GILBERT STRANG: OK. More about eigenvalues and eigenvectors. Well, actually, it's going to be the same thing about eigenvalues and eigenvectors but I'm going to use matrix notation.

So, you remember I have a matrix A, 2 by 2 for example. It's got two eigenvectors. Each eigenvector has its eigenvalue. So I could write the eigenvalue world that way. I want to write it in matrix form. I want to create an eigenvector matrix by taking the two eigenvectors and putting them in the columns of my matrix.

If I have n of them, that allows me to give one name. The eigenvector matrix, maybe I'll call it V for vectors. So that's A times V. And now, just bear with me while I do that multiplication of A times the eigenvector matrix.

So what do I get? I get a matrix. That's 2 by 2. That's 2 by 2. You get a 2 by 2 matrix. What's the first column? The first column of the output is A times the first column of the input. And what is A times x1? Well, A times x1 is lambda 1 times x1. So that first column is lambda 1 x1. And A times the second column is Ax2, which is lambda 2 x2. So I'm seeing lambda 2 x2 in that column. OK.

Matrix notation. Those were the eigenvectors. This is the result of A times V. But I can look at this a little differently. I can say, wait a minute, that is my eigenvector matrix, x1 and x2-- those two columns-- times a matrix. Yes.

Taking this first column, lambda 1 x1, is lambda 1 times x1, plus 0 times x2. Right there I did a matrix multiplication. I did it without preparing you for it. I'll go back and do that preparation in a moment. But when I multiply a matrix by a vector, I take lambda 1 times that one, 0 times that one. I get lambda 1 x1, which is what I want.

Can you see what I want in the second column here? The result I want is lambda 2 x2. So I want no x1's, and lambda 2 of that column. So that's 0 times that column, plus lambda 2, times that column. Are we OK?

So, what do I have now? I have the whole thing in a beautiful form, as this A times the eigenvector matrix equals, there is the eigenvector matrix again, V. And here is a new matrix that's the eigenvalue matrix. And everybody calls that-- because those are lambda 1 and lambda 2. So the natural letter is a capital lambda. That's a capital Greek lambda there, the best I could do.

So do you see that the two equations written separately, or the four equations or the n equations, combine into one matrix equation. This is the same as those two together. Good. But now that I have it in matrix form, I can mess around with it. I can multiply both sides by V inverse.

If I multiply both sides by V inverse I discover-- well, shall I multiply on the left by V inverse? Yes, I'll do that. If I multiply on the left by V inverse that's V inverse AV. This is matrix multiplication and my next video is going to recap matrix multiplication.

So I multiply both sides by V inverse. V inverse times V is the identity. That's what the inverse matrix is. V inverse, V is the identity. So there you go. Let me push that up. That's really nice. That's really nice. That's called diagonalizing A. I diagonalize A by taking the eigenvector matrix on the right, its inverse on the left, multiply those three matrices, and I get this diagonal matrix. This is the diagonal matrix lambda.

Or other times I might want to multiply by both sides here by V inverse coming on the right. So that would give me A, V, V inverse is the identity. So I can move V over there as V inverse. That's what it amounts to.

I multiply both sides by V inverse. So this is just A and this is the V, and the lambda, and now the V inverse. That's great.

So that's a way to see how A is built up or broken down into the eigenvector matrix, times the eigenvalue matrix, times the inverse of the eigenvector matrix. OK. Let me just use that for a moment. Just so you see how it connects with what we already know about eigenvalues and eigenvectors. OK.

So I'll copy that great fact, that A is V lambda, V inverse. Oh, what do I want to do? I want to look at A squared. So if I look at A squared, that's V lambda V inverse times another one. Right? There's an A, there's an A. So that's A squared.

Well, you may say I've made a mess out of A squared, but not true. V inverse V is the identity. So that it's just the identity sitting in the middle. So the V at the far left, then I have the lambda, and then I have the other lambda-- lambda squared-- and then the V inverse at the far right. That's A squared.

And if I did it n times, I would have A to the n-th what would be the lambda to the n-th power V inverse. What is this? What is this saying about? This is A squared. How do I understand that equation? To me that says that the eigenvalues of A squared are lambda squared. I'm just squaring each eigenvalue.

And the eigenvectors? What are the eigenvectors of A squared? They're the same V, the same vectors, x1, x2, that went into v. They're also the eigenvectors of A squared, of A cubed, of A to the n-th, of A inverse.

So that's the point of diagonalizing a matrix? Diagonalizing a matrix is another way to see that when I square the matrix, which is usually a big mess, looking at the eigenvalues and eigenvectors it's the opposite of a big mess. It's very clear. The eigenvectors are the same as for A. And the eigenvalues are squares of the eigenvalues of A.

In other words, we can take the n-th power and we have a nice notation for it. We learned already that the n-th power has the eigenvalues to the n-th power, and the eigenvectors the same. But now I just see it here. And there it is for the n-th power.

So if I took the same matrix step 1,000 times, what would be important? What controls the thousandth power of a matrix? The eigenvectors stay. They're just set. It would be the thousandth power of the eigenvalue.

So if this is a matrix with an eigenvalue larger than 1, then the thousandth power is going to be much larger than one. If this is a matrix with eigenvalues smaller than 1, there are going to be very small when I take the thousandth power. If there's an eigenvalue that's exactly 1, that will be a steady state. And 1 to the thousandth power will still be 1 and nothing will change.

So, the stability. What happens as I multiply, take powers of a matrix, is a basic question parallel to the question what happens with a differential equation when I solve forward in time? I think of those two problems as quite parallel. This is taking steps, single steps, discrete steps. The differential equation is moving forward continuously. This is a difference between hop, hop, hop in the discrete case and run forward continuously in the differential case. In both cases, the eigenvectors and the eigenvalues are the guide to what happens as time goes forward.

OK. I have to do more about working with matrices. Let me come to that next. Thanks.