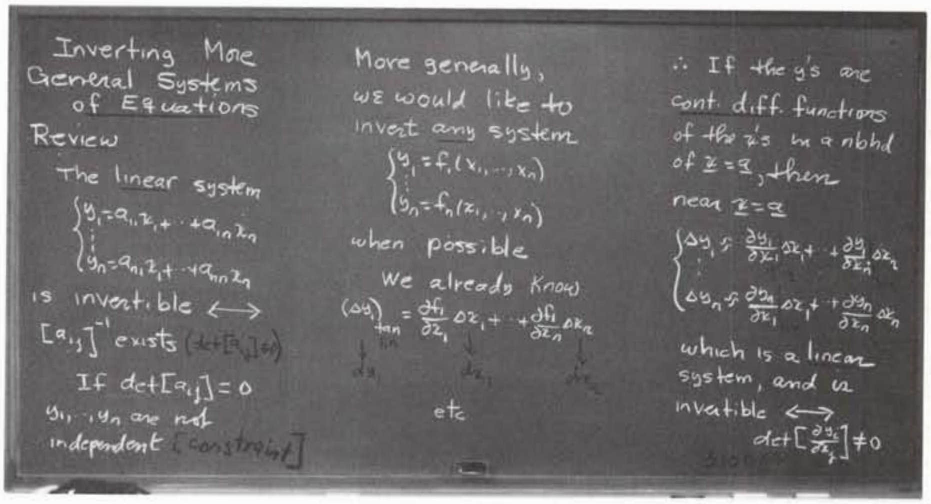
Unit 5: The Total Differential Revisited

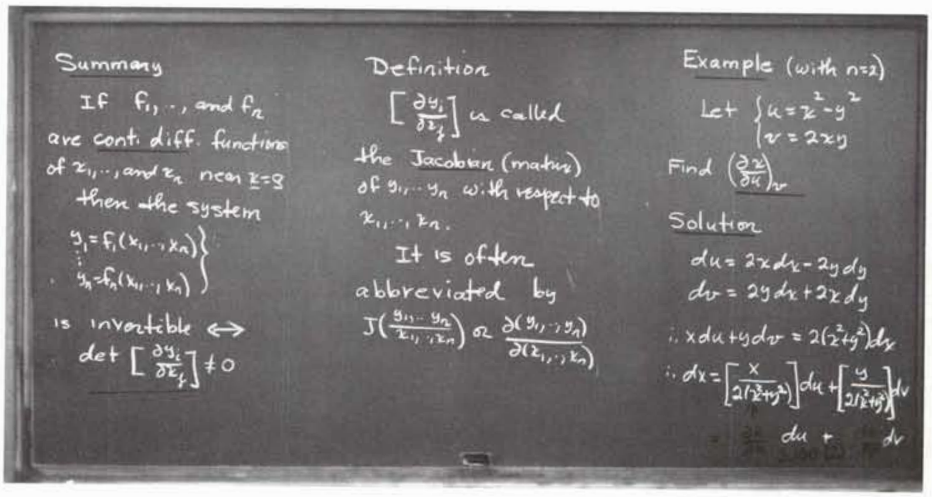
1. Preface 4.040

Note that we ended Block 3 with a discussion of the total differential. We then motivated Block 4 by pointing out that in terms of differentials, nonlinear systems of equations could be approximated by linear systems, and this was our "excuse" for developing matrix (linear) algebra. Our aim in this unit is to return to the notion of the total differential, armed with our knowledge of matrix algebra, and see what new light can be shed on this study.

2. Lecture 4.040

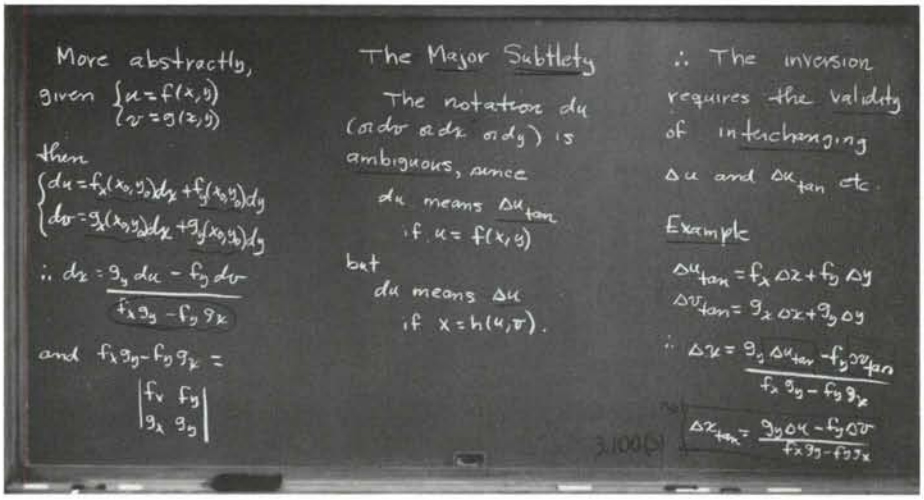


a.



b.

