK, let me go over across here to the one with 10. OK, so now my equation is $1y'' + 6y' + 10y = 0$, remembering prime means derivative. OK, so again, I try $y = e^{\lambda t}$. I plug it in. When I have two derivatives, bring down λ squared. One derivative brings down λ . No derivatives leaves the 10, equals 0. That's my equation for λ . Ha! I don't know how to factor that one. And in fact, I better use the quadratic formula just to show what happens here.

So the quadratic formula will be the two roots λ . Can I remember that dumb formula? Minus r plus or minus the square root of r squared minus $4km$, all divided by m . I got the 2's and the 4's out of it by taking r to be 3 here.

OK, so it's minus 3 plus or minus the square root of r squared is 9 minus 1 times 10. k and m is 10 divided by 1. Ha! You see something different's going on here. I have the square root of a negative number. Over there, if I wrote out that square root, you would have seen the square root of plus 1. That gave me minus 3 plus 1 or minus 3 minus 1. That was the minus 2 and minus 4.

Now I'm different. Now seeing the square root of minus 1, so this is minus 3 plus or minus i . So I see the solution y of t . You see, this is i here, the square root of minus 1. We can deal with that. It's a complex number, an imaginary number. And the combination minus 3 plus i is a complex number, and we have to accept that that's our λ . So I have any multiple of c , and the λ here is minus 3 plus i . And the second solution is $e^{-3 - i}t$.

I found the general solution. We could say done, except you might feel, well, how did imaginary-- what are we going to do with these imaginary numbers here? How did they get in this perfectly real differential equation? Well, they slipped in because the solutions were not real numbers. The solutions were minus 3 plus or minus i .

But we can get real again. So this is one way to write the answer, but I just want to show you using the earlier lecture, using the beautiful fact that Euler discovered. So now let me complete this example by remembering Euler's great formula for e to the it . Because you see we have

an e to the minus $3t$, perfectly real, decaying. The spring is slowing down because of air resistance. But we also have an e to the it . That's what Euler's formulas about and Euler's formula says that e to the it is the cosine of t plus i times the sine of t .

So it's through Euler's formula that these oscillations are coming in. The direct method led to an e to the it . But the next day, Euler realized that e to the minus it -- or probably being Euler, it didn't take till next day-- will be minus i sine t . So both e to the it and e to the minus it , they both can get replaced by sines and cosines. So in place of e to the it , I'll put that. In place of e to the minus it , I put that one.

The final result is-- can I just jump to that? The final result is that with some different constants, we have the cosine-- oh! let me not forget e to the minus $3t$. That's part of this answer. I'm damping this out by e to the minus $3t$, this resistance r , times cosine of t and the e to the minus $3t$ sine of t .

OK, that's good. General solution, back to a real numbers. It describes a damped oscillation. It's damped out. It's slowing down. Rather, it's decaying. The amplitude is-- the spring is like-- it's like having a shock absorber or something. It's settling down to the center point pretty fast. But as it settles, it's goes back across that center point, oscillates across.

OK, now you might finally ask, the last step of this lecture, where do powers of t come in? Where does t come in? So far we've seen exponentials come in. We've seen sines and cosines come in. Can I do a last example just here in the corner, which will be y double prime, $6y$ prime, and now this time instead of 8 or 10 , I'm going to take $9y$ equals 0 .

OK, can we use this as example 3, which we can solve? You know that I'm going to try y equals to λt . Let me substitute that. I'll get λ squared coming from two derivatives, λ coming from one derivative, 9 coming from no derivatives. I've got my quadratic equation that's supposed to give me two λ s.

Little problem here. When I factor this, it factors into $\lambda + 3$ squared equals 0 . So the answer, the λ , is minus 3 twice. Twice! The two λ s happen to hit the same value: minus 3 .

OK, we don't have any complex stuff here. It's two real values that happen to coincide. And when that happens, well, minus 3 tells us that a solution e to the minus $3t$ works. Where do we get the other solution? We want two solutions to a second order equation. I can't just use e to

the minus $3t$ again. I need a second solution, and it shows up at this-- it's typical of math that it shows-- something special happens at this special situation of a double root, and the solutions will be e^{-3t} and $t e^{-3t}$. That's the last step, and maybe I just put it-- well, where am I going to put it? I'll bring this down and just put it here.

So now I'm solving this particular problem y is a multiple of e^{-3t} . Good, but I can't just repeat it for the second one. So the second one, it just turns out that then is when a little factor t appears. You might like to substitute that for practice with the product rule. If you substitute that in the differential equation, you'll find everything cancels, and it works.

So the conclusion is linear constant coefficient differential equations are completely solved by trying $e^{\lambda t}$ and finding that number λ . If it's a real number, we have exponentials. If it's an imaginary number, we have sines and cosines. If it's a repeated number, we have an extra factor t showing up. That's the exceptional event in that particular case.

So there you go. Differential equations with constant coefficients we can handle. You can handle. Thank you.

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