

Product and Quotient Rules

Goal To find the derivative of $y = f(x)g(x)$ from $\frac{df}{dx}$ and $\frac{dg}{dx}$

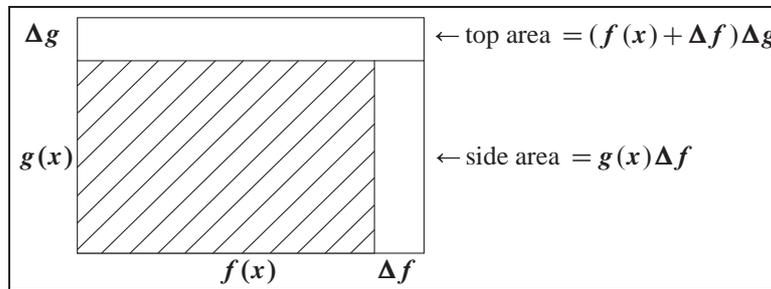
Idea Write $\Delta y = f(x + \Delta x)g(x + \Delta x) - f(x)g(x)$ by separating Δf and Δg

That same Δy is $f(x + \Delta x)[g(x + \Delta x) - g(x)] + g(x)[f(x + \Delta x) - f(x)]$

$$\frac{\Delta y}{\Delta x} = f(x + \Delta x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x} \quad \text{Product Rule} \quad \frac{dy}{dx} = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

Example $y = x^2 \sin x$ Product Rule $\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

A picture shows the two unshaded pieces of $\Delta y = f(x + \Delta x)\Delta g + g(x)\Delta f$



Example $f(x) = x^n$ $g(x) = x$ $y = f(x)g(x) = x^{n+1}$

Product Rule $\frac{dy}{dx} = x^n \frac{dx}{dx} + x \frac{dx^n}{dx} = x^n + xnx^{n-1} = (n+1)x^n$

The correct derivative of x^n leads to the correct derivative of x^{n+1}

Quotient Rule If $y = \frac{f(x)}{g(x)}$ then $\frac{dy}{dx} = \left(g(x)\frac{df}{dx} - f(x)\frac{dg}{dx} \right) / g^2$

EXAMPLE $\frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = (\cos x(\cos x) - \sin x(-\sin x)) / \cos^2 x$

This says that $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x$ (Notice $(\cos x)^2 + (\sin x)^2 = 1$)

EXAMPLE $\frac{d}{dx} \left(\frac{1}{x^4} \right) = \frac{x^4 \text{ times } 0 - 1 \text{ times } 4x^3}{x^8} = \frac{-4}{x^5}$ This is nx^{n-1}

Prove the Quotient Rule $\Delta y = \frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)} = \frac{f + \Delta f}{g + \Delta g} - \frac{f}{g}$

Write this Δy as $\frac{g(f + \Delta f) - f(g + \Delta g)}{g(g + \Delta g)} = \frac{g\Delta f - f\Delta g}{g(g + \Delta g)}$

Now divide that Δy by Δx As $\Delta x \rightarrow 0$ we have the Quotient Rule

Practice Questions

1. Product Rule: Find the derivative of $y = (x^3)(x^4)$. Simplify and explain.
2. Product Rule: Find the derivative of $y = (x^2)(x^{-2})$. Simplify and explain.
3. Quotient Rule: Find the derivative of $y = \frac{\cos x}{\sin x}$.
4. Quotient Rule: Show that $y = \frac{\sin x}{x}$ has a maximum (zero slope) at $x = 0$.
5. Product and Quotient! Find the derivative of $y = \frac{x \sin x}{\cos x}$.
6. $g(x)$ has a minimum when $\frac{dg}{dx} = 0$ and $\frac{d^2g}{dx^2} > 0$. The graph is bending up
 $y = \frac{1}{g(x)}$ has a *maximum* at that point: Show that $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

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Resource: Highlights of Calculus
Gilbert Strang

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