Lecture 4.040 continued



3. (Optional) Review Thomas, Section 15.8 (You may have become "rusty" in your knowledge of the total differential during our excursion into matrix algebra.)

4. (Optional) Do Exercises 4.5.1, 4.5.2, and 4.5.3 (These are optional, not because they introduce advanced topics, but because they review previous material. You may want to try these to make sure that you have sufficient computational skill in the handling of differentials.)

5. Exercises

4.5.1

Let $w = f(x,y) = x^2 - y^2$.

a. Use Δw_{tan} to approximate $f(3.001, 1.99)$. Compare this with the exact value of $f(3.001, 1.99)$ to determine the error in the approximate value.

b. Use the method of part (a) to "approximate" $f(7,5)$, assuming that $(7,5)$ is "sufficiently close" to $(3,2)$.

c. Determine e_1 and e_2 specifically in terms of h and k in the expression $f(3 + h, 2 + k) = f_x(3,2)h + f_y(3,2)k + e_1h + e_2k$ and use this result to check the approximation obtained in part (a).

4.5.2

Let $w = f(x,y)$ where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

show that $f_x(0,0)dx + f_y(0,0)dy$ is <u>not</u> a good approximation for Δw no matter how small a (non-zero) neighborhood of $(0,0)$ we pick.

4.5.3

Let $\begin{cases} u = x^2 - y^2 \\ v = 2\,xy \end{cases}$. Express dy in terms of du and dy and thus

(continued on the next page)

4.5.3 continued

determine y_u and y_v.

4.5.4(L)

Define $\underline{f}:E^2 \to E^2$ by $\underline{f}(x,y) = (x^2 - y^2, 2xy)$

a. Compute (i) $\underline{f}(3,2)$, (ii) $\underline{f}$ (3.001, 1.99), (iii) $\underline{f}(3 + h, 2 + k)$.

b. Use differentials in a neighborhood of (3,2) to approximate $\underline{f}(3.001, 1.99)$.

4.5.5

(If you are convinced that you understand that our results are not limited to 1-, 2-, or 3-dimensional space you may omit this exercise since it is a bit time-consuming computationally. On the other hand, if you have the time it might be rewarding to see how our results apply analytically even when pictures are unavailable.)

Define $\underline{f}:E^4 \to E^4$ by

$$\underline{f}(x_1, x_2, x_3, x_4) = (x^2_1 + x^2_2 + x^2_3 + x^2_4, x_1x_2x_3x_4, x^3_1 + x^3_2 + x_3x_4, x^2_1 + x_2x_3 + x_1x^2_4).$$

a. Use differentials in a neighborhood of (1,1,1,1) to approximate $\underline{f}(1.001, 1.001, 1.001, 1.001)$.

b. Compute $\underline{f}(1.001, 1.001, 1.001, 1.001)$ exactly and use this result to find the error in the approximation of part (a).

4.5.6(L)

With $\underline{f}$ as in Exercise 4.5.4(L), use differentials to approximate the point (x,y) near (3,2) for which $\underline{f}(x,y) = (5.00052, 12.00026)$, assuming, of course, that there is such a point.

4.5.7(L)

Let x, y, and z be independent variables and define u, v, and w by $u = \frac{1}{2}x^2 + y^2 + z^2$, $v = 3x^2y + 4z^3$, and $w = 4xyz$. Express du, dv, dw in terms of dx, dy, dz in a neighborhood of the point $x = y = z = 1$. Then solve this system for dx, dy, and dz in terms of du, dv, and dw. Use this result to determine x_u, x_v, x_w, y_u, y_v, y_w, z_u, z_v, and z_w at the point $\underline{f}(1,1,1)$.

4.5.8(L)

Let $\underline{f}:E^2 \to E^2$ be defined by $\underline{f}(x,y) = (\underbrace{e^x \sin y}_{u}, \underbrace{e^x \cos y}_{v})$.

a. Show that for each $(x,y) \in E^2$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \neq 0 .$$

b. By comparing $\underline{f}(x_0,y_0)$ and $\underline{f}(x_0,y_0 + 2\pi)$ show that $\underline{f}$ is not 1-1 on E^2.