Topic 3

Lagrangian Continuum Mechanics Variables for General Nonlinear Analysis

Contents:

- **The principle of virtual work in terms of the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors**
- **Deformation gradient tensor**
- **Physical interpretation of the deformation gradient**
- **Change of mass density**
- **Polar decomposition of deformation gradient**
- **Green-Lagrange strain tensor**
- **Second Piola-Kirchhoff stress tensor**
- **Important properties of the Green-Lagrange strain and 2nd Piola-Kirchhoff stress tensors**
- **Physical explanations of continuum mechanics variables**
- **Examples demonstrating the properties of the continuum mechanics variables**

Textbook: Sections 6.2.1, 6.2.2

Examples: 6.5, 6.6, 6.7, 6.8, 6.10, 6.11, 6.12, 6.13, 6.14

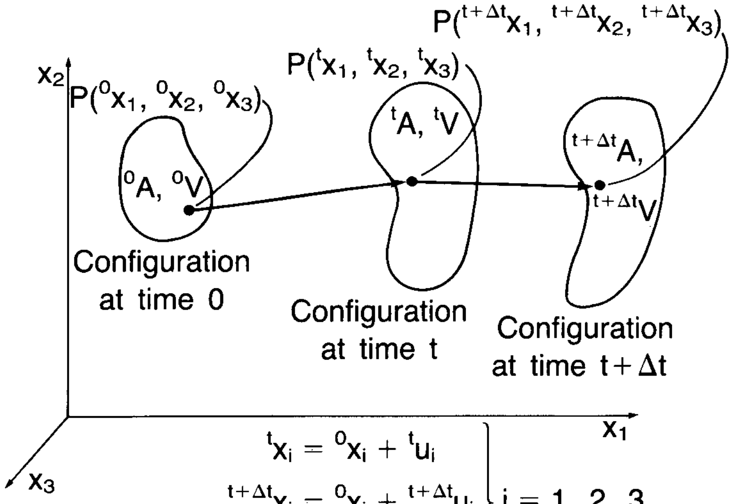
CONTINUUM MECHANICS FORMULATION

For
- Large displacements
- Large rotations
- Large strains

Hence we consider a body subjected to arbitrary large motions,

We use a Lagrangian description.

Transparency 3-1



$$\left.\begin{aligned} {}^{t}x_i &= {}^{0}x_i + {}^{t}u_i \\ {}^{t+\Delta t}x_i &= {}^{0}x_i + {}^{t+\Delta t}u_i \\ u_i &= {}^{t+\Delta t}u_i - {}^{t}u_i \end{aligned}\right\} i = 1,\ 2,\ 3$$

Transparency 3-2

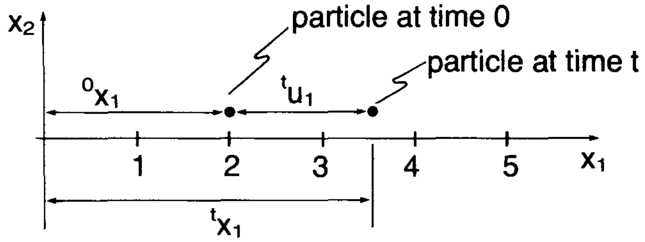
Transparency 3-3

Regarding the notation we need to keep firmly in mind that

— the Cartesian axes are stationary.

— the unit distances along the x_i-axes are the same for ${}^0x_i$, ${}^tx_i$, ${}^{t+\Delta t}x_i$.

Example:



Transparency 3-4

PRINCIPLE OF VIRTUAL WORK

Corresponding to time $t+\Delta t$:

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\; {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

where

$$ {}^{t+\Delta t}\mathcal{R} = \int_{^{t+\Delta t}V} {}^{t+\Delta t}f_i^B\, \delta u_i\; {}^{t+\Delta t}dV + \int_{^{t+\Delta t}S} {}^{t+\Delta t}f_i^S\, \delta u_i^S\; {}^{t+\Delta t}dS$$

Transparency 3-5

and

${}^{t+\Delta t}\tau_{ij}$ = Cauchy stresses (forces/unit area at time $t+\Delta t$)

$$\delta_{t+\Delta t}e_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial^{t+\Delta t}x_j} + \frac{\partial \delta u_j}{\partial^{t+\Delta t}x_i}\right)$$

= variation in the small strains referred to the configuration at time $t+\Delta t$

Transparency 3-6

We need to rewrite the principle of virtual work, using new stress and strain measures:

- We cannot integrate over an unknown volume.
- We cannot directly work with increments in the Cauchy stresses.

