PROFESSOR: Ladies and gentlemen, welcome to this lecture on nonlinear finite element analysis of solids and structures.

In the previous lectures, we talked about the basic process of the incremental solution that is used in finite element analysis. And we talked about quite some important continuum mechanics variables that we are employing in nonlinear finite element analysis. This was really preparatory work for the discussion of a general formulation that is very widely used to analyze nonlinear systems using finite elements. And this formulation is called the total Lagrangian formulation. This is the formulation that I'd like to discuss with you now in this lecture.

The total Lagrangian formulation is a formulation that refers the stress and the strain variables at time t plus delta t to the original configuration at time zero. Total, in fact, means reference to the original configuration here. And notice we, of course, encountered this equation already earlier in the early lectures.

We have on the left hand side again the total internal virtual work. And on the right hand side, the total external virtual work all at time t plus delta t. We have here the second Piola-Kirchhoff stress tensor. Here we have the Green-Lagrange strain tensor. And on the right hand side, of course, all the body forces, surface forces et cetera would enter.

We discussed in a previous lecture how the right hand side is calculated in general and what it all contains. We also discussed that this equation here is quite equivalent to this equation. And in this equation, we have the Cauchy stress tensor operating on the virtual strain tensor, an infinitesimally small strain tensor. And this product is integrated over the volume at time t plus delta t.

On the right hand side, we have the same quantity as up there. Now notice that we really want to solve, of course, this equation here in finite element analysis. But we discussed earlier that in order to do so effectively, we need to introduce a new stress measure, a new strain measure, and that, of course, brings us then to this equation up here.

Let us recall briefly some of the important points that we made earlier in the other lectures. We said that this equation here, applied at time t plus delta t, is an expression of the equilibrium, compatibility, and the stress-strain law at time t plus delta t. Let's just look at this equation once more, where these quantities enter. Well, if equilibrium is satisfied locally everywhere in the continuum, then this equation must be holding for any virtual displacements that satisfy the displacement boundary conditions, and corresponding virtual strains.

Notice that the virtual displacements enter on the right hand side, and these here are the corresponding virtual strains. The compatibility enters in here because we would, of course, use compatible virtual displacement, and we would calculate also the stresses from compatible displacements. The stress-strain law enters in the calculation of the stress here from the strains given. And once again we apply this at time t plus delta t.

We also noted earlier that we will be using an incremental solution. Namely, that the solution at time t plus delta t will be calculated from the solution at time t by incrementing the displacements by ui. Notice that we will assume, of course, that the tui's are known. That the displacementts at time t are known, and that these are the unknown quantities that we're looking for. Of course, that gives us then the unknown quantities that we actually want to solve for.

Our goal is, for the finite element solution to linearize the equation of the principle of virtual work, so as to obtain a set of equations, finite element equations, that read as follows. Here we have a stiffness matrix, a tangent stiffness matrix. Here we have an incremental displacement vector.

And on the right hand side, we have a load vector, a vector of nodal point forces corresponding to the externally applied loads at time t plus delta t. And here we have a nodal point force vector that is equivalent to the current element stresses, in this particular instance at time t, as indicated by the superscript t here. Now this incremental displacement vector, of course, will be added to the displacement that we know already, and that correspond to time t. And that will give us then a first estimate for the solution, displacement solution, at time t plus delta t.

Notice this is also summarized down here. Notice, however, that this vector here is only an approximation to the actual vector that we're looking for, because of the linearization process that leads us to this system of equations. The equation tku equals r minus f is, of course, applicable for a single element as well as a total assemblage of elements.

In other words, this equation here, tk delta u1 equals t plus delta r minus tf, would be applicable to an element in which n then would mean just the degrees of freedom, the number of degrees of freedom of that element. And also for a total structure in which n then, of course means the total number of structural degrees of freedom. The tk matrix, the r vector, and the f vector would be constructed using the direct stiffness method the way we are used to it in the linear elastic analysis.

In other words, there is nothing new here to be discussed really, as far as a construction of these matrices go, when we talk about a total element assemblage from the individual element matrices. However, an important point is that we cannot simply linearize the principle of virtual work when it is written in this form here. And the reason was, of course, that we cannot integrate over an unknown volume. The volume t plus delta t is unknown, or the volume at time t plus delta t is unknown, and we cannot directly increment the Cauchy stresses the way we discussed it already earlier.

To linearize then, we need to choose a known reference configuration. And one known reference configuration that is a very natural one to use really, is the one corresponding to time zero. In this case, if we use that reference configuration, we talk about the total Lagrangian formulation. If we use as a reference configuration the configuration corresponding to time t, then we talk about the updated Lagrangian formulation.

I'd like to make one point here. Namely, that of course we have calculated already from all the configurations from time zero to time t. So you may ask yourself, why is he not choosing as a reference configuration, for example, the configuration t minus delta t? In other words, one configuration prior to time t as a possibility. Well, it would be a possibility, but if you choose to do so, you would lose the advantages that you have in the updated Lagrangian formulation, and the advantages that you would have in the total Lagrangian formulation.

Of course, there are differences here and we will talk about the advantages and disadvantages of either of these formulations. And you would be left with that choice of t minus delta t reference configuration, with basically all the disadvantages only. And therefore, we generally want to choose either this one or that one as the reference configuration.

Let us now discusses the total Lagrangian formulation. In the total Lagrangian formulation, we use the reference configuration zero, and therefore we talk about this stress. We use this stress and that strain, this was a second Piola-Kirchhoff stress referred to the configuration at time zero. This is the Green-Lagrange strain referred to the configuration at time zero.

The principle of virtual work once again, originally in this form, is now written in this form. We have had that already before on viewgraphs, and I don't think I need to go through those details again. But the important point is that this is the starting point of the total Lagrangian formulation.

The formulation proceeds as follows. We know the solution at time t, therefore this stress, this displacement derivative, all the static and kinematic variables in fact, are known corresponding to the configuration at time t. And we can now decompose the stress corresponding to time t plus delta t into one known value and one unknown value.

Similarly we proceed with the strain measures. The strain measures, the Green-Lagrange strain at time t plus delta t is decomposed into one value that we know already, and an increment. And, of course, remember these two increments are unknown. They are unknown, whereas these are known because we have calculated already the configuration corresponding to time t.

In terms of displacements, we can directly develop that the Green-Lagrange strain at time t is written as shown here. We went over these different components already earlier. Notice once again that we have a product of displacement derivatives here, that this is, of course, a nonlinear term.

And once again, as I pointed out in an earlier lecture, we have nothing neglected here. You see only quadratic terms. There are no cubic terms or higher-order terms. We have nothing neglected. This is a strain measure that holds for any amount of deformation, any amount of strain.

Then we can also write this same strain, Green-Lagrange strain, corresponding to time t plus delta t, and this would be the result. Notice whatever we had here as t superscript has now become a t plus delta t superscript. That's in fact the only difference. If you subtract from this strain, that strain, we get the increment, and that increment is written out right here.

Now, there are some interesting points regarding this incremental strain. This here is an increment in the strain that is linear in the displacement components. Similarly this one. Notice this one here, is, of course a product, product and may look, at first sight, as a nonlinear term. When you see products of displacements, you think of a nonlinearity. However, if you look closer, you find that there is a t here. Therefore, this part is known.

Since this part is known, and this part is unknown, this total term is really linear in the incremental displacement. In fact, the incremental displacements don't even go in here, because we're differentiating here the displacements at time t with respect to the original coordinates.

Therefore, this total term here is really still linear in ui. Linear in the incremental displacements. We call this part here, as given here, the initial displacement effect. The initial displacement effect because the initial displacements go into these derivatives here. We are differentiating the displacement at time t with respect to the original coordinates.

This term, on the other hand here, is nonlinear in ui. It is nonlinear because here we're taking one term that depends on ui, and we multiply it by another term that depends on ui. Notice this a differentiation of uk, the incremental displacement, into the k direction with respect to the original coordinates. Coordinate axis xj here particular.

So this is here, clearly a nonlinear term in ui. A quadratic term, as a matter of fact. We will look at this strain, of course, a little bit more just now. But keep in mind that there is one linear term that contains an initial displacement effect and a nonlinear term. I might add here that, of course, this initial displacement effect is zero if the initial displacements are zero.

In other words, in our incremental solution, for example, if we are just starting the solution process, then the initial displacements are still zero. This term drops out.

We note now regarding the formulation that the variation on the total Green-Lagrange strain tensor corresponding to time t plus delta t, is really equal to the variation in the increment of the Green-Lagrange strain tensor, from time t to time t plus delta t. In this picture we show what we mean by that.

Here on the left hand side in black, you have the body in its original configuration. It moves to a configuration at time t. And then to a configuration, here shown in green, that corresponds to time t plus delta t. Notice here we have the displacement corresponding to time t, and then the displacement corresponding from time t to time t plus delta t. The total displacement, of course, here is the displacement from time zero to time t plus delta t.

What we are doing, of course in the principle of virtual work, is to impose a variation in displacements about the configuration at time t plus delta t. And that variation is here indicated by the blue line. And if we impose this variation here, since the displacements at time t are constant, surely whatever these displacements were should not matter. And that is, in fact, expressed by this relationship here.

If we vary here, since the displacements to ui are constant, they do not affect the variation on the strains. And therefore, we can simply vary the incremental strain, and we find that variation is indeed equal to the variation of the total strain. Of course, you can also prove that to yourself mathematically by just going through the arithmetic. In other words, applying the variation to this quantity, and then applying the variation to that quantity, which I've given you on the previous viewgraph, and you would find that indeed you get identically the same expressions.

We can define a linear strain term, linear strain increment, as 0 eij, and the nonlinear strain increment as 0 eta ij. Notice that once again, here we have the quadratic terms. And, of course, when we add these two terms together, we get back our total strain increment from time t to time t plus delta t. If we take a variation on that increment, that is equivalent to taking a variation on the linear term and the nonlinear term.

Notice that so far we have only talked about continuum mechanics. Of course we want to use the continuum mechanics principles that we discussed so far in the finite element discretization. But notice that these strain terms here that we talked about, are really continuum mechanics variables.

There is an interesting observation. Namely, if we talk about this strain measure and that strain measure, rather the linear incremental strains and the nonlinear incremental strains, linear and nonlinear in ui, then if we apply these in finite element analysis, we have to remember that the displacements are interpolated in terms of nodal point variables. And what we're really looking for in finite element analysis are the nodal points variables. In isoparametric finite element analysis, which is the final element procedure that we will be using extensively, we interpolate the internal element displacements, tui, in terms of the nodal point displacements.

Let me define a little bit more what this is here. This is, of course, the displacement from time zero to time t into the i's direction. We are summing here over all the finite element nodal points, the N nodal points. We have interpolation functions, hk, that we will talk more about later on. And we have here the nodal point displacement at time t into the i direction of the nodal point k. This k goes with that k. And we're summing over all the nodal points.

Now notice that this is the linear relationship in isoparametric finite element analysis of solids, I should say. This is a linear relationship. And therefore, our linear strain terms that we talked earlier about, being linear in the incremental continuum mechanics displacements, are also linear in the incremental nodal point displacements.

Similarly, this term here, which was nonlinear in the increments of displacements within the domain, from a continuum mechanics point of view, will also be nonlinear in the nodal point displacement increments. However, in the formulation of structural elements, with the incremental displacements, in fact also the total displacements, I interpolated using nodal point displacements and nodal point rotations.

In other words, we also deal with nodal point rotations in order to calculate the total displacements of the element. And in that case, we have to recