Linear Approximation and Newton's Method

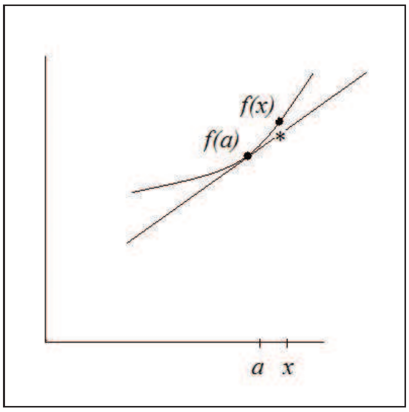
Start at $x = a$ with known $f(a) =$ height and $f'(a) =$ slope

KEY IDEA $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$ when x is near a

Tangent line has slope $f'(a)$

Solve for $f(x)$

$\boldsymbol{f(x) \approx f(a) + (x - a)\, f'(a)}$



$\approx$ means "approximately"
curve $\approx$ line near $x = a$

Examples of linear approximation to $f(x)$

1. $f(x) = e^x$ $\quad f(0) = e^0 = 1$ and $f'(0) = e^0 = 1$ are known at $a = 0$

Follow the tangent line $\quad e^x \approx 1 + (x - 0)1 = \mathbf{1 + x}$

$1 + x$ is the linear part of the series for e^x

2. $f(x) = x^{10}$ and $f'(x) = 10x^9$ $\quad f(1) = 1$ and $f'(1) = 10$ known at $a = 1$

Follow the tangent line $x^{10} \approx \mathbf{1 + (x - 1)10}$ $\quad$ near $x = 1$

Take $x = 1.1$ $\quad (1.1)^{10}$ is approximately $1 + 1 = 2$

Newton's Method (looking for x to nearly solve $f(x) = 0$)

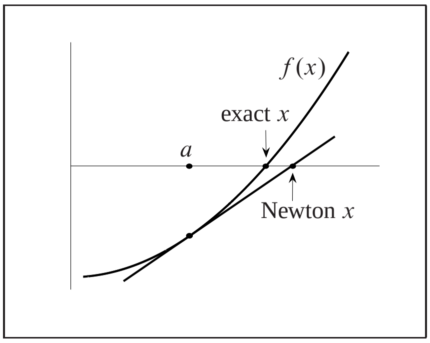
Go back to $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$

$f(a)$ and $f'(a)$ are again known

Solve for x when $f(x) = 0$

$\boldsymbol{x - a \approx -\dfrac{f(a)}{f'(a)}}$ $\qquad$ **Newton x**

Line crossing near curve crossing



Examples of Newton's Method Solve $f(x) = x^2 - 1.2 = 0$

1. $a = 1$ gives $f(a) = 1 - 1.2 = -\mathbf{.2}$ and $f'(a) = 2a = \mathbf{2}$

Tangent line hits 0 at $x - 1 = -\dfrac{(-.2)}{2}$ Newton's x will be 1.1

2. For a better x, Newton starts again from that point $a = 1.1$

Now $f(a) = 1.1^2 - 1.2 = \mathbf{.01}$ and $f'(a) = 2a = \mathbf{2.2}$

The new tangent line has $x - 1.1 = -\dfrac{.01}{2.2}$ For this x, x^2 is very close to 1.2

Practice Questions

1. The graph of $y = f(a) + (x-a)f'(a)$ is a straight _____

At $x = a$ the height is $y =$ _____

At $x = a$ the slope is $dy/dx =$ _____

This graph is t_____t to the graph of $f(x)$ at $x = a$

For $f(x) = x^2$ at $a = 3$ this linear approximation is $y =$ _____

2. $y = f(a) + (x-a)f'(a)$ has $y = 0$ when $x - a =$ _____

Instead of the curve $f(x)$ crossing 0, Newton has tangent line y crossing 0

$f(x) = x^3 - 8.12$ at $a = 2$ has $f(a) =$ _____ and $f'(a) = 3a^2 =$ _____

Newton's method gives $x - 2 = -\dfrac{f(a)}{f'(a)} =$ _____

This Newton $x = 2.01$ nearly has $x^3 = 8.12$. It actually has $(2.01)^3 =$ _____ .

Resource: Highlights of Calculus
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