Topic 2

Basic Considerations in Nonlinear Analysis

Contents:

- **The principle of virtual work in general nonlinear analysis, including all material and geometric nonlinearities**
- **A simple instructive example**
- **Introduction to the finite element incremental solution, statement and physical explanation of governing finite element equations**
- **Requirements of equilibrium, compatibility, and the stress-strain law**
- **Nodal point equilibrium versus local equilibrium**
- **Assessment of accuracy of a solution**
- **Example analysis: Stress concentration factor calculation for a plate with a hole in tension**
- **Example analysis: Fracture mechanics stress intensity factor calculation for a plate with an eccentric crack in tension**
- **Discussion of mesh evaluation by studying stress jumps along element boundaries and pressure band plots**

Textbook: Section 6.1

Examples: 6.1, 6.2, 6.3, 6.4

References: The evaluation of finite element solutions is studied in

Sussman, T., and K. J. Bathe, "Studies of Finite Element Procedures—On Mesh Selection," *Computers & Structures, 21,* 257–264, 1985.

Sussman, T., and K. J. Bathe, "Studies of Finite Element Procedures—Stress Band Plots and the Evaluation of Finite Element Meshes," *Engineering Computations,* to appear.

IN THIS LECTURE

- WE DISCUSS THE PRINCIPLE OF VIRTUAL WORK USED FOR GENERAL NONLINEAR ANALYSIS
- WE EMPHASIZE THE BASIC REQUIREMENTS OF MECHANICS
- WE GIVE EXAMPLE ANALYSES
  - PLATE WITH HOLE
  - PLATE WITH CRACK

Markerboard 2-1

Transparency 2-1

THE PRINCIPLE OF VIRTUAL WORK

$$\int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = {}^t\mathcal{R}$$

where

$${}^t\mathcal{R} = \int_{^tV} {}^t f_i^B\, \delta u_i\, {}^t dV + \int_{^tS} {}^t f_i^S\, \delta u_i^S\, {}^t dS$$

${}^t\tau_{ij}$ = forces per unit area at time t (Cauchy stresses)

$$\delta_t e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial {}^t x_j} + \frac{\partial \delta u_j}{\partial {}^t x_i}\right)$$

