

Unit 5: The Total Differential Revisited

1. Preface 4.040

Note that we ended Block 3 with a discussion of the total differential. We then motivated Block 4 by pointing out that in terms of differentials, nonlinear systems of equations could be approximated by linear systems, and this was our "excuse" for developing matrix (linear) algebra. Our aim in this unit is to return to the notion of the total differential, armed with our knowledge of matrix algebra, and see what new light can be shed on this study.

2. Lecture 4.040

**Inverting More General Systems of Equations**

**Review**

The linear system

$$\begin{cases} y_1 = a_{11}z_1 + \dots + a_{1n}z_n \\ \vdots \\ y_n = a_{n1}z_1 + \dots + a_{nn}z_n \end{cases}$$

is invertible  $\leftrightarrow$   $[a_{ij}]^{-1}$  exists ( $\det[a_{ij}] \neq 0$ )

If  $\det[a_{ij}] = 0$   
 $y_1, \dots, y_n$  are not independent (constraint)

More generally, we would like to invert any system

$$\begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ y_n = f_n(x_1, \dots, x_n) \end{cases}$$

when possible

We already know

$$(\Delta y)_i = \frac{\partial f_i}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f_i}{\partial x_n} \Delta x_n$$

etc

$\therefore$  If the  $y$ 's are cont. diff. functions of the  $x$ 's in a nbhd of  $\bar{x} = \bar{z}$ , then near  $\bar{x} = \bar{z}$

$$\begin{cases} \Delta y_1 \approx \frac{\partial y_1}{\partial x_1} \Delta x_1 + \dots + \frac{\partial y_1}{\partial x_n} \Delta x_n \\ \vdots \\ \Delta y_n \approx \frac{\partial y_n}{\partial x_1} \Delta x_1 + \dots + \frac{\partial y_n}{\partial x_n} \Delta x_n \end{cases}$$

which is a linear system, and is invertible  $\leftrightarrow$   $\det \left[ \frac{\partial y_i}{\partial x_j} \right] \neq 0$

a.

**Summary**

If  $f_1, \dots, f_n$  are cont. diff. functions of  $x_1, \dots, x_n$  near  $\bar{x} = \bar{z}$  then the system

$$\begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ y_n = f_n(x_1, \dots, x_n) \end{cases}$$

is invertible  $\leftrightarrow$   $\det \left[ \frac{\partial y_i}{\partial x_j} \right] \neq 0$

**Definition**

$\left[ \frac{\partial y_i}{\partial x_j} \right]$  is called the **Jacobian (matrix)** of  $y_1, \dots, y_n$  with respect to  $x_1, \dots, x_n$ .

It is often abbreviated by  $J \left( \frac{y_1, \dots, y_n}{x_1, \dots, x_n} \right)$  or  $\frac{\partial (y_1, \dots, y_n)}{\partial (x_1, \dots, x_n)}$

**Example (with  $n=2$ )**

Let  $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

Find  $\left( \frac{\partial x}{\partial u} \right)_v$

**Solution**

$$\begin{aligned} du &= 2x dx - 2y dy \\ dv &= 2y dx + 2x dy \\ \therefore x du + y dv &= 2(x^2 + y^2) dx \\ \therefore dx &= \left[ \frac{x}{2(x^2 + y^2)} \right] du + \left[ \frac{y}{2(x^2 + y^2)} \right] dv \\ &= du + dv \end{aligned}$$

b.

Lecture 4.040 continued

More abstractly,  
 given  $\begin{cases} u = f(x, y) \\ v = g(x, y) \end{cases}$   
 then  
 $\begin{cases} du = f_x(x, y)dx + f_y(x, y)dy \\ dv = g_x(x, y)dx + g_y(x, y)dy \end{cases}$   
 $\therefore dx = \frac{g_y du - f_y dv}{f_x g_y - f_y g_x}$   
 and  $f_x g_y - f_y g_x = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}$

The Major Subtlety  
 The notation  $dx$   
 (or  $dx$  or  $dy$ ) is  
 ambiguous, since  
 $dx$  means  $\Delta x_{tan}$   
 if  $x = f(x, y)$   
 but  
 $dx$  means  $\Delta x$   
 if  $x = h(u, v)$ .

$\therefore$  The inversion  
 requires the validity  
 of interchanging  
 $\Delta u$  and  $\Delta x_{tan}$  etc.

Example  
 $\Delta u_{tan} = f_x \Delta x + f_y \Delta y$   
 $\Delta v_{tan} = g_x \Delta x + g_y \Delta y$   
 $\therefore \Delta x = \frac{g_y \Delta u_{tan} - f_y \Delta v_{tan}}{f_x g_y - f_y g_x}$   
 $\Delta x_{tan} = \frac{g_y \Delta u - f_y \Delta v}{f_x g_y - f_y g_x}$

Study Guide  
Block 4: Matrix Algebra  
Unit 5: The Total Differential Revisited

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3. (Optional) Review Thomas, Section 15.8 (You may have become "rusty" in your knowledge of the total differential during our excursion into matrix algebra.)
4. (Optional) Do Exercises 4.5.1, 4.5.2, and 4.5.3 (These are optional, not because they introduce advanced topics, but because they review previous material. You may want to try these to make sure that you have sufficient computational skill in the handling of differentials.)

5. Exercises

4.5.1

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Let  $w = f(x,y) = x^2 - y^2$ .

- a. Use  $\Delta w_{\tan}$  to approximate  $f(3.001, 1.99)$ . Compare this with the exact value of  $f(3.001, 1.99)$  to determine the error in the approximate value.
- b. Use the method of part (a) to "approximate"  $f(7,5)$ , assuming that  $(7,5)$  is "sufficiently close" to  $(3,2)$ .
- c. Determine  $e_1$  and  $e_2$  specifically in terms of  $h$  and  $k$  in the expression  $f(3+h, 2+k) = f_x(3,2)h + f_y(3,2)k + e_1h + e_2k$  and use this result to check the approximation obtained in part (a).

4.5.2

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Let  $w = f(x,y)$  where

$$f(x,y) = \begin{cases} \frac{2xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

show that  $f_x(0,0)dx + f_y(0,0)dy$  is not a good approximation for  $\Delta w$  no matter how small a (non-zero) neighborhood of  $(0,0)$  we pick.

4.5.3

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Let  $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$ . Express  $dy$  in terms of  $du$  and  $dv$  and thus

(continued on the next page)

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4.5.3 continued

determine  $y_u$  and  $y_v$ .

4.5.4(L)

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Define  $\underline{f}: E^2 \rightarrow E^2$  by  $\underline{f}(x,y) = (x^2 - y^2, 2xy)$

- a. Compute (i)  $\underline{f}(3,2)$ , (ii)  $\underline{f}(3.001, 1.99)$ , (iii)  $\underline{f}(3+h, 2+k)$ .
- b. Use differentials in a neighborhood of  $(3,2)$  to approximate  $\underline{f}(3.001, 1.99)$ .

4.5.5

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(If you are convinced that you understand that our results are not limited to 1-, 2-, or 3-dimensional space you may omit this exercise since it is a bit time-consuming computationally. On the other hand, if you have the time it might be rewarding to see how our results apply analytically even when pictures are unavailable.)

Define  $\underline{f}: E^4 \rightarrow E^4$  by

$$\underline{f}(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2, x_1x_2x_3x_4, \\ x_1^3 + x_2^3 + x_3x_4, x_1^2 + x_2x_3 + x_1x_4^2).$$

- a. Use differentials in a neighborhood of  $(1,1,1,1)$  to approximate  $\underline{f}(1.001, 1.001, 1.001, 1.001)$ .
- b. Compute  $\underline{f}(1.001, 1.001, 1.001, 1.001)$  exactly and use this result to find the error in the approximation of part (a).

4.5.6(L)

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With  $\underline{f}$  as in Exercise 4.5.4(L), use differentials to approximate the point  $(x,y)$  near  $(3,2)$  for which  $\underline{f}(x,y) = (5.00052, 12.00026)$ , assuming, of course, that there is such a point.

4.5.7(L)

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Let  $x$ ,  $y$ , and  $z$  be independent variables and define  $u$ ,  $v$ , and  $w$  by  $u = \frac{1}{2}x^2 + y^2 + z^2$ ,  $v = 3x^2y + 4z^3$ , and  $w = 4xyz$ . Express  $du$ ,  $dv$ ,  $dw$  in terms of  $dx$ ,  $dy$ ,  $dz$  in a neighborhood of the point  $x = y = z = 1$ . Then solve this system for  $dx$ ,  $dy$ , and  $dz$  in terms of  $du$ ,  $dv$ , and  $dw$ . Use this result to determine  $x_u$ ,  $x_v$ ,  $x_w$ ,  $y_u$ ,  $y_v$ ,  $y_w$ ,  $z_u$ ,  $z_v$ , and  $z_w$  at the point  $\underline{f}(1,1,1)$ .

4.5.8(L)

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Let  $\underline{f}: E^2 \rightarrow E^2$  be defined by  $\underline{f}(x,y) = (\underbrace{e^x \sin y}_u, \underbrace{e^x \cos y}_v)$ .

- a. Show that for each  $(x,y) \in E^2$

$$\begin{vmatrix} \partial(u,v) \\ \partial(x,y) \end{vmatrix} \neq 0.$$

- b. By comparing  $\underline{f}(x_0, y_0)$  and  $\underline{f}(x_0, y_0 + 2\pi)$  show that  $\underline{f}$  is not 1-1 on  $E^2$ .

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