We introduce:

${}_0^t\underline{S}$ = 2nd Piola-Kirchhoff stress tensor

${}_0^t\underline{\varepsilon}$ = Green-Lagrange strain tensor

Transparency 3-7

The 2nd Piola-Kirchhoff stress tensor:

$${}^t_0S_{ij} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_tx_{i,m}\, {}^t\tau_{mn}\, {}^0_tx_{j,n}$$

The Green-Lagrange strain tensor:

$${}^t_0\varepsilon_{ij} = \frac{1}{2}\left({}^t_0u_{i,j} + {}^t_0u_{j,i} + {}^t_0u_{k,i}\, {}^t_0u_{k,j}\right)$$

where $${}^0_tx_{i,m} = \frac{\partial\, {}^0x_i}{\partial\, {}^tx_m}\ , \quad {}^t_0u_{i,j} = \frac{\partial\, {}^tu_i}{\partial\, {}^0x_j}$$

Transparency 3-8

Note: We are using the indicial notation with the summation convention.

For example,

$$\begin{aligned} {}^t_0S_{11} = \frac{{}^0\rho}{{}^t\rho} \,[\, & {}^0_tx_{1,1}\, {}^t\tau_{11}\, {}^0_tx_{1,1} \\ & + {}^0_tx_{1,1}\, {}^t\tau_{12}\, {}^0_tx_{1,2} \\ & + \ldots \\ & + {}^0_tx_{1,3}\, {}^t\tau_{33}\, {}^0_tx_{1,3}\,] \end{aligned}$$

Transparency 3-9

Using the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors, we have

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV = \int_{{}^0V} {}_0^tS_{ij}\, \delta\, {}_0^t\varepsilon_{ij}\, {}^0 dV$$

This relation holds for all times

$\Delta t, 2\Delta t, \ldots, t, t+\Delta t, \ldots$

Transparency 3-10

To develop the incremental finite element equations we will use

$$\int_{{}^0V} {}_0^{t+\Delta t}S_{ij}\, \delta\, {}_0^{t+\Delta t}\varepsilon_{ij}\, {}^0 dV = {}^{t+\Delta t}\mathcal{R}$$

- We now integrate over a known volume, ${}^0V$.
- We can incrementally decompose ${}_0^{t+\Delta t}S_{ij}$ and ${}_0^{t+\Delta t}\varepsilon_{ij}$, i.e.

$$ {}_0^{t+\Delta t}S_{ij} = {}_0^tS_{ij} + {}_0S_{ij}$$

$$ {}_0^{t+\Delta t}\varepsilon_{ij} = {}_0^t\varepsilon_{ij} + {}_0\varepsilon_{ij}$$

Transparency 3-11

Before developing the incremental continuum mechanics and finite element equations, we want to discuss

- some important kinematic relationships used in geometric nonlinear analysis
- some properties of the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors

Transparency 3-12

To explain some important properties of the 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor, we consider the

Deformation Gradient Tensor

- This tensor captures the straining and the rigid body rotations of the material fibers.
- It is a very fundamental quantity used in continuum mechanics.