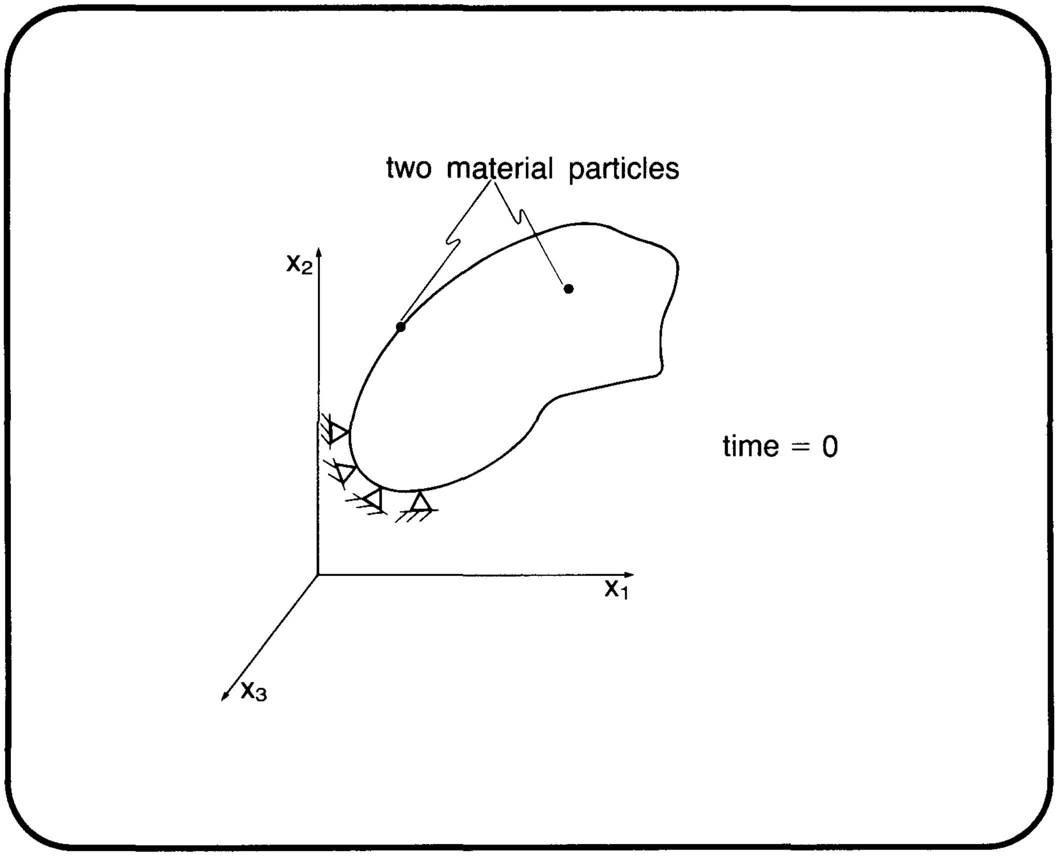
Transparency 2-2

and

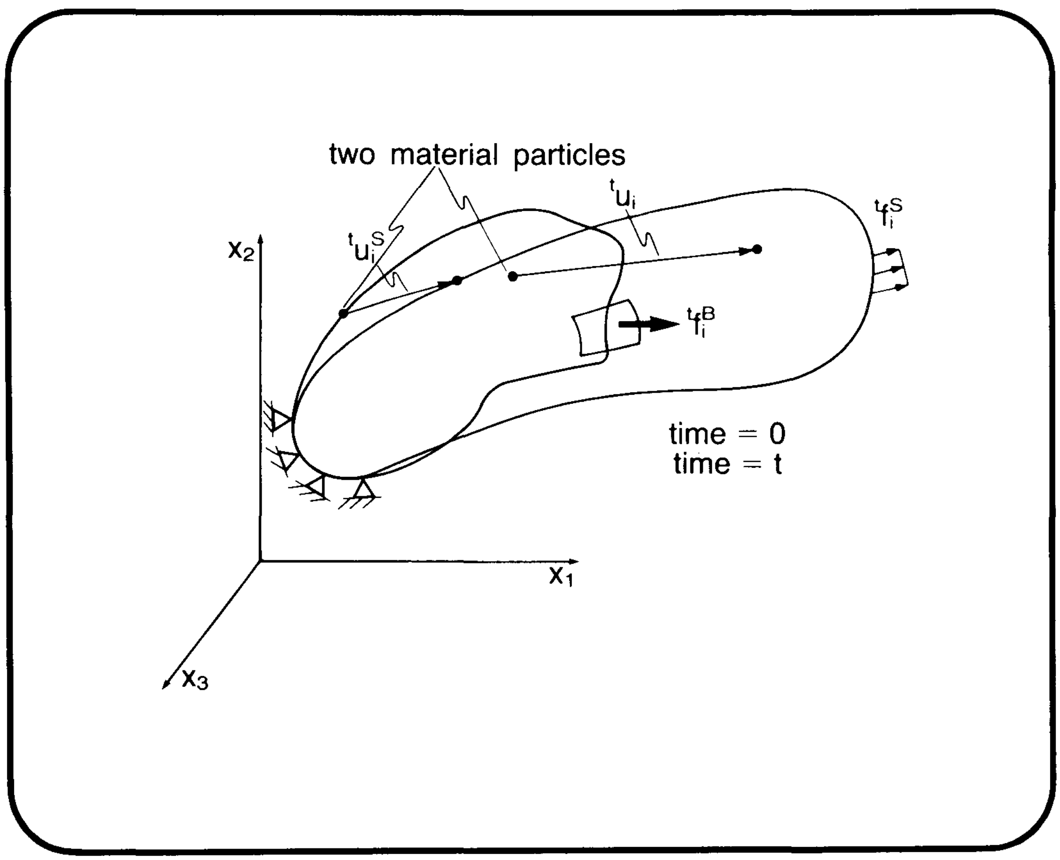
δu_i, $\delta_t e_{ij}$ = virtual displacements and corresponding virtual strains

${}^tV$, ${}^tS$ = volume and surface area at time t

${}^t f_i^B$, ${}^t f_i^S$ = externally applied forces per unit current volume and unit current area

