PROFESSOR: Ladies and gentlemen, welcome to this lecture on non-linear finite element analysis. In the previous lectures, I introduced you to non-linear finite element analysis, to the solution methods that we're using in non-linear finite element analysis, using primarily physical concepts. We are now ready to discuss the mathematical basis of the procedures that we're using to obtain solutions. Of course, this material is very complex, and there is a lot we could discuss. But let us have a go at it.

We have a body that undergoes large displacements, large rotations, and large strains. So therefore, we consider the body to undergo very large motions. We will use a Lagrangian description of the motion of the body. Here, on this viewgraph, I have prepared schematically the body that we are looking at. Here the body is in its configuration at time zero. Notice that the area is, the surface area of the body is described as 0 A, the volume is described as 0 V, the 0 being a left superscript.

I should point out, at this point, that you will be encountering now a lot of superscripts and subscripts, and that actually makes the discussion somewhat difficult. And for that reason, I like to take now, on this viewgraph, just a minute to explain to you why we have a particular superscript, a particular subscript, in that particular location, and what it means to us. So, a left superscript here, as shown on A and V, means at time indicated by that, at the time indicated by that superscript. So here we have 0 A, 0 V once again, area at time 0, volume at time 0.

A particular point here, point P, has the coordinates 0 X 1, 0 X 2, 0 X 3. Now, notice that these upper 0s here correspond to the time, as I just pointed out, whereas 1, 2, and 3, of course, correspond to the coordinate axis.

This is a particular material particle. This material particle will move with the body from time 0 to time t as shown on this viewgraph. Of course, at this particular time now, the surface area of the body has become t A, t now meaning surface area at time t. And the volume of the body is t V. Notice that the coordinates of this material particle now are t x 1, t x 2, t x 3.

We will assume, later on in our actual solution, that the solution up to time t is known, in other words, that the area, the volume, in fact, the stresses, the strains, at time t are known. And that we will want to march ahead one time increment, to time, t plus delta t. Notice the volume of the body, now, is given by t plus delta t V. The surface area is given by t plus delta t A. The coordinates of the material particle have now become t plus delta t x 1, t plus delta t x 2, and t plus delta t x 3.

Well, there's one very important point. And that important point is that we are considering that the Cartesian coordinate axis in which the motion of the body is measured remains stationary. You should always keep that in mind. In other words, here we have the Cartesian axis, x 1, x 2, x 3, and those remain stationary, whereas the body moves through those axes, through the, well, system of axes, And we want to, of course, solve for that motion.

With the particle motion described by the coordinates that I just mentioned, we can now define the following quantities down here. We can say that t x i, the coordinates at time t of the material particle, are equal to 0 x i-- both of those quantities, of course, we know already now what they are-- plus t u i. t u i are just the displacements from time 0 to time t.

Similarly for the situation from time 0 to time t plus delta t. Notice that these are now the displacements from time 0 to time t plus delta t. And we can, of course, talk also about the incremental displacements. The incremental displacements from time t plus delta t, or rather from time t, I should say, to time t plus delta t. And those incremental displacements are given by u i.

Of course, the lower right subscript always refers to the axis. In other words, i goes from one to two to three. For the u's and for the x's.

Well, I mentioned already that the Cartesian axes are stationary. And I have prepared this viewgraph to point it out once more very strongly. The unit distances along the xi-axis are the same for 0 x i, t x i, and t plus delta t x i. That is also a very important point.

Let's look at this very simple example. Here we have an x 2- axis, and we have the x 1-axis along here. Notice the units along this axis are 1, 2, 3, and so on. The material particle has initially the coordinates, 2, say. Of course, then, this being the length of 0 x 1. This particle, at time 0, moves to time t by t u 1, the way we just said it earlier. And of course, the sum of these two quantities is equal to t x 1.

Notice that t x 1 is measured along the same axis, with the same unit distances, as 0 x 1. And therefore, for this reason, we now use just x 1 here, without the left superscript, to denote the coordinate axis.

By the way, in my book, the textbook for this video course, I have frequently put here, a t or 0. I feel now, it is better, actually, to drop that superscript, because this description, the way I just explained it to you, I think, is even better.

Well, let us talk now, then, about the principle of virtual work. And now, we want to look at the principle of virtual work applied to time t plus delta t. The reason being, of course, that we want to solve for the static and kinematic variables at time t plus delta t.

We assume that we know the solution from time 0 to time t. We discussed in the previous lectures already the principle. I'd like to now just review it once more, because it's just of utmost importance to very well grasp, understand what the principle says, and how we apply it then.

Here we have now the Cauchy stresses. I defined the Cauchy stresses earlier as the force per unit area. The i j, of course, mean coordinate axis, i and j. In three dimensional analysis, going from 1 to 3. t plus delta t means the Cauchy stress at time t plus delta t. These are the real stresses that we want to solve for.

Here we have virtual strains. These are infinitesimally small strains, referred to the configuration at time t plus delta t. This t plus delta t means a reference to the configuration at time t plus delta t.

And here we have, of course, the t plus delta t d V, because we're integrating over that volume which, of course, is unknown in the incremental solution. This is the internet virtual work, which must be equal to the external virtual work. Here is again, the script R. And this script R, at time t plus delta t, is defined down here in the viewgraph.

We're integrating over the volume, at time t plus delta t, the body forces. And these are the real body forces, of forces per unit volume at time t plus delta t, multiplied by the virtual displacements, delta u i. And the integration, of course, goes over the volume at time t plus delta t. We're adding to this part the surface forces, forces per unit area, at time t plus delta t, multiplied by the virtual displacements on the surface. This capital S stands for "on the surface." And this product is integrated over the total surface of the body at time t plus delta t.

We assume that we know this part once we have imposed a virtual displacement. And therefore, we can calculate the right-hand side. Corresponding to that virtual displacement we obtain virtual strains. And therefore, we can calculate also the left-hand side. And once again, the principle of virtual work says, that the left-hand side, the internal virtual work, must be equal to the right-hand side, the external virtual work, for any arbitrary virtual displacements that satisfy the displacement boundary conditions. Of course, remember, as I pointed out earlier, in an earlier lecture, that the virtual strains must correspond to the virtual displacements. And here we have the virtual strains, once again.

On the viewgraph we saw this stress, and I call it the Cauchy stress. Here, you have it spelled out once more. It's the force per unit area at time t plus delta t. We also talked about the virtual strains. And the virtual strains here, this was the quantity R, defined as shown on the right-hand side.

Let me once again point strongly out that we are differentiating here, with respect to the coordinates, at time t plus delta t. Of course, the j and the i coordinate. And also, please recognize that this strain is really the infinitesimal strain tensor that you are very well familiar with in infinitesimal analysis, in infinitesimal displacement analysis, I should say, except for one difference that we're using here, the unknown coordinates at time t plus delta t. And that is spelled out down here, that we are using a variation in the small strains referred to the configuration, at time t plus delta t.

So really, if you are familiar with the principle of virtual work, as applied in infinitesimal displacement analysis, then you recognize that the principle of virtual work now applied to large deformation analysis, large displacement analysis, is quite the same, except that we are applying it to the current geometry, the current configuration of the body.

In my textbook, actually, I have a little note, a footnote that you might also look at, in the beginning of Chapter Six, where I discuss this principle of virtual work, that one way to look at it is that you see a body and, you see the body in front of yourself moving through the stationary coordinate frame-- once again, the Cartesian coordinate frame is stationary. You see it move, you see the body move through that coordinate frame. And there is somebody standing, taking a picture. And that picture is taken at time t plus delta t.

And now you have a picture. And you apply the principle of virtual work to that particular configuration which has been captured in that picture. That's perhaps one way to look at it, at least one way that I like to look at it.

In order to solve, or to work with the principal of virtual work, you need to rewrite the principal, really, because we cannot integrate over an unknown volume, and we cannot directly work with increments in the Cauchy stresses. The volume t plus delta t, remember, is unknown. And the reason why we cannot directly work, or simply work with increments in Cauchy stresses is because a Cauchy stress is always referred to the current geometry, current configuration. So if we talk about the Cauchy stress at time t, then it's the force per unit area at time t. And the Cauchy stress at time t plus delta t is the force per unit area at time t plus delta t.

Now you can't add a quantity that is referred to time t to a quality that is referred to time t plus delta t, because the areas, the reference areas, have changed. And we have to somehow take care of that in our formulation.

And for that reason, we introduce these two new quantities. The t 0 s is the 2nd Piola-Kirchhoff stress tensor. And the t 0 epsilon is the Green-Lagrange strain tensor. We introduce these quantities because these are well known quantities. If you were to look into continuum mechanics texts, they have been described in many, many books. And they are well known for some time. But we have extracted, basically, this information from the continuum mechanics literature, because it's a convenient, these are convenient stress and strain measures to work with in large deformation, finite element analysis.

So, this is the stress and that is the strain we will be using. And let me say the following now. This t here, the upper t, means stress at time t, in the configuration at time t. This lower 0 means that the stress is referred to the configuration at time 0. Similarly for the Green-Lagrange strain. Upper left, the configuration in which the strain, or stress measure is measured, and the lower subscript means, to which the configuration, to which the measure is referred to. So this is something to keep in mind, because we will be encountering this quite a bit.

Here is the definition of the stress tensor, t 0 s i j. Once again, the stress in configuration t, referred to the configuration at time 0. i j, of course, being the Cartesian coordinate components. Here we have the mass density ratio. The mass density at time 0, in configuration 0, if you like. The mass density at time t, in configuration at time t. And here we have a new quantity, which is actually the inverse of the deformation gradient. This is a component of the inverse of the deformation gradient. We will talk about this quantity quite a bit just now.

Here we have the Cauchy stress. And here we have another component such as that one. This is the 2nd Piola-Kirchhoff stress. Once again, I will talk about this piece of information quite a bit just now. The Green-Lagrange strain tensor is defined as shown here, t 0 epsilon i j, i j, of course, being the components into the

Cartesian coordinate axes, 1/2 t 0 u i comma j plus t zero u j comma i plus t 0 u k comma i times t zero u k j.

Notice that here we would select a particular i and j, say i equals 1, j equals 2. You would substitute those i and j numbers in here, so to say, and here as well. And notice that this k then, would actually run over all possibilities. k going from 1, then to 2, then to 3. In other words, there is a summation involved here. By the way, similarly, we have a summation involved here, because m and n would run over all the possibilities.

I will talk just now a little bit more about it. But let's first now look at how this particular piece of information here is defined. It's defined as a partial of 0 x i with respect to t x m. Notice that this coordinate, of course, is given, is assumed to be given. And that one here is also now assumed to be given.

Notice, of course, that t x m is equal to 0 x m plus a displacement that has occurred. Notice that this differentiation here is defined as given here, partial t u i with respect to the original coordinates. So, this lower 0 here, this lower 0 here refers to the fact that we're differentiating with respect to the original coordinates. This upper t here means that we are differentiating the displacement at time t.

Notice, of course, that i and j are dummy indices here. So, in fact, this one here is what I'm considering. But at the same time, I'm considering that one, if I just switch around i and j.

So these are the definitions of these two quantities. And I mentioned briefly already that we are summing in these definitions. For example, t s 1 1 would be calculated from the formulas that I gave you as 0 rho over t rho-- this, of course, is a scalar-- times this product here. What do we do here? We are selecting the 1 as that 1 here. That 1 here refers to that 1. And then we are letting these components here, these components here, and that one here, run over all the possibilities.

Notice, here we have 1 2. That is this 1 here, and that 1, 2. And here we have 3, 3 3, and 3. So in other words, these first two are fixed. These first two here are fixed, this 1 and that 1. They correspond to this 1, to these two 1s. But the other components vary over all the possibilities, namely, 1, to 2, to 3. This is here the summation convention, which is abundantly used in continuum mechanics. And you might be very well familiar with it.

Using this 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors-- and that is really the important point now-- we can rewrite the basic equation of the principle of virtual work in the form given here. Here on the left-hand side, we have the Cauchy stresses times the variations on the infinitesimal strains, or the small, infinitesimally virtual small strains. We integrate this product over the current volume.

And we mentioned already that we cannot deal with this integration very well. Well, we now rewrite, we can rewrite this integral as shown here, 2nd Piola-Kirchhoff stresses times variations in the Green-Lagrange strains. And we integrate this, we are integrating this product over the original volume of the body.

And that is the clue. We know this volume. We asked in general, increment analysis. We would not know this volume if, for example, we only have calculated the response up to t minus delta t. So, if we know this volume we can directly deal with this integration.

And we will see also that since these stress components are always referred to the same volume, the same area, we can also incrementally decompose these stresses. And the same holds for the strains.

Notice this relation, of course, holds for all times, delta t, 2 delta t, t, t plus delta t. The way we talked about the principal of virtual work just now, we really had, on the left-hand side here, t plus delta t. Well, you would just exchange this t, that t, and that t, and that t here, to a t plus delta t. And similarly, of course, on the right-hand side as well. I chose at the preparation of the viewgraph to simply call it t, because after all, this equation holds anyways, for any of those particular times.

To develop, then, the incremental finite element equations, we use the principle of virtual work in this form. If we assume that at, that the response has been calculated from time 0 to time t, we apply the principle of virtual work at time t plus delta t. We integrate, now, over a known volume, as I pointed out. And, we can directly decompose the stresses at time t plus delta t, and the Green-Lagrange strains at time t plus delta t as shown in these equations.

Now, notice that here we have the stress at time t plus delta t referred to the configuration at time 0 being equal to the stress at time t referred to the configuration at time 0 plus an incremental that is also referred to time 0. And this is what we would like to see. Similarly for the Green-Lagrange strain. These two items make if for us possible to work very well with the principle of virtual work in this particular form.

Before developing, however, the incremental continuum mechanics equations, I'd like to discuss with you now some important kinematic relationships that we use in general non-linear analysis. The 2nd Piola-Kirchhoff stress and Green-Lagrange strains have some very interesting, properties that I think I'd like to share with you in a discussion.

To explain some of these properties, we want to introduce the deformation gradient tensor. The deformation gradient tensor is actually something very, very basic in continuum mechanics. And so, let us spend a little bit of time on this tensor, to get also a bit of a physical feel for what it means, what it stands for, and what can we do with it. The tensor captures the straining of the body and the rotation of the body. And, as I mentioned, it is really a very fundamental quantity used in continuum mechanics.

The definition of the deformation gradient is as follows. Here, we have t 0 x being equal to a matrix, a large matrix, which, of course, is written in a Cartesian coordinate system. And notice what we're doing in the elements of that matrix. We take the partial of t x 1 with respect to 0 x 1. In other words, we're taking the coordinates at time t and differentiate them with respect to the original coordinates. Here, the 1 component with respect to the 1 component. Here, the t x 1 with respect to the 0 x 2. And, t x 1 with respect to 0 x 3. Like that, we get a three by three matrix.

Using indicial notation, we can simply say that an element of that matrix, t 0 X i j, capital X, because the matrix is defined as the capital t 0 X, being equal partial of t X i with respect to 0 x j. Of course, i j are those i j's here. And, you will also find in the textbook the following notation here, t 0 x i j. Notice, when I have a comma here, we denote the differentiation by that comma, and we use a little x. That is different from using the capital X here, with no comma, because that capital X simply denotes a component of, let's go up here once more, the t 0 x of that matrix.

So this is a mathematical definition, really, of the deformation gradient. And notice that in order to calculate the elements of this matrix, of course what you need to be given is this t x 1, t x 2, t x 3, as a function of 0 x 1, 0 x 2, and 0 x 3. If you are given the deformation field, t x i as a function of the 0 x j's, then you can calculate directly the components of the three by three matrix.

Another way to write the deformation gradient is via this equation. t 0 x can be written as 0 del, del 0. This here is a gradient operator. Let's look down here. We have defined it here. Times t x, or this gradient operator operates, I should say, on t x capital T. This capital T means transposed. And this t x capital T is given right here. Notice what we're listing in this vector here now, in this row vector, are simply the coordinates at time t, t x 1, t x 2, t x 3.

Notice that there is a transpose here. That transpose is simply introduced to make sure that this right-hand side is equal to what we have defined on the previous viewgraph this to be. If you leave out that capital T, you would not quite get what we actually have defined on the previous viewgraph. So that's why we put that capital T in there. It would be a nice exercise for you to just actually substitute from here into there, plug into there, and write out the three by three matrix, to make sure that you actually get what's given here on the left-hand side the way we defined it on the previous viewgraph.

The deformation gradient, as I briefly mentioned, describes the deformations, the rotations, and stretches of the material fibers. Let's look at some examples now. For example, if we have d 0 x here, that is the material fiber in its original configuration. Of course, that original configuration is going to be deformed, moved through the stationary coordinate system. That fiber moves into this situation here. It becomes d t x. The deformation gradient gives us this element as a function of that element, as given in this equation. In other words, d t x is equal to t 0 x times d 0 x.

If you know this piece of information, and you have calculated the deformation gradient, you directly get d t x. That's what the deformation gradient does for us.

Well, an example. It's always nice to look at some examples to demonstrate what we mean by these theoretical formulas. And here we have a very simple example. It's a one dimensional deformation.

We start off with this piece of material, a rectangular piece of material. And that piece of material is stretched out from time 0 to time t into this red configuration. Quite a bit of stretching, but the motion is simple because it's a uni-axial motion. Notice, the original length of this piece of material is 1, and this side here moves out 0.5.

Of course, the material particles here will move. For example, a particle that was originally here, the black little lines there, will move over to the right, and it becomes this red little line. This black line moves over to become that red line. Notice the displacement of this black line is given by this blue arrow here. This black line moves over to there, and the displacement is given by that blue arrow.

If we know the displacements of all these particles here, we know the deformation field. Vice versa, if the deformation field is given, and now we look at this equation here, we know the displacements of all of the particles along. And t x 1 is equal to 0 x 1 for this particular deformation field plus 0.5 0 x 1 squared.

Let's look at what this means. It tells that if I plug in a particular coordinate on the right-hand side of a material particle in its original configuration, then I'm going to get the final coordinate of that same material particle, that same material particle.

Well, the deformation gradient is defined via this relationship, and it's therefore given as 1 plus 0 x 1. Notice, we have only motion in one direction, so we only calculate this one component.

Let's consider, quite physically, a material particle initially at x 1 equals 0.8. In other words, x 1 is this direction, 0 x 1 is equal to 0.8. Remember, x 1 just gives us the coordinate direction. We now put a 0 on it to signify that 0.8 is the position of the particle initially. That particle moves with the deformation field that I've been given to this point here, and t x 1 is then 1.120.

The same can, of course, be repeated for an adjacent material particle. In other words, let us look now at a material particle that is just a little bit to the right-hand side, namely 0 x 1 being 0.850, slightly to the right-hand side of the other material particle that we looked at. This material particle, with the deformation field that we have been prescribing, moves to t x 1, this number here. We can now compute t 0 x 1 1, the element of the deformation gradient as this difference here. And the numbers are plugged in here. And we obtain 1.82.

Of course this is an estimate, because we did not take a differential. If we actually calculate the deformation gradient the way it's being defined, we get 1.80, which is quite close to the estimate here. The reason for that difference, of course, is that 0 x 1, for this particle, is not close enough, close enough to the earlier particle, the other particle that we also looked at. In other words, 0.850 is not close enough to 0.8, in order to get a very close relationship here between these two numbers.

Let's look at another example, a two dimensional deformation example. Here now, we have as the original configuration, this black body here, black outline shown, in the coordinate system x 1 X 2. Once again, remember the coordinate system is stationary. And let us look at two material fibers, this one and that one. This one does not carry a hat. That one carries a hat. So these are two different material fibers.

These fibers move to here from time 0 to time t. This fiber, d 0 x becomes d t x. That fiber, d 0 x hat, becomes d t x hat. In other words, of course, this here in red shown, is the configuration at time t.

Now, notice that these fibers here have particular coordinates, 0 x 1, 0 x 2. You can think of these coordinates to be at the beginning of that fiber. Of course, these are differential fibers. They are very, very small.

These coordinates move over to t x 1, t x 2. In other words, the particle from here moves to here and takes on these coordinates. We also, say, know, that the whole-- I mean we also know, the whole deformation field. And therefore we can directly calculate the deformation gradient, the way I explained it, by taking the differentiation of the current coordinates with respect to the original coordinates. And those differentiations give us these elements.

If we do so, we have the relationships between the d 0 x and d t x given right down here. Notice, d 0 x has these components in the x 1 x 2 coordinate frame. And you can see, for example, this 0.866, the component in the x 1 direction, this 0.5 is the component into the x 2 direction. If we multiply this out, we get d t x. And d t x, shown here, has no component into the x 2 direction, but has a component of 0.75 long into the x 1 direction.

What this tells us, is that if we know the motion of the body from time 0 to time t, and particularly at time t, and if we have calculated the deformation gradient the way we have defined it, then we can relate how the original fibers, or the fibers in the original configuration will rotate and stretch. And that relationship is given via the deformation gradient, as surely shown here quite physically. Because this fiber, originally here, has rotated into a new position, and has, in this particular case, been compressed.

So, this relationship tells us how the fiber has moved, rotated, and compressed. It does not tell us how much the movement was. That is not given by this relationship. That we have already, because we have to have that information to calculate t 0 x. But what this information, this equation gives us, is how much the fiber has rotated and compressed, or stretched.

Well, let's look at another fiber, the one that we had on the earlier viewgraph. Here, another fiber. And if we recognize, of course, that for that material particle here, the same material particle still, we still have the same deformation gradient. We can take this fiber, or rather the vectorial representation of that fiber, as given here, multiply this out, and we directly find out what has happened to that infinitesimal fiber in the motion that we're considering.

Notice, in this particular case, the components of this fiber are 0 in this direction, 1.5 into that direction. And it has moved to here, and has taken on components 1 and 1 into each of the coordinate directions.

The deformation gradient is very useful for various measures, computations that we perform in finite element analysis. And one such computation is to calculate the mass density ratios. In other words, we have to assess how the mass density of the body, as it moves through the space, changes. Let us look at how we do that.

Here we have our stationary coordinate system, x 1, x 2, x 3. And in that stationary coordinate system, I'd like to focus our attention onto a differential volume, a very small volume. I call that 0 d V curl. A curl is there because it is a specific volume that we now want to look at. Notice that the volume that I'm talking about here is spanned out by the differential. Differential is d 0 x curl 1, 2 and 3 not shown. But this is the infinitesimal volume at time 0.

Now, in the motion from time 0 to time t, of course this volume will change. And at time t, it looks like this. Notice that these differentials here have changed. We show just one, d 0 x curl one, has gone over into d t x curl 1, the t now denoting time t.

And this is the volume obtained from, that can be calculated from these vectors shown here. In fact, this is how we calculate this volume, and can therefore relate, also, the mass densities.

Let us look at some of the basic equations. First of all, we can say that the mass in this volume, this differential volume that we have focused our attention on, and that I have given a curl for that reason, that that mass must be preserved. And this means that this relationship must hold. Volume times mass density at time t must be equal to volume times mass density at time 0.

We can show, then, that this relationship holds. And I will talk about that just now it a little bit more. And these two relationships, then, clearly give us that this is how the mass density changes. In other words, the mass density at time t can directly be calculated from the mass density at time 0 and the determinant of the deformation gradient tensor. Notice, in three dimensional analysis, this is a three by three matrix. And you would have to calculate this matrix, at the particular point that you're focusing your attention on, and take the determinant of that matrix.

Let us look at this relationship here, because that is a very basic relationship. It tells how the volume, the differential volumes change as a function of time. And to show you how we arrive at this relationship, I've just compiled here a few viewgraphs to give you the proof.

Initially, we can say that d 0 x curl 1 is given via this relationship. Notice, this is the length of this vector, simply. And, notice that I had aligned this vector with the x 1 coordinate axis, therefore these 0s. Similar, d 0 x curl 2 was aligned with the x 2 coordinate axis, and therefore these two 0s. And the length of the vector is d s 2. d x curl 3 is given similarly, but aligned, this vector is aligned in the 3 direction, and the length was given as, is given as d s 3. Hence, by simple calculation, of course, we find directly that the volume, the differential volume that in the original configuration, is given via this equation.

But, if we now recognize that we can express d t x curl i via this relationship here, and that is nothing else than applying the deformation gradient tensor the way we already talked about. We looked already at examples where we put in a particular fiber, original configuration, and calculated how this fiber stretched, deformed from time 0 to time t. I just gave you some examples. And we use that information now, here, to express the d t x curl i in terms of d 0 x curl i. Of course, for i equals 1, 2, 3.

And now we use here a formula that you must have encountered some time ago, in your studies of mathematics. If you look at your old math books, I'm sure you can extract that formula, which simply says that the volume given, spanned by these vectors, is given via this product here, as a cross product of these two vectors, and the dot product of the resulting two vectors. This, of course, gives you a vector here, and you dot this vector with this vector, and you get, directly, a number, which is the volume of interest.

However, if we now multiply this out here, you find, directly, that this is the answer. In other words, the deformation gradient comes in right there. The determinant of the deformation gradient comes in here. Of course, it comes in here because you would substitute from here into there. And once again, since this product here is nothing else than the original volume, we're home.

We have proven that d t V curl is equal to the determinant of t 0 x times 0 d V curl, the determinate of the deformation gradient in this relationship between the differential volumes. It's really an interesting proof, and maybe you want to think a little bit more about it, and even go through this arithmetic here, to really reinforce the understanding of what we are doing here.

Let's look at a simple example to just exemplify once again what we just talked about. Here we have a little example. A piece of material 1 long, that in its original configuration is 1 long, I should say. We are stretching that piece of material up to time t, as shown. We assume uniform stretching. We assume plain strain conditions. In other words, the strain through the thickness is equal to 0 here, in this particular case. And the deformation field is assumed to be as shown here.

Then, with the deformation field given, clearly we can calculate this deformation gradient. Here, this component is simply obtained by differentiating t x 1 with respect to 0 x 1. If you do so, you get a 1 here plus 0.25, giving you a 1.25.

Because there is no shearing in the material, and no rotation, these off diagonal terms are 0. And, because there is no stretching through this direction, and through the thickness direction, we have 1s right there. To take the determinant of this matrix is rather simple. You just take the product of the diagonal terms, and you directly obtain 1.25 for its determinant.

Now, let us plug this into our formula. And here we have the result of that formula. It tells us 0 rho is equal to 1.25 t rho for any particle, right here, because this relationship is independent of the 0 x 1 component. Well, of course, this is also a result of the uniform stretching.

Notice that this makes physical sense. If we have taken a piece of material up here-- once more let us look at it-- that does not shrink here, that does not stretch or shrink in this normal direction as well, then surely the mass density has to decrease as we stretch it. And in fact, this is being shown right down here, t rho is equal to 0 rho divided by 1.25. So, the result makes sense.

We use also the inverse deformation gradient. And here, on this viewgraph, I've just summarized a few equations regarding this inverse deformation gradient. Here, we now look at the equation, where on the left-hand side we have the material fiber at time 0, and on the right-hand side is the material fiber at time t. In other words, d 0 x on the left-hand side, and d t x on the right-hand side. The inverse deformation gradient stands right here.

Mathematically, we can show that the inverse deformation gradient is nothing else than t 0 x inverse. Of course, notice that 0 t x would be calculated by differentiating the original coordinates with respect to the coordinates at time t. However, we can also calculate this matrix directly by taking this matrix here, where we differentiate the coordinates at time t with respect to the original coordinates. And then, having obtained this matrix, calculated this matrix, we simply invert it.

Well the proof that indeed holds is given down here in green. We simply take the top equation and substitute for d t x, as shown here, our relationship that we had already earlier. And, if we then, well, look at this equation, we can put brackets around here. And since d 0 x is equal to d 0 x for any material fiber, this must be the identity matrix, which already completes the proof.

An important theorem in continuum mechanics is the polar decomposition theorem, which tells that the deformation gradient, t 0 x, can always be decomposed into a rotation matrix and a stretch matrix. The rotation matrix is an orthogonal matrix, of course, meaning that R, transposed R, is the identity matrix. And the stretch matrix is a symmetric matrix.

This can always be done for any x. And the proof, actually, is given in the textbook. Please look at your study guide, where I indicate where you can find the proof in the textbook. And it might be quite good to actually go through that proof in some detail, so as to reinforce your understanding of what this theorem really tells. The important point, once again, is that we can always write the x matrix into this form as R times u.

And to give you an example of what this means, let's look at this one here. Here we have, in the coordinate frame x 1 x 2, a piece of material 3 long 2 wide. This is the original configuration of the material. And that material moves from time 0 to time t, as shown here. First it moves, it is being stretched into the x 1 direction and into the x 2 direction to take on the configuration shown in red. And then it rotates to take on the configuration in green. And that, indeed, is the configuration at time t.

Let us calculate the x matrix corresponding to that configuration, and see how it would be decomposed into R times u. Well, we have done that down here. Notice, this is the x matrix, t 0 x. These are the components of that matrix. And here we have the R matrix, t 0 R. And here we have the t 0 u. In other words, this is the rotation matrix, and this is the stretch matrix.

Notice this stretch matrix is indeed symmetric. Let's look at the components in the stretch matrix a little bit closer. 1.33 is nothing else than 4 divided by 3, in other words, the stretching into the x 1 direction. 1.5, down here, is nothing else than the stretching into x 2 direction, namely 2 goes over into 3. 3 divided by 2, 1.5.

Notice that these off diagonal elements are 0, the reason being that there has been no shearing from the original configuration to the red configuration. The rotation matrix really expresses mathematically the movement of this red piece of material into the green configuration.

Notice that the entries in this rotation matrix are nothing else than cosines and sines of this 30-degree rotation. For example, here you have the cosine of 30 degrees. This is the sine of 30 degrees. In fact, it's the rotation of this material fiber right there, from here into there.

Well, this is the rotation matrix and stretch matrix for this example. And we could easily construct other examples. In fact, you find in the textbook some more examples that you might want to study regarding this particular phenomenon.

Using the deformation gradient, we can describe, we can define a Cauchy-Green deformation tensor. There is a right Cauchy-Green deformation tensor. There is also a left Cauchy-Green deformation tensor which, however, we will not be using. So, when I talk about the Cauchy-Green deformation tensor, I really mean the right Cauchy-Green deformation tensor.

And it's defined as shown here. We take the deformation gradient, transposed, times the deformation gradient. Now, notice the deformation gradient is always a three by three matrix, in a three dimensional coordinate space. However, it can be a non-symmetric or symmetric matrix. Generally, it is a non-symmetric matrix. It would be a symmetric matrix if there has been no rotation points on it. But, in general, it's non-symmetric.

Notice that this non-symmetric matrix, multiplied here, as shown by the non-symmetric matrix transposed, makes c symmetric, a very important information. Well, indeed, if we substitute for x here, we find that c is nothing else than t 0 u squared. In other words, the stretch matrix squared, since R is orthogonal. R goes in here. But R transposed R, of course, is the identity matrix. And therefore c is simply u squared.

Notice that t 0 c is independent of the rotation, via this equation here, and therefore we say that c is invariant under a rigid body rotation. In other words, a rigid body rotation of a material fiber does not enter into the elements of c.

Let's look at an example. Two dimensional motion, the original configuration for the piece of material that I like to consider is shown here. It moves from time 0 to time t, to the red configuration. And then, in the increment of time delta t to that green configuration. But this movement over time delta t is simply a rigid body motion of 90 degrees. Well, we can calculate t 0 x corresponding to this configuration. Here it's done. And, by the formula that I gave you, you get directly t 0 c by taking x transposed times x. You get this matrix here.

Well, now we can also calculate t 0 x. How do we obtain, by the way, this matrix? There is a subtle point. You obtain this matrix simply by pre-multiplying this matrix by a rotation matrix corresponding to this 90-degree rotation. This is the answer.

And if we now take this matrix, transposed by the matrix itself, in other words this product x transpose x, we get the c matrix. And we notice that the c matrices are the same for both of these configurations. In other words, this simply exemplifies that the rigid body rotation from red to green here did not enter into the elements of the c. They did not change from time t to time t plus delta t.

We use this c matrix to define a strain measure, the Green-Lagrange strain measure. It's, of course, a strain measure that is very well known, described in many, many textbooks. And it is, however, very useful for finite element analysis, as I briefly pointed out already earlier.

This is the definition of the Green-Lagrange strain tensor. Notice, this here is a three by three matrix. And we are taking this three by three matrix. We subtract the identity matrix. The identity matrix, of course, being just 1s on the diagonal, and 0s on all the off diagonal elements. And there is a 1/2 here that gives us the definition of the Green-Lagrange strain tensor.

There are a number of very interesting properties that this strain tensor displays. And I've listed them here. We can see that this strain tensor is symmetric. We can see that this strain tensor does not change from time t to time t plus delta t if the only motion from time t to time t plus delta t was a rigid body motion. And we can see directly that the Green-Lagrange strain tensor is 0 if, from time 0 to time t, the only motion was a rigid body motion.

One can prove these particular items, these statements, and I would like to encourage you to do so. While you are doing it, I will do the same here. And then we share our experiences regarding these proofs just afterwards.

Well, here I am giving now to you some information regarding these proofs. t 0 epsilon is symmetric because t 0 c is symmetric. And remember, we take here a symmetric matrix from which we subtract a symmetric matrix, and clearly, the result must be a symmetric matrix. A very simple proof really.

Regarding the second item that I mentioned earlier, for rigid body motion, we know that from time t to time t plus delta t, we have that x at time t plus delta t can be written as R times t 0 x. In other words, here I'm talking about the deformation gradient at time t plus delta t, and here about the deformation gradient at time t. And they are related merely by a rotation matrix.

Now, if you substitute from here into this equation, but apply the time t plus delta t, you obtain, of course, as we saw actually on the earlier example, that this c matrix does not change. And therefore, also the Green-Lagrange stain tensor does not change. In fact, all of this information was really addressed in the earlier examples that we discussed.

Also, for rigid body motion, we have that the c matrix is simply the identity matrix, because remember, the c matrix, Cauchy-Green deformation tensor, does not change with rigid body rotation. It is equal to the identity matrix. And if you then takes this c matrix and substitute right up there, on the top, you get that the Green-Lagrange strain tensor is 0.

So, 3 properties, really, that we want to keep in mind regarding the Green-Lagrange strain tensor when we later on use it.

Another definition of the Green-Lagrange strain tensor is given on this viewgraph. In fact, this right-hand side that I'm showing here is simply obtained by substituting, for the deformation gradient, the identity matrix plus the deformation while the displacement derivatives, with respect to the coordinates. In other words, what you want to do, basically, is to substitute for t 0 x, in terms of displacements.

And if you go through and write all the information out, you end up that the t 0 epsilon i j component of the Green-Lagrange strain tensor is given by this equation here. Notice that we have here i, j, the same subscript that you see here, too. Of course, we're differentiating with respect to the j component here. Here we have j, i, and here k, i k,j. We're summing over k here.

Notice also, that this part here is non-linear in the displacements. This part is non-linear in the displacements because these are, so to say, quadratic terms. This here, this part, is linear in displacements.

From time to time, students approach me and say to me, well, you must have neglected something in the definition, or in this Green-Lagrange strain tensor. Because you're only going up to quadratic terms. you must have neglected cubic terms, and high order terms.

Well, the answer there is, we have really neglected nothing. The point is, that we have defined this tensor to be given by this equation. Of course, my earlier definition was in terms of deformation gradients, in terms of the deformation gradient. Now we actually have here substituted displacements. But it's still a definition.

This is, in fact, the same tensor. The same components here would be calculated using the earlier definition than this one here. Same thing. And, as I say, it holds for any deformation, any amount of straining and stretching.

Let's look at one example, a very simple example, where we have a simple four-node element that we are pulling out. Originally, it it is in the black configuration. It goes over into the red configuration.

If we calculate t 0 epsilon 1 1 from the formula that I just showed you, you get these two terms. This here is the engineering strain term that we are very well familiar with. And this is here the non-linear term, because it's a quadratic term. Non-linear in the displacement, of course.

If we plot this information, as shown here on the graph, notice we're showing here t delta over L, the original length, where t delta is the displacement. And we have plotting here the epsilon 1 1 component. Actually, we should have said here-- let me make this correction right now here-- we should have had, really have had here a t 0 epsilon 1 1. The t 0 was missing there.

Well, if we plot, in other words, this component here as a function of these elements here, of the t delta over 0 L, we get this red line, the red line. Notice the non-linearality in that strain. The engineering strain, which is only the first term here, gives us a straight line.

Notice that this strain here increases quite rapidly as you pull out, and it decreases in magnitude here, with this curvature, as you push in. So there's no symmetry in the strain when you extend or compress a piece of material. In the engineering strain component, of course, we see a symmetry, because we have a straight line.

Let us look at another example to calculate once the Green-Lagrange strain. Here, we have the original configuration of a four-node element. And that four-node element, shown black here, goes over into this configuration at time t, stretched into two directions, and then, at time t plus delta t, it has rotated by 45 degrees. The deformation gradient corresponding to time t is shown here. The Cauchy-Green deformation tensor shown here.

Notice no off diagonal elements. Because the piece of material, the four-node element-- you may think of this as a four-node element, of course-- has simply been stretched and compressed. There's no shearing. Therefore, we have 0 terms here. And, if we calculate the Green-Lagrange strain tensor, we get this answer. Once again, no shearing components in this tensor as well.

If we now do the same for the configuration at time t plus delta t, the green configuration, we obtain this deformation gradient, this Cauchy-Green deformation tensor, and this is a Green-Lagrange strain tensor. Where we notice that the Cauchy-Green deformation tensor has not changed, and the Green-Lagrange strain tensor has also not changed.

And this is what, of course, we have proven already earlier, that from time t to time t plus delta t, the Green-Lagrange strain tensor and the Cauchy-Green deformation tensor, both of those do not change. However for rigid body motion, the x tensor, the deformation gradient, does change, look at here, because this deformation gradient is affected by the rotation.

Let's look at another example, because all these examples really enrich our understanding of what these kinematic quantities stand for, and what they give us. Here, we look at the example of a simple shear deformation. Originally, the element that we are looking at was in this configuration. And, it is sheared over. You can think of this top line being simply moved over horizontally into the red configuration.

Notice that the original length here is 1. This length is also 1. Notice that the movement over is t delta. And we calculate the deformation gradient once again. The deformation gradient now being given by via these components.

Let's try to explain these components a bit. Well, this here is really the differentiation of t x 1 with respect to 0 x 1. Now, t x 1 has not changed, and therefore, this is 1. t x 1 is equal to 0 x 1 in other words, and therefore, this is 1. Notice that the same holds also for partial t x 2 with respect to 0 x 2.

Notice that this element here stands for partial t x 1 with respect to 0 x 2. And there, we look at partial t x 1, a change into this direction as we walk that direction, and that gives us this t delta, t delta over this length here. In other words, this element is nothing else than this length divided by that length. And it's, of course, the same for any material particle over this domain. Because all of these material particles have been sheared over.

Well, if you now take x transposed times x, you get this matrix here. And you identify directly these terms, if we then calculate by the formula given, the Green-Lagrange strain, this is the result.

Notice that here we have now shearing components. And notice also one very interesting phenomena, a component appearing here. This is an interesting component, because it is really an enormous strain component in this direction. And now there's an epsilon 1, 1, epsilon 2, 2, I should have said, component, an epsilon 2, 2 component that is non-zero in this particular case,

In a material non-linear-only formulation, where we only include infinitesimal displacements, in other words, this part, of course, would be 0. And that is also signified by the square term. In other words, if indeed the motion is small in this direction, then this would become 0, at least approximately 0 when compared to these terms, and it means that the Green-Lagrange strain tensor reduces to the infinitesimal strain tensor that we are so familiar with.

The 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensors are energetically conjugate. This is a very important statement. And this is the reason why we are working with these two tensors, the 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor. In fact, what is happening here, is that this product here, which is the virtual work at time t per current volume, per unit current volume, that this here is given, of course, by the Cauchy stress times the small, or the virtual, infinitesimal strain, is defined this way.

And here, we have the virtual work at time t per unit original volume. And that is given by the 2nd Piola-Kirchhoff stress times the variation in the Green-Lagrange strain.

We already had, earlier, that the integration of this over the current volume is equal to the integration of this quantity over the original volume. Well, this is here once more, something to be kept in mind. But the important point is that the energy, that these two quantities are energetically conjugate, and this is expressed via this product here.

Let's look at the 2nd Piola-Kirchhoff stress tensor now. The 2nd Piola-Kirchhoff stress tensor in indicial notation is defined as shown here. And we had already that definition early on a viewgraph. In matrix notation, we can directly say this is the way you can also write it.

Notice that the entries in this three by three matrix-- I'm always thinking about three components, or rather, three dimensional coordinate space-- the entries here are nothing else than those elements. The entries of this matrix, of course, are those elements here. And similarly here, this is the scalar that we talked about earlier already.

If we solve from here for Cauchy stresses, we directly obtain these two equations. Notice that in this matrix notation now, we have the deformation gradient. Whereas in the earlier equation that we looked at here, we had the inverse deformation gradient. Well, these are relations that we want to keep in mind, and that we will be referring to quiet abundantly later on.

Let us look at some properties of the 2nd Piola-Kirchhoff stress tensor. First of all, t 0 s is symmetric. While that can be seen directly by looking at the definition, you might want to prove it to yourself with a small example as well. t 0 s is invariant under a rigid body motion, translation, and/or rotation. Hence, t 0 s only changes when the material is actually deformed. This is a very important statement, important fact, I should say, and we want to look at an example just now.

t 0 s has no direct physical interpretation. Well, if you look at some books, you'll see some pictures regarding t 0 s, sorry, regarding the 2nd Piola-Kirchhoff stress tensor. In fact, I have some pictures in my book, in an example there. But, these pictures are quite far fetched, and they really don't give as deep a physical insight as one would like to have with a stress measure. I personally like to now take almost the attitude of saying, well, I accept that there is no real strong physical interpretation of this stress measure.

It is used because it's a convenient stress measure to deal with. We would not, in an engineering analysis, really print it out in a computer program. As I said earlier, we would, of course, want to have the Cauchy stress as the stress measure with which we want to design our structure.

I look at it as a measure that is convenient to work with in the engineering analysis. It's interesting, of course, from a theoretical point of view to look at the elements and to see how large they are, how small they are, when compared to the Cauchy stress, the real physical stress that we're interested in. But, we don't really need to look for a strong physical interpretation.

I know we are engineers. Engineers like to see pictures, like to understand physically what's happening. And I can assure you that I've tried myself quite hard to get physical interpretation of this definition, of this 2nd Piola-Kirchhoff stress tensor. But, I have not arrived with many good results. The best I could do is given in my textbook, in the book that you are using as a textbook.

Let's look at an example here. The example of a four-node element being stretched and rotated as shown here, and then rotated as shown here. Notice that from time t, shown as the red configuration, to time t plus delta t, we have only a rigid body motion, a rotation of 60 degrees. Of course, the material would be subjected to some Cauchy stresses, and Cauchy stresses in this configuration and Cauchy stresses in this configuration.

Let us see how the Cauchy stresses, and how the 2nd Piola-Kirchhoff stresses evolve during this motion. And on this viewgraph here, we have summarized that information.

At time t the deformation gradient is as shown here. The Cauchy stresses, say, are like this. Of course, we would have to introduce the material law, et cetera, et cetera. Let us assume that the Cauchy stresses are like that. And then, given the Cauchy stresses, the deformation gradient, we calculate the 2nd Piola-Kirchhoff stress tensor.

Let's say that this is now, has now been calculated. And we do the same at time t plus delta t. We know that there has, of course, been additional motion from time t to time t plus delta t, which means that these elements here will change. And you should now be in a position to calculate this tensor, because you know there is only a rigid body motion that is being applied. And then, knowing this tensor, we can directly also calculate the 2nd Piola-Kirchhoff stress tensor. And we can calculate the Cauchy stress tensor.

So we can really calculate all of these. And what do we see? This is the really important point. We see that the 2nd Piola-Kirchhoff stress tensor has not changed from time t to time t plus delta t. This is the important information that I mentioned earlier. However, the Cauchy stress tensor, of course, has changed. The Cauchy stress tensor has changed.

One can, of course, prove that the 2nd Piola-Kirchhoff stress tensor does not change when the material is only subjected to a rigid body motion. And such proof is actually given in the textbook. But this is a simple application, and it exemplifies to you what I mean by that statement. It's an important statement that you should keep in mind as we go along with our further discussion of the mathematical basis of non-linear finite element analysis.

This brings us then to the end of this lecture. I have tried to give you some of the mathematical bases, some of the ingredients, rather, that we are using in finite element analysis. We have discussed stress and strain measures that we will be encountering further in the next lectures. Thank you very much for your attention.