Transparency 3-13

The deformation gradient is defined as

$$ {}_0^t\underline{X} = \begin{bmatrix} \frac{\partial\,{}^tx_1}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_1}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_1}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^tx_2}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_2}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_2}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^tx_3}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_3}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_3}{\partial\,{}^0x_3} \end{bmatrix} $$

in a Cartesian coordinate system

Using indicial notation,

$$ {}_0^tX_{ij} = \frac{\partial\,{}^tx_i}{\partial\,{}^0x_j} = {}_0^tx_{i,j} $$

Transparency 3-14

Another way to write the deformation gradient:

$$ {}_0^t\underline{X} = ({}_0\underline{\nabla}\,{}^t\underline{x}^T)^T $$

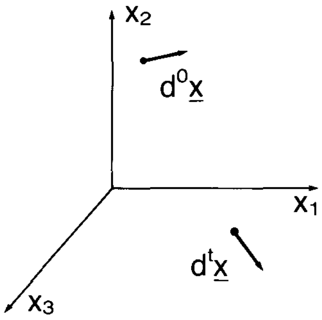
where

$$ {}_0\underline{\nabla} = \begin{bmatrix} \frac{\partial}{\partial\,{}^0x_1} \\ \frac{\partial}{\partial\,{}^0x_2} \\ \frac{\partial}{\partial\,{}^0x_3} \end{bmatrix} , \quad {}^t\underline{x}^T = [{}^tx_1 \quad {}^tx_2 \quad {}^tx_3] $$

the gradient operator

Transparency 3-15

The deformation gradient describes the deformations (rotations and stretches) of material fibers:

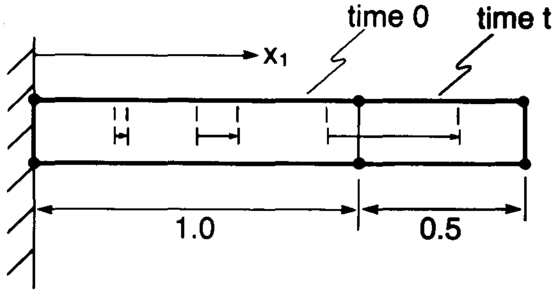


The vectors $d^0\underline{x}$ and $d^t\underline{x}$ represent the orientation and length of a material fiber at times 0 and t. They are related by

$$d^t\underline{x} = {}_0^t\underline{X}\ d^0\underline{x}$$

Transparency 3-16

Example: One-dimensional deformation



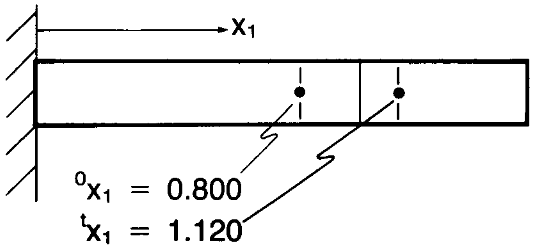
Deformation field: ${}^t x_1 = {}^0 x_1 + 0.5({}^0 x_1)^2$

$$\Big\downarrow$$

$${}_0^t X_{11} = \frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1} = 1 + {}^0 x_1$$

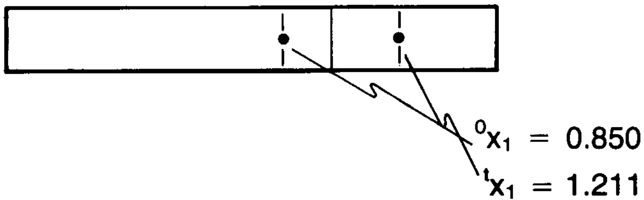
Transparency 3-17

Consider a material particle initially at $x_1 = 0.8$:



Transparency 3-18

Consider an adjacent material particle:



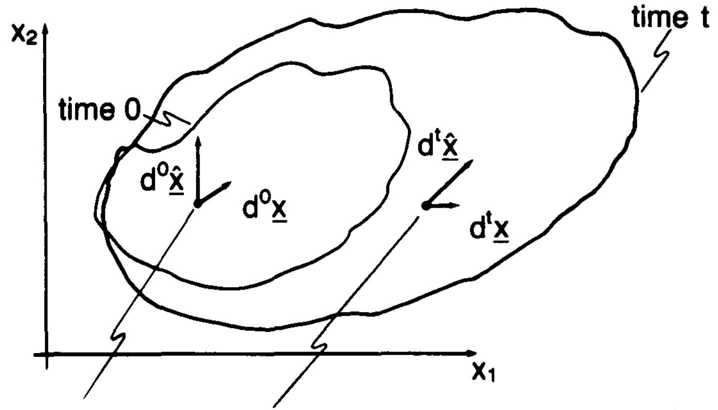
Compute ${}^t_0X_{11}$:

$$\frac{\Delta {}^tx_1}{\Delta {}^0x_1} = \frac{1.211 - 1.120}{.850 - .800} = 1.82 \leftarrow \text{Estimate}$$

$$\left. {}^t_0X_{11} \right|_{{}^0x_1 = 0.8} = 1.80$$

Transparency 3-19

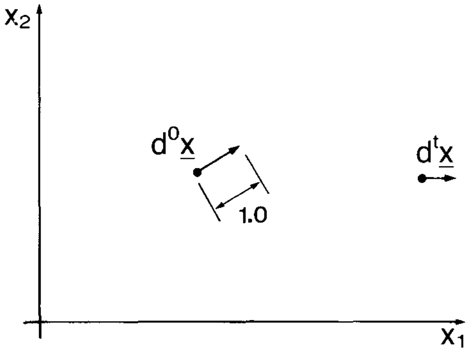
Example: Two-dimensional deformation



$$\overline{({}^0x_1,\ {}^0x_2)} \rightarrow \overline{({}^tx_1,\ {}^tx_2)}:\ {}_0^t\underline{X} = \begin{bmatrix} .481 & .667 \\ -.385 & .667 \end{bmatrix}$$

Transparency 3-20

Considering $d^0\underline{x}$,

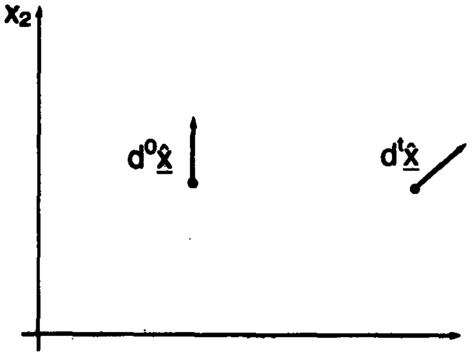


$$\begin{array}{ccc} d^t\underline{x} & = & {}_0^t\underline{X} \qquad d^0\underline{x} \end{array}$$

$$\begin{bmatrix} .75 \\ 0 \end{bmatrix} = \begin{bmatrix} .481 & .667 \\ -.385 & .667 \end{bmatrix} \begin{bmatrix} .866 \\ .5 \end{bmatrix}$$

Transparency 3-21

Considering $d^0\underset{\sim}{x}$,

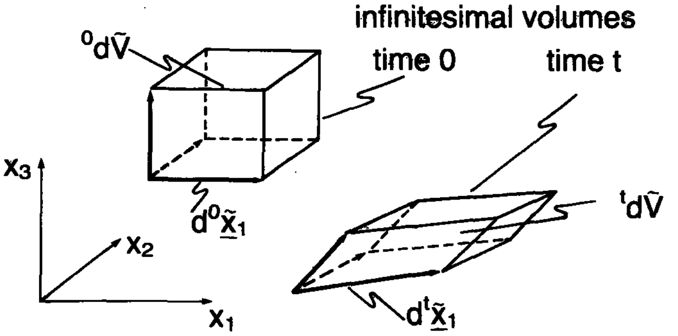


$$d^t\underset{\sim}{x} = {}^t_0\underset{\sim}{X}\; d^0\underset{\sim}{x}$$

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}.481 & .667\\ -.385 & .667\end{bmatrix}\begin{bmatrix}0\\1.5\end{bmatrix}$$

Transparency 3-22

The mass densities ${}^0\rho$ and ${}^t\rho$ may be related using the deformation gradient:



Three material fibers describe each volume.

Transparency 3-23

For an infinitesimal volume, we note that mass is conserved:

$$ {}^{t}\rho \, {}^{t}d\tilde{V} = {}^{0}\rho \, {}^{0}d\tilde{V} $$

volume at time t

volume at time 0

However, we can show that

$$ {}^{t}d\tilde{V} = \det {}_{0}^{t}\underline{X} \; {}^{0}d\tilde{V} $$

Hence

$$ {}^{0}\rho = {}^{t}\rho \; \det {}_{0}^{t}\underline{X} $$

Transparency 3-24

Proof that ${}^{t}d\tilde{V} = \det {}_{0}^{t}\underline{X} \; {}^{0}d\tilde{V}$:

$$ d{}^{0}\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ds_1 \; ; \; d{}^{0}\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ds_2 $$

$$ d{}^{0}\underline{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ds_3 $$

Hence ${}^{0}d\tilde{V} = ds_1 \, ds_2 \, ds_3$.

Transparency 3-25

But $d\,{}^t\underline{x}_i = {}^t_0\underline{X}\, d\,{}^0\underline{x}_i$; i = 1, 2, 3

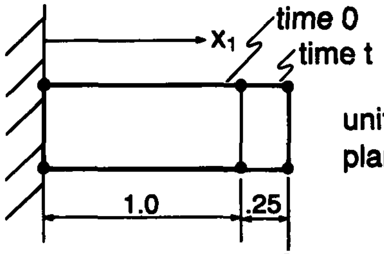
and $${}^t d\tilde{V} = (d\,{}^t\underline{x}_1 \times d\,{}^t\underline{x}_2)\cdot d\,{}^t\underline{x}_3$$

$$= \det {}^t_0\underline{X}\; ds_1\, ds_2\, ds_3$$

$$= \det {}^t_0\underline{X}\; {}^0 d\tilde{V}$$

Transparency 3-26

Example: One-dimensional stretching



uniform stretching
plane strain conditions

Deformation field: ${}^t x_1 = {}^0 x_1 + 0.25\,{}^0 x_1$

Deformation gradient: $${}^t_0\underline{X} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \det {}^t_0\underline{X} = 1.25$$

Hence ${}^0\rho = 1.25\,{}^t\rho$ (${}^t\rho < {}^0\rho$ makes physical sense)

Transparency 3-27

We also use the inverse deformation gradient:

$$d\,{}^{0}\underline{x} = {}^{0}_{t}\underline{X}\; d\,{}^{t}\underline{x}$$

MATERIAL FIBER AT TIME 0

MATERIAL FIBER AT TIME t

Mathematically, ${}^{0}_{t}\underline{X} = ({}^{t}_{0}\underline{X})^{-1}$

Proof:

$$\begin{aligned} d\,{}^{0}\underline{x} &= {}^{0}_{t}\underline{X}\,({}^{t}_{0}\underline{X}\; d\,{}^{0}\underline{x}) \\ &= ({}^{0}_{t}\underline{X}\;{}^{t}_{0}\underline{X})\; d\,{}^{0}\underline{x} \\ &= \underline{I}\; d\,{}^{0}\underline{x} \end{aligned}$$

Transparency 3-28

An important point is:

$$ {}^{t}_{0}\underline{X} = {}^{t}_{0}\underline{R}\;{}^{t}_{0}\underline{U}$$

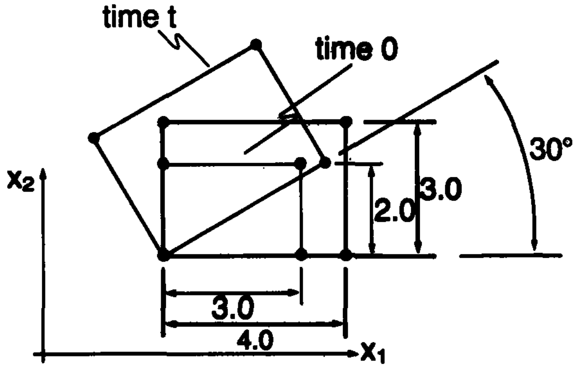
Polar decomposition of ${}^{t}_{0}\underline{X}$:

${}^{t}_{0}\underline{R}$ = orthogonal (rotation) matrix

${}^{t}_{0}\underline{U}$ = symmetric (stretch) matrix

We can always decompose ${}^{t}_{0}\underline{X}$ in the above form.

Example: Uniform stretch and rotation



$${}_0^t\underline{X} = {}_0^t\underline{R}\;{}_0^t\underline{U}$$

$$\begin{bmatrix} 1.154 & -0.750 \\ 0.667 & 1.299 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} \begin{bmatrix} 1.333 & 0 \\ 0 & 1.500 \end{bmatrix}$$

Transparency 3-29

Using the deformation gradient, we can describe the (right) Cauchy-Green deformation tensor

$${}_0^t\underline{C} = {}_0^t\underline{X}^T\;{}_0^t\underline{X}$$

This tensor depends only on the stretch tensor ${}_0^t\underline{U}$:

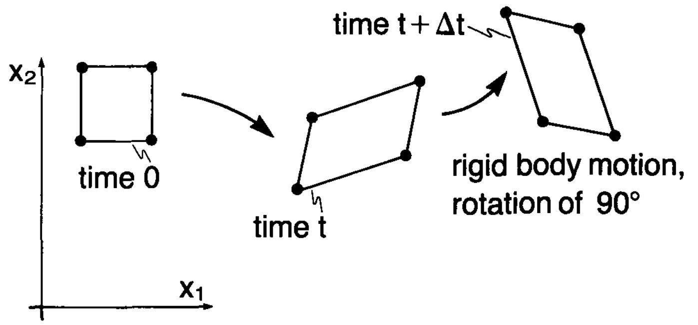
$$\begin{aligned} {}_0^t\underline{C} &= ({}_0^t\underline{U}^T\;{}_0^t\underline{R}^T)\;({}_0^t\underline{R}\;{}_0^t\underline{U}) \\ &= ({}_0^t\underline{U})^2 \quad \text{(since } {}_0^t\underline{R} \text{ is orthogonal)} \end{aligned}$$

Hence ${}_0^t\underline{C}$ is invariant under a rigid body rotation.

Transparency 3-30

Transparency 3-31

Example: Two-dimensional motion



$${}^{t}_{0}\underline{X} = \begin{bmatrix} 1.5 & .2 \\ .5 & 1 \end{bmatrix} \qquad {}^{t+\Delta t}_{0}\underline{X} = \begin{bmatrix} -.5 & -1 \\ 1.5 & .2 \end{bmatrix}$$

$${}^{t}_{0}\underline{C} = \begin{bmatrix} 2.5 & .8 \\ .8 & 1.04 \end{bmatrix} \qquad {}^{t+\Delta t}_{0}\underline{C} = \begin{bmatrix} 2.5 & .8 \\ .8 & 1.04 \end{bmatrix}$$

Transparency 3-32

The Green-Lagrange strain tensor measures the stretching deformations. It can be written in several equivalent forms:

1) ${}^{t}_{0}\underline{\varepsilon} = \frac{1}{2}\left({}^{t}_{0}\underline{C} - \underline{I}\right)$

From this,

- ${}^{t}_{0}\underline{\varepsilon}$ is symmetric.
- For a rigid body motion between times t and t+Δt, ${}^{t+\Delta t}_{0}\underline{\varepsilon} = {}^{t}_{0}\underline{\varepsilon}$.
- For a rigid body motion between times 0 and t, ${}^{t}_{0}\underline{\varepsilon} = \underline{0}$.

Transparency 3-33

- ${}_0^t\underline{\varepsilon}$ is symmetric because ${}_0^t\underline{C}$ is symmetric

$${}_0^t\underline{\varepsilon} = \frac{1}{2}({}_0^t\underline{C} - \underline{I})$$

- For a rigid body motion from t to t+Δt, we have

$${}_0^{t+\Delta t}\underline{X} = \underline{R}\ {}_0^t\underline{X}$$

$${}_0^{t+\Delta t}\underline{C} = {}_0^t\underline{C} \Rightarrow {}_0^{t+\Delta t}\underline{\varepsilon} = {}_0^t\underline{\varepsilon}$$

- For a rigid body motion

$${}_0^t\underline{C} = \underline{I} \Rightarrow {}_0^t\underline{\varepsilon} = \underline{0}$$

Transparency 3-34

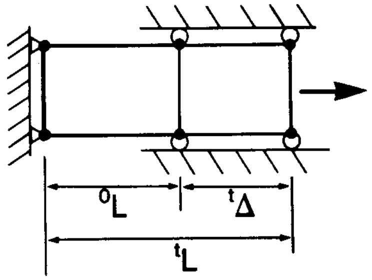
$$2)\ {}_0^t\varepsilon_{ij} = \frac{1}{2}\left(\underbrace{{}_0^tu_{i,j} + {}_0^tu_{j,i}}_{\substack{\text{LINEAR IN}\\ \text{DISPLACEMENTS}}} + \underbrace{{}_0^tu_{k,i}\,{}_0^tu_{k,j}}_{\substack{\text{NONLINEAR IN}\\ \text{DISPLACEMENTS}}}\right)$$

where $${}_0^tu_{i,j} = \frac{\partial\,{}^tu_i}{\partial\,{}^0x_j}$$

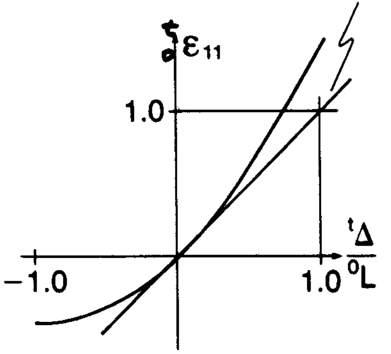
Important point: This strain tensor is exact and holds for any amount of stretching.

Transparency 3-35

Example: Uniaxial strain

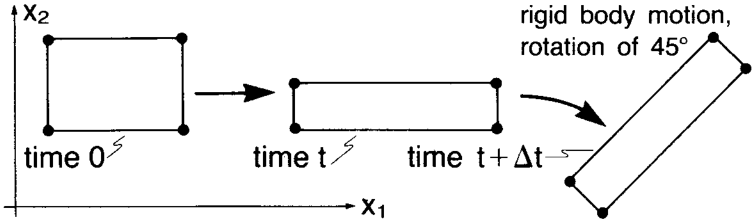


$$ {}^t_0\varepsilon_{11} = \underset{\text{engineering strain}}{\frac{{}^t\Delta}{{}^0L}} + \frac{1}{2}\left(\frac{{}^t\Delta}{{}^0L}\right)^2 $$



Transparency 3-36

Example: Biaxial straining and rotation

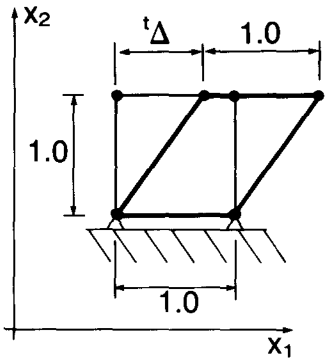


$$ {}^t_0\underline{X} = \begin{bmatrix} 1.5 & 0 \\ 0 & .5 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ \ 0}\underline{X} = \begin{bmatrix} 1.06 & -.354 \\ 1.06 & .354 \end{bmatrix} $$

$$ {}^t_0\underline{C} = \begin{bmatrix} 2.25 & 0 \\ 0 & .25 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ \ 0}\underline{C} = \begin{bmatrix} 2.25 & 0 \\ 0 & .25 \end{bmatrix} $$

$$ {}^t_0\underline{\varepsilon} = \begin{bmatrix} .625 & 0 \\ 0 & -.375 \end{bmatrix} \qquad {}^{t+\Delta t}_{\ \ 0}\underline{\varepsilon} = \begin{bmatrix} .625 & 0 \\ 0 & -.375 \end{bmatrix} $$

<u>Example</u>: Simple shear



$${}^{t}_{0}\underline{X} = \begin{bmatrix} 1.0 & {}^{t}\Delta \\ 0 & 1.0 \end{bmatrix}$$

$${}^{t}_{0}\underline{C} = \begin{bmatrix} 1.0 & {}^{t}\Delta \\ {}^{t}\Delta & 1.0 + ({}^{t}\Delta)^2 \end{bmatrix}$$

$${}^{t}_{0}\underline{\varepsilon} = \begin{bmatrix} 0 & {}^{t}\Delta/2 \\ {}^{t}\Delta/2 & ({}^{t}\Delta)^2/2 \end{bmatrix}$$

For small displacements, ${}^{t}_{0}\underline{\varepsilon}$ is approximately equal to the small strain tensor.