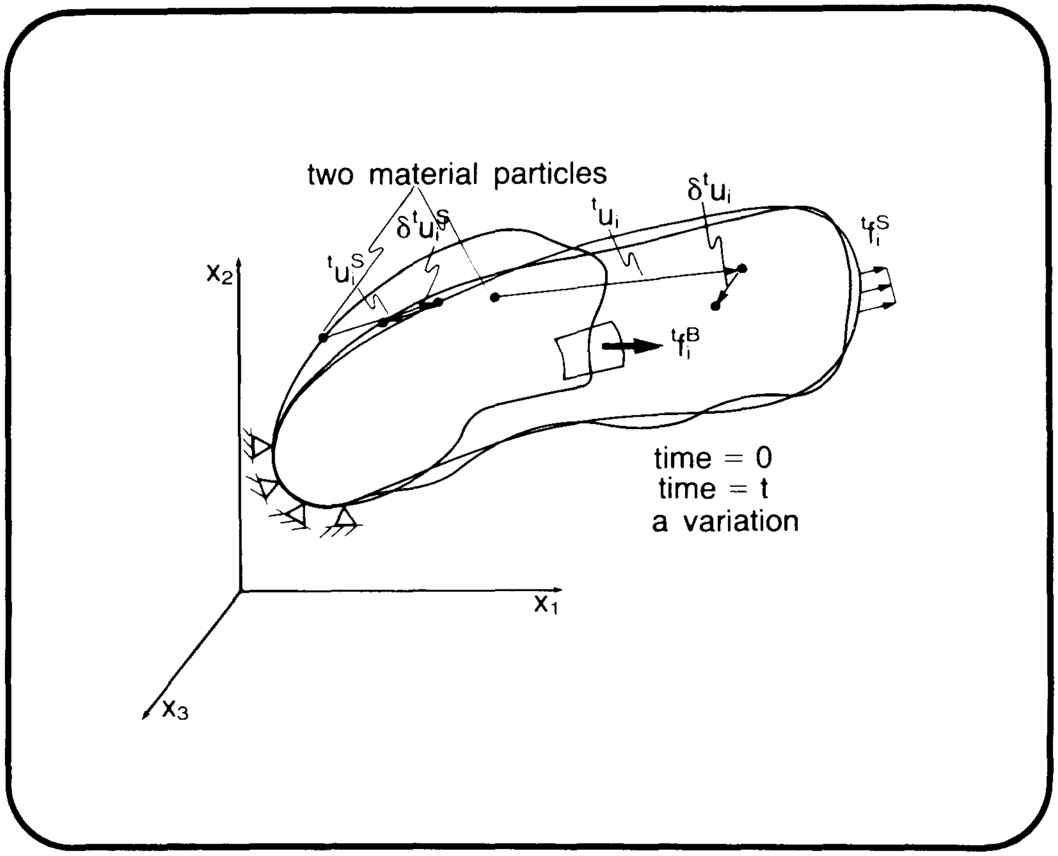


Transparency 2-3

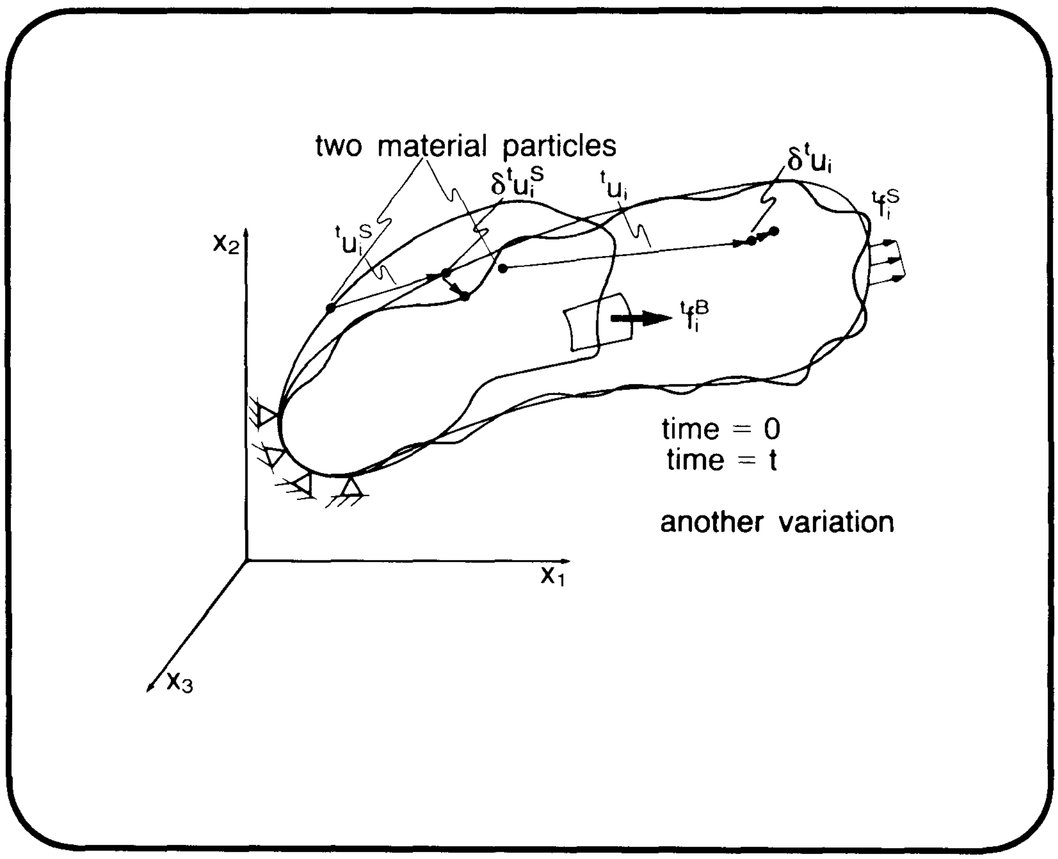


Transparency 2-4

Transparency 2-5



Transparency 2-6



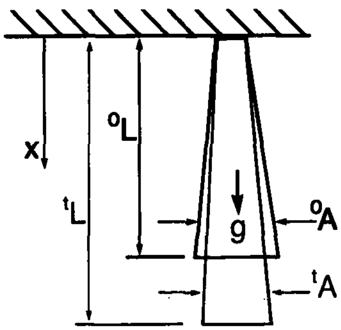
Note: Integrating the principle of virtual work by parts gives

- Governing differential equations of motion
- Plus force (natural) boundary conditions

just like in infinitesimal displacement analysis.

Transparency 2-7

Example: Truss stretching under its own weight



Assume:

- Plane cross-sections remain plane
- Constant uniaxial stress on each cross-section

We then have a one-dimensional analysis.

Transparency 2-8

Transparency 2-9

Using these assumptions,

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = \int_{{}^tL} {}^t\tau\, \delta_t e\, {}^tA\, {}^t dx\,,$$

$${}^t\mathcal{R} = \int_{{}^tL} {}^t\rho g\, \delta u\, {}^tA\, {}^t dx$$

Hence the principle of virtual work is now

$$\int_{{}^tL} {}^t\tau\, {}^tA\, \delta_t e\, {}^t dx = \int_{{}^tL} {}^t\rho g\, {}^tA\, \delta u\, {}^t dx$$

where

$$\delta_t e = \frac{\partial \delta u}{\partial\, {}^t x}$$

Transparency 2-10

We now recover the differential equation of equilibrium using integration by parts:

$$\int_{{}^tL} \left[\frac{\partial}{\partial\, {}^t x} ({}^t\tau\, {}^tA) + {}^t\rho g\, {}^tA \right] \delta u\, {}^t dx - [({}^t\tau\, {}^tA)\delta u]\Big|_0^{{}^tL} = 0$$

Since the variations δu are arbitrary (except at $x = 0$), we obtain

$$\frac{\partial}{\partial\, {}^t x} ({}^t\tau\, {}^tA) + {}^t\rho g\, {}^tA = 0\,, \qquad ({}^t\tau\, {}^tA)\Big|_{{}^tL} = 0$$

THE GOVERNING DIFFERENTIAL EQUATION

THE FORCE (NATURAL) BOUNDARY CONDITION

FINITE ELEMENT APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = \int_{{}^tV} {}^t f_i^B\, \delta u_i\, {}^t dV + \int_{{}^tS} {}^t f_i^S\, \delta u_i^S\, {}^t dS$$

↓

BY THE FINITE ELEMENT METHOD

↓

$$\delta \underline{U}^T\, {}^t\underline{F} = \delta \underline{U}^T\, {}^t\underline{R}$$

Transparency 2-11

- Now assume that the solution at time t is known. Hence ${}^t\tau_{ij}$, ${}^tV$, . . . are known.
- We want to obtain the solution corresponding to time $t+\Delta t$ (i.e., for the loads applied at time $t+\Delta t$).
- The principle of virtual work gives for time $t+\Delta t$

$$^{t+\Delta t}\underline{F} = {}^{t+\Delta t}\underline{R}$$

Transparency 2-12

Transparency 2-13

To solve for the unknown state at time t+Δt, we assume

$$^{t+\Delta t}\underline{F} = {}^{t}\underline{F} + {}^{t}\underline{K}\,\Delta\underline{U}$$

Hence we solve

$$^{t}\underline{K}\,\Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^{t}\underline{F}$$

and obtain

$$^{t+\Delta t}\underline{U} \doteq {}^{t}\underline{U} + \Delta\underline{U}$$

Transparency 2-14

More generally, we solve

$$^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}$$

$$^{t+\Delta t}\underline{U}^{(i)} = {}^{t+\Delta t}\underline{U}^{(i-1)} + \Delta\underline{U}^{(i)}$$

using

$$^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}\,, \qquad {}^{t+\Delta t}\underline{U}^{(0)} = {}^{t}\underline{U}$$

Transparency 2-15

- Nodal point equilibrium is satisfied when the equation

$$ {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} = \underline{0} $$

is satisfied.

- Compatibility is satisfied provided a compatible element layout is used.

- The stress-strain law enters in the calculation of ${}^{t}\underline{K}$ and ${}^{t+\Delta t}\underline{F}^{(i-1)}$.

Transparency 2-16

Most important is the appropriate calculation of ${}^{t+\Delta t}\underline{F}^{(i-1)}$ from ${}^{t+\Delta t}\underline{U}^{(i-1)}$.

The general procedure is:

$$ {}^{t+\Delta t}\underline{U}^{(i-1)} \xrightarrow{\text{gives}} \text{strains} \xrightarrow{\text{gives}} \text{stresses} \xrightarrow{\text{gives}} {}^{t+\Delta t}\underline{F}^{(i-1)} $$

(CONSTITUTIVE RELATIONS ENTER)

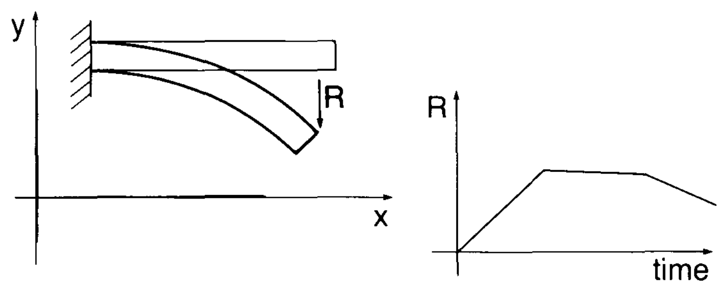
Note:

$$ {}^{t+\Delta t}\underline{\sigma}^{(i-1)} = {}^{t}\underline{\sigma} + \int_{{}^{t}\underline{e}}^{{}^{t+\Delta t}\underline{e}^{(i-1)}} \underline{C}\, d\underline{e} $$

Transparency 2-17

Here we assumed that the nodal point loads are independent of the structural deformations. The loads are given as functions of time only.

Example:



Transparency 2-18

WE SATISFY THE BASIC REQUIREMENTS OF MECHANICS:

Stress-strain law

Need to evaluate the stresses correctly from the strains.

Compatibility

Need to use compatible element meshes and satisfy displacement boundary conditions.

Transparency 2-19

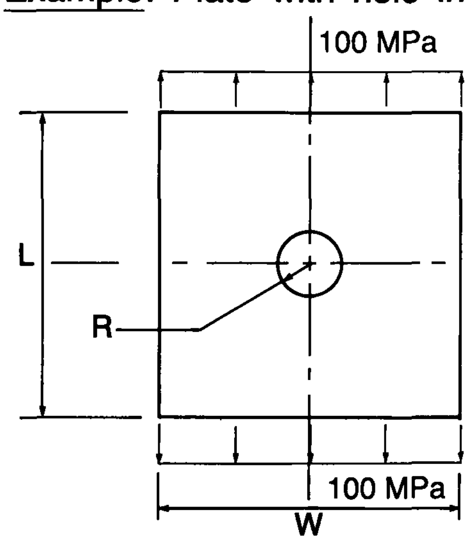
Equilibrium

- Corresponding to the finite element nodal point degrees of freedom (global equilibrium)
- Locally if a fine enough finite element discretization is used

Check:
— Whether the stress boundary conditions are satisfied
— Whether there are no unduly large stress jumps between elements

Transparency 2-20

Example: Plate with hole in tension



$E = 207000$ MPa
$\nu = 0.3$
$\sigma_y = 740$ MPa
$E_T = 2070$ MPa

$R = 0.01$ m
$L = W = 0.1$ m
thickness $= 0.01$ m

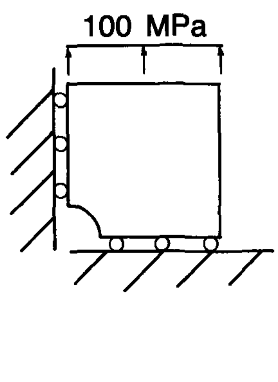
Transparency 2-21

Purpose of analysis:

To accurately determine the stresses in the plate, assuming that the load is small enough so that a linear elastic analysis may be performed.

Transparency 2-22

Using symmetry, we only need to model one quarter of the plate:



Transparency 2-23

Accuracy considerations:

Recall, in a displacement-based finite element solution,

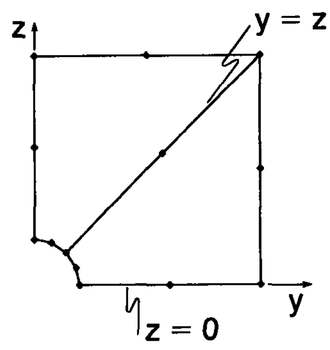
- Compatibility is satisfied.
- The material law is satisfied.
- Equilibrium (locally) is only approximately satisfied.

We can observe the equilibrium error by plotting stress discontinuities.

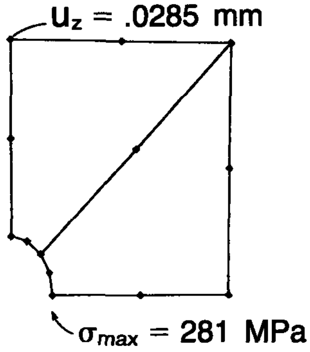
Transparency 2-24

Two element mesh: All elements are two-dimensional 8-node isoparametric elements.

Undeformed mesh:

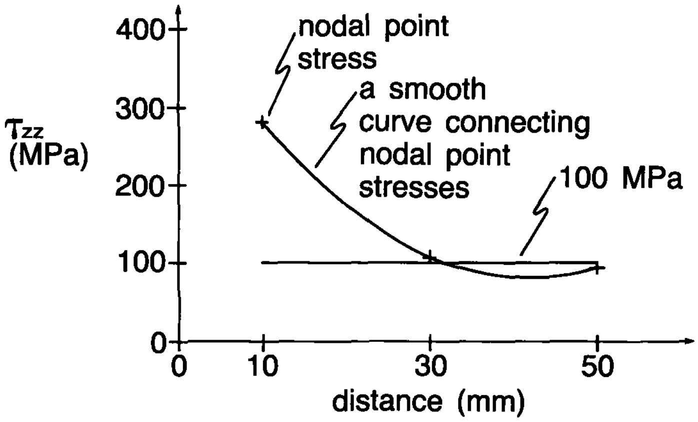


Deformed mesh (displacements amplified):



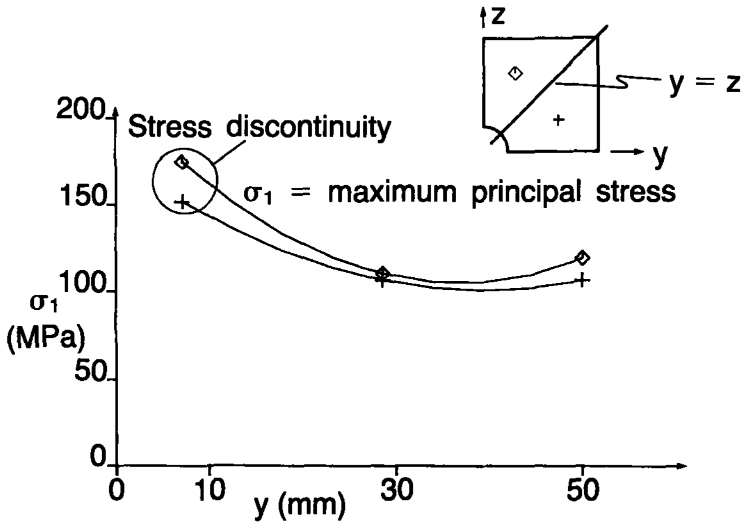
Transparency 2-25

Plot stresses (evaluated at the nodal points) along the line z=0:



Transparency 2-26

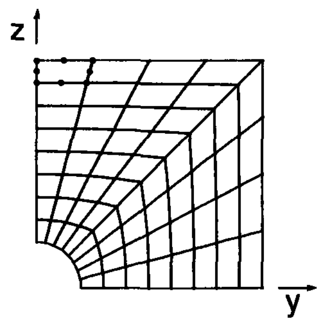
Plot stresses along the line y = z:



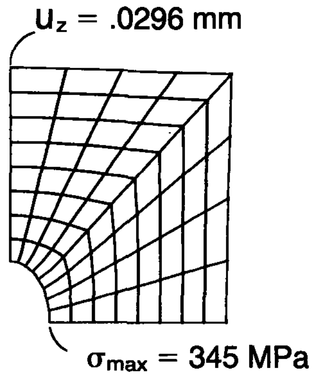
Sixty-four element mesh: All elements are two-dimensional 8-node isoparametric elements.

Transparency 2-27

Undeformed mesh:

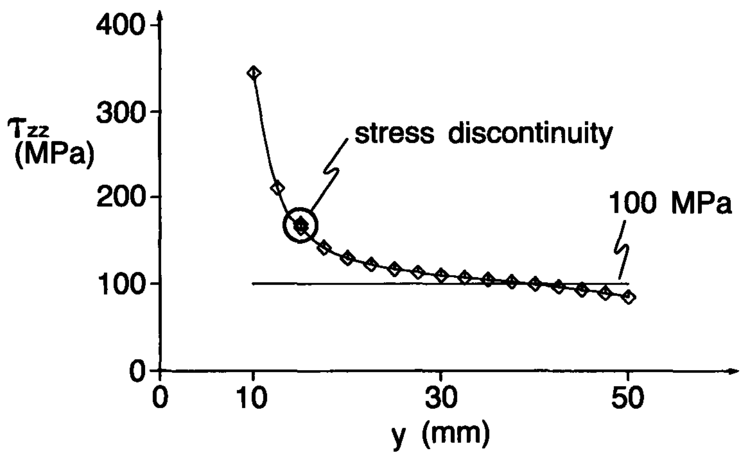


Deformed mesh (displacements amplified):



Plot stresses along the line z=0:

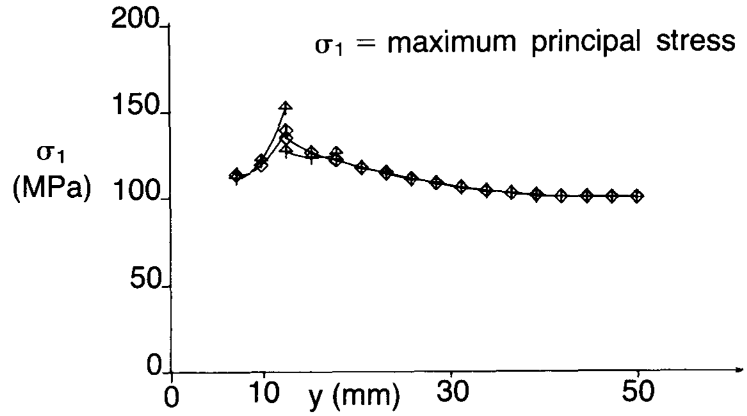
Transparency 2-28



Transparency 2-29

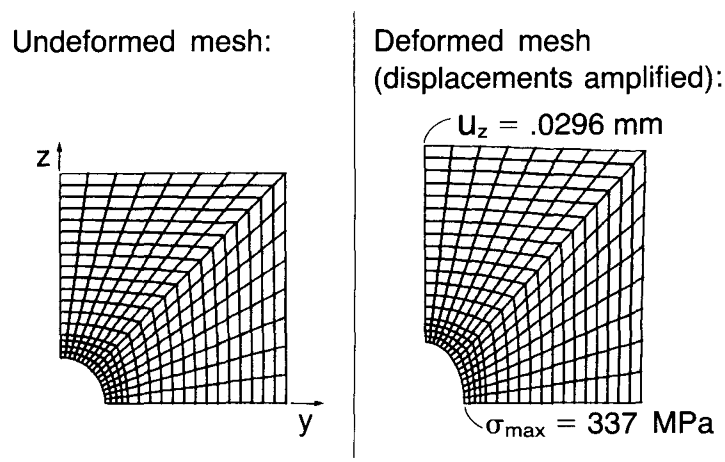
Plot stresses along the line y = z:

The stress discontinuities are negligible for y > 20 mm.

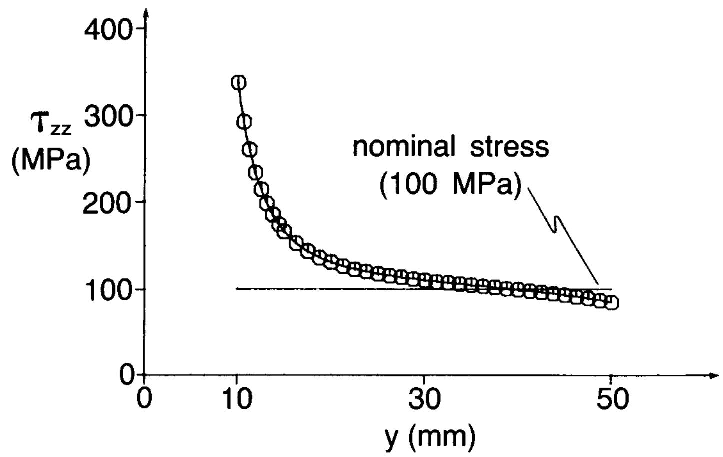


Transparency 2-30

288 element mesh: All elements are two-dimensional 8-node elements.



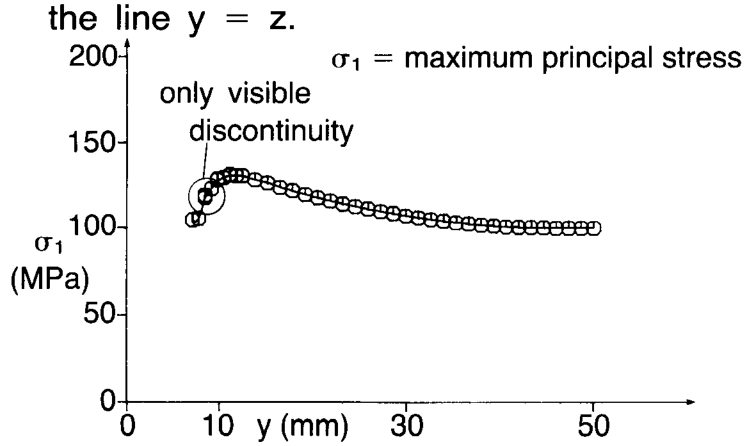
Plot stresses along the line z = 0:



Transparency 2-31

Plot stresses along the line y = z:

- There are no visible stress discontinuities between elements on opposite sides of the line y = z.



Transparency 2-32

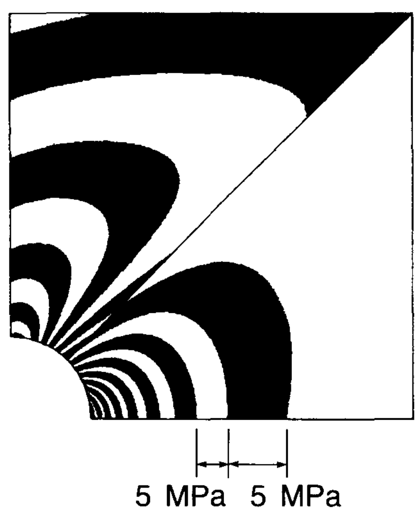
Transparency 2-33

- To be confident that the stress discontinuities are small everywhere, we should plot stress jumps along each line in the mesh.
- An alternative way of presenting stress discontinuities is by means of a pressure band plot:
  - — Plot bands of constant pressure where

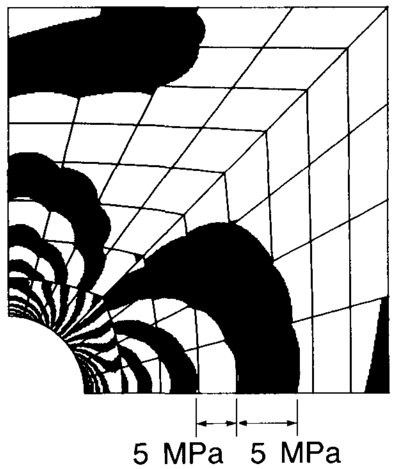
$$\text{pressure} = \frac{-(\tau_{xx} + \tau_{yy} + \tau_{zz})}{3}$$

Transparency 2-34

Two element mesh: Pressure band plot

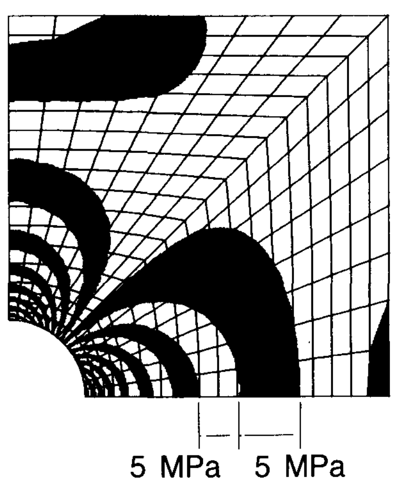


Sixty-four element mesh: Pressure band plot



Transparency 2-35

288 element mesh: Pressure band plot



Transparency 2-36

Transparency 2-37

We see that stress discontinuities are represented by breaks in the pressure bands. As the mesh is refined, the pressure bands become smoother.

— The stress state everywhere in the mesh is represented by one picture.

— The pressure band plot may be drawn by a computer program.

— However, actual magnitudes of pressures are not directly displayed.

Transparency 2-38

Summary of results for plate with hole meshes:

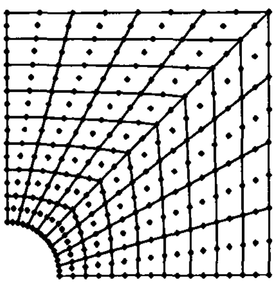
Number of elements	Degrees of freedom	Relative cost	Displacement at top (mm)	Stress concentration factor
2	20	0.08	.0285	2.81
64	416	1.0	.0296	3.45
288	1792	7.2	.0296	3.37

Transparency 2-39

- Two element mesh cannot be used for stress predictions.
- Sixty-four element mesh gives reasonably accurate stresses. However, further refinement at the hole is probably desirable.
- 288 element mesh is overrefined for linear elastic stress analysis. However, this refinement may be necessary for other types of analyses.

Transparency 2-40

Now consider the effect of using 9-node isoparametric elements. Consider the 64 element mesh discussed earlier, where each element is a 9-node element:



Will the solution improve significantly?

Transparency 2-41

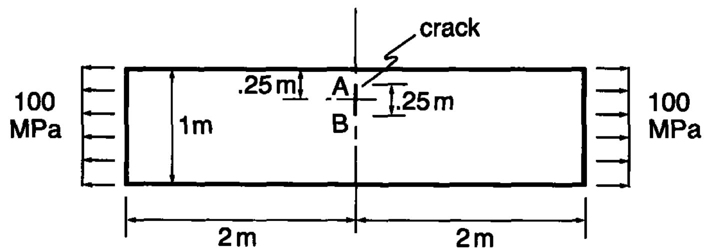
No, the answers do not improve significantly:

	Sixty-four 8-node elements	Sixty-four 9-node elements
Number of degrees of freedom	416	544
Displacement at top (mm)	.029576	.029577
Stress concentration factor	3.452	3.451

The stress jump and pressure band plots do not change significantly.

Transparency 2-42

Example: Plate with eccentric crack in tension



$E = 207000$ MPa thickness = 0.01 m
$\nu = 0.3$ plane stress
$K_c = 110$ MPa $\sqrt{m}$

- Will the crack propagate?

Background:

Assuming that the theory of linear elastic fracture mechanics is applicable, we have

K_I = stress intensity factor for a mode I crack

K_I determines the "strength" of the $1/\sqrt{r}$ stress singularity at the crack tip.

$K_I > K_C$ — crack will propagate

(K_C is a property of the material)

Transparency 2-43

Computation of K_I: From energy considerations, we have for plane stress situations

$$K_I = \sqrt{EG} \ , \ G = -\frac{\partial \Pi}{\partial A}$$

where Π = total potential energy

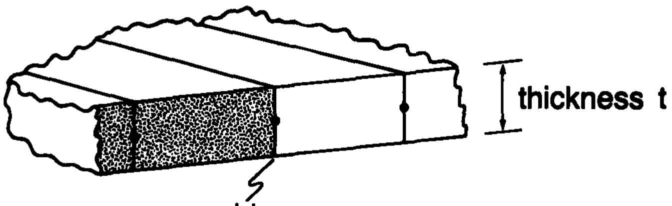
A = area of the crack surface

G is known as the "energy release rate" for the crack.

Transparency 2-44

Transparency 2-45

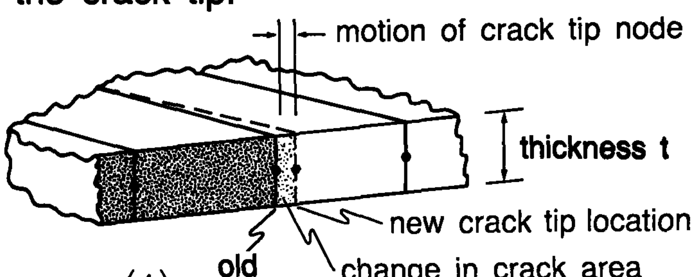
In this finite element analysis, each crack tip is represented by a node. Hence the change in the area of the crack may be written in terms of the motion of the node at the crack tip.



$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial \ell}\left(\frac{1}{t}\right)$$

Transparency 2-46

In this finite element analysis, each crack tip is represented by a node. Hence the change in the area of the crack may be written in terms of the motion of the node at the crack tip.



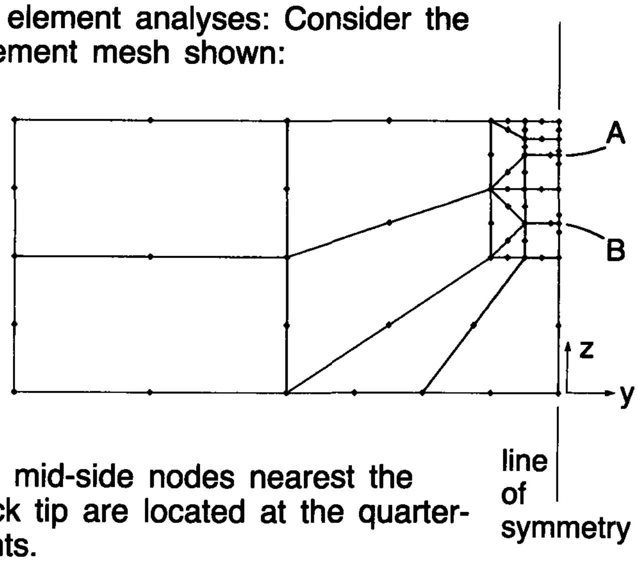
$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial \ell}\left(\frac{1}{t}\right)$$

The quantities $\frac{\partial \Pi}{\partial \ell}$ may be efficiently computed using equations based on the chain differentiation of the total potential with respect to the nodal coordinates describing the crack tip. This computation is performed at the end of (but as part of) the finite element analysis.

See T. Sussman and K. J. Bathe, "The Gradient of the Finite Element Variational Indicator with Respect to Nodal Point Coordinates . . . ", Int. J. Num. Meth. Engng. Vol. 21, 763–774 (1985).

Transparency 2-47

Finite element analyses: Consider the 17 element mesh shown:

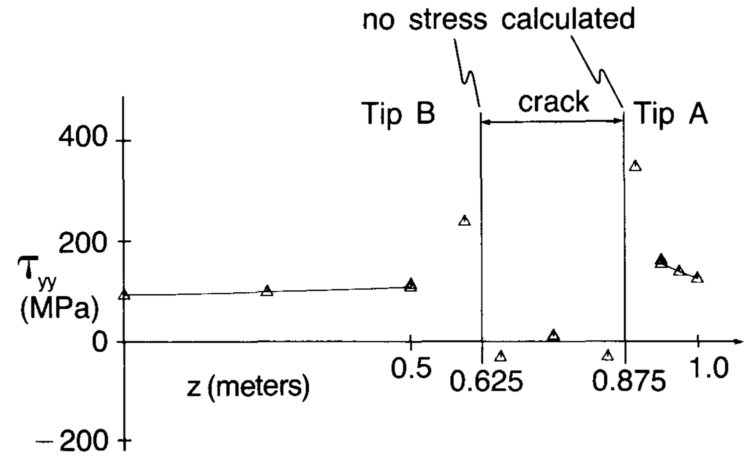


- The mid-side nodes nearest the crack tip are located at the quarter-points.

Transparency 2-48

Transparency 2-49

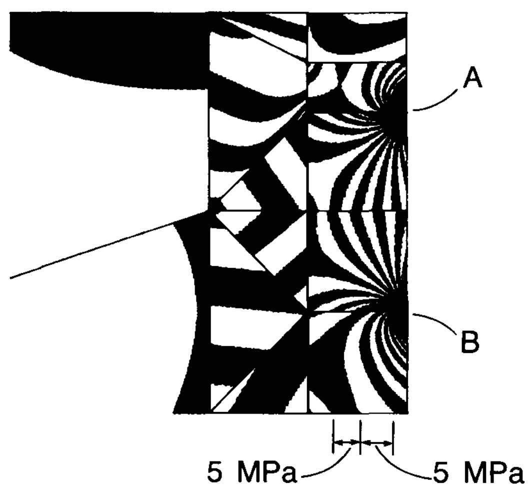
Results: Plot of stresses on line of symmetry for 17 element mesh.



Transparency 2-50

Pressure band plot (detail):

- The pressure jumps are larger than 5 MPa.



Based on the pressure band plot, we conclude that the mesh is too coarse for accurate stress prediction.

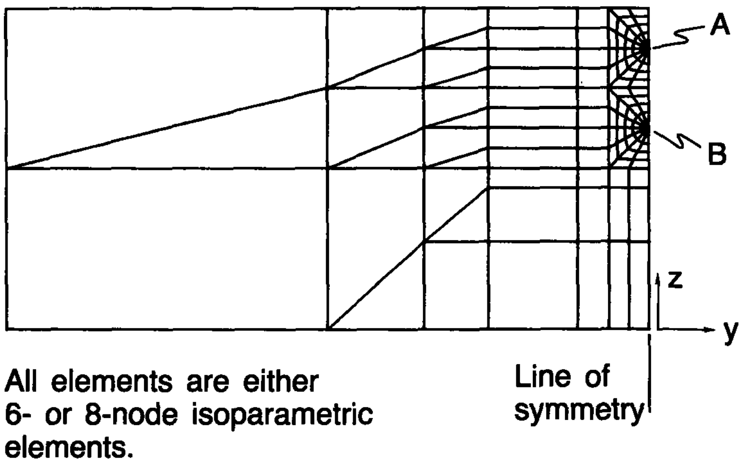
However, good results are obtained for the stress intensity factors (when they are calculated as described earlier):

$K_A = 72.6$ MPa$\sqrt{m}$ (analytical solution $= 72.7$ MPa$\sqrt{m}$)

$K_B = 64.5$ MPa$\sqrt{m}$ (analytical solution $= 68.9$ MPa$\sqrt{m}$)

Transparency 2-51

Now consider the 128 element mesh shown:

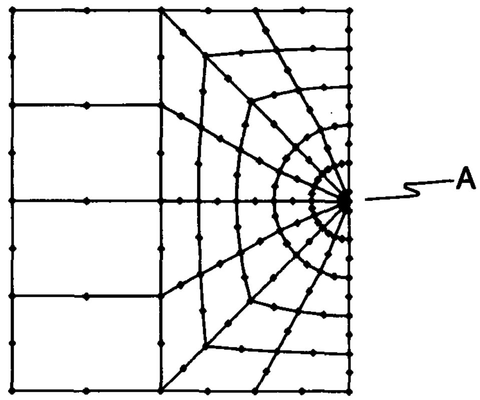


All elements are either 6- or 8-node isoparametric elements.

Transparency 2-52

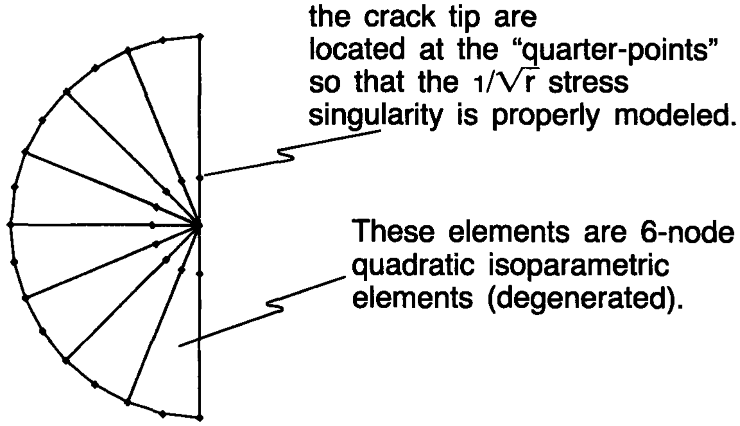
Transparency 2-53

Detail of 128 element mesh:

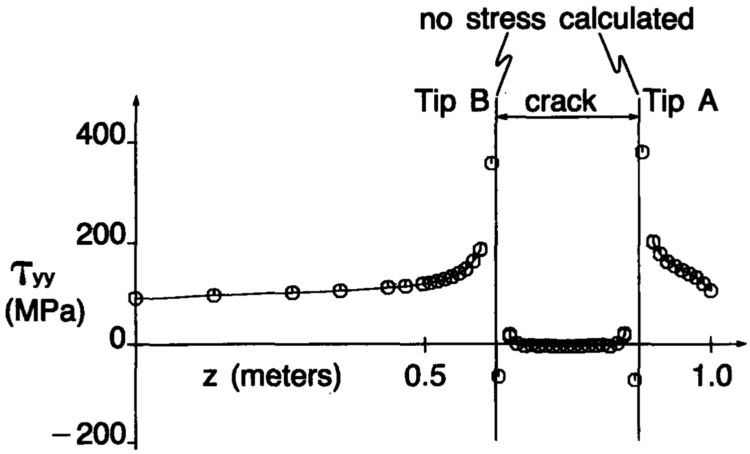


Transparency 2-54

Close-up of crack tip A:



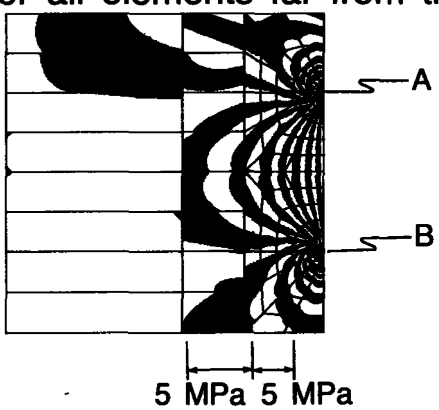
Results: Stress plot on line of symmetry for 128 element mesh.



Transparency 2-55

Pressure band plot (detail) for 128 element mesh:

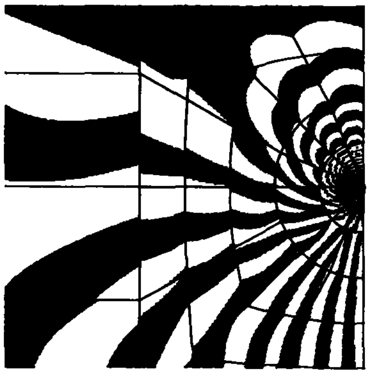
- The pressure jumps are smaller than 5 MPa for all elements far from the crack tips.



Transparency 2-56

Transparency 2-57

A close-up shows that the stress jumps are larger than 5 MPa in the first and second rings of elements surrounding crack tip A.



A

Transparency 2-58

Based on the pressure band plot, we conclude that the mesh is fine enough for accurate stress calculation (except for the elements near the crack tip nodes).

We also obtain good results for the stress intensity factors:

$K_A = 72.5\ \text{MPa}\sqrt{\text{m}}$ (analytical solution $= 72.7\ \text{MPa}\sqrt{\text{m}}$)
$K_B = 68.8\ \text{MPa}\sqrt{\text{m}}$ (analytical solution $= 68.9\ \text{MPa}\sqrt{\text{m}}$)

We see that the degree of refinement needed for a mesh in linear elastic analysis is dependent upon the type of result desired.

- Displacements — coarse mesh
- Stress intensity factors — coarse mesh
- Lowest natural frequencies and associated mode shapes — coarse mesh
- Stresses — fine mesh

General nonlinear analysis — usually fine mesh

Transparency 2-59