Transparency 3-37

The 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor are <u>energetically</u> <u>conjugate</u>:

${}^{t}\tau_{ij}\,\delta_t e_{ij}$ = Virtual work at time t per unit <u>current</u> volume

${}^{t}_{0}S_{ij}\,\delta\,{}^{t}_{0}\varepsilon_{ij}$ = Virtual work at time t per unit <u>original</u> volume

where ${}^{t}_{0}S_{ij}$ is the 2nd Piola-Kirchhoff stress tensor.

Transparency 3-38

Transparency 3-39

The 2nd Piola-Kirchhoff stress tensor:

$${}^t_0S_{ij} = \frac{{}^0\rho}{{}^t\rho}\; {}^0_tx_{i,m}\; {}^tT_{mn}\; {}^0_tx_{j,n}$$ — INDICIAL NOTATION

$${}^t_0\underline{S} = \frac{{}^0\rho}{{}^t\rho}\; {}^0_t\underline{X}\; {}^t\underline{T}\; {}^0_t\underline{X}^T$$ — MATRIX NOTATION

Solving for the Cauchy stresses gives

$${}^tT_{ij} = \frac{{}^t\rho}{{}^0\rho}\; {}^t_0x_{i,m}\; {}^t_0S_{mn}\; {}^t_0x_{j,n}$$ — INDICIAL NOTATION

$${}^t\underline{T} = \frac{{}^t\rho}{{}^0\rho}\; {}^t_0\underline{X}\; {}^t_0\underline{S}\; {}^t_0\underline{X}^T$$ — MATRIX NOTATION

Transparency 3-40

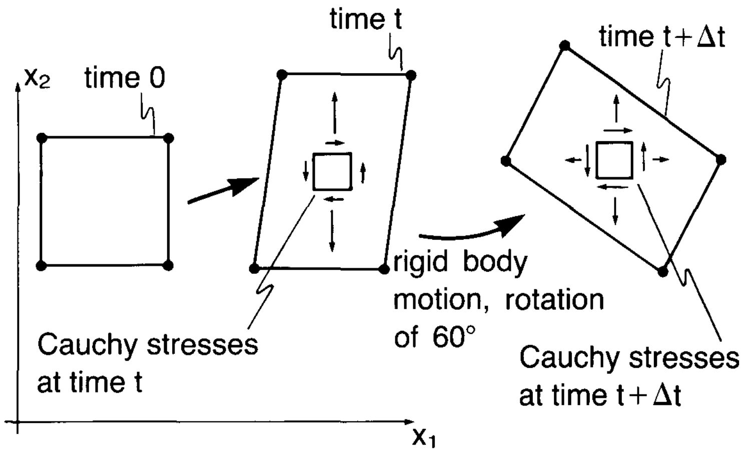
Properties of the 2nd Piola-Kirchhoff stress tensor:

- ${}^t_0\underline{S}$ is symmetric.
- ${}^t_0\underline{S}$ is invariant under a rigid-body motion (translation and/or rotation).

  Hence ${}^t_0\underline{S}$ changes only when the material is deformed.
- ${}^t_0\underline{S}$ has no direct physical interpretation.

Transparency 3-41

Example: Two-dimensional motion



Transparency 3-42

At time t,	At time t+Δt,
${}^t_0\underline{X} = \begin{bmatrix} 1 & .2 \\ 0 & 1.5 \end{bmatrix}$	${}^{t+\Delta t}_{\ \ \ 0}\underline{X} = \begin{bmatrix} .5 & -1.20 \\ .866 & .923 \end{bmatrix}$
${}^t\underline{\tau} = \begin{bmatrix} 0 & 1000 \\ 1000 & 2000 \end{bmatrix}$	${}^{t+\Delta t}\underline{\tau} = \begin{bmatrix} 634 & -1370 \\ -1370 & 1370 \end{bmatrix}$
${}^t_0\underline{S} = \begin{bmatrix} -346 & 733 \\ 733 & 1330 \end{bmatrix}$	${}^{t+\Delta t}_{\ \ \ 0}\underline{S} = \begin{bmatrix} -346 & 733 \\ 733 & 1330 \end{bmatrix}$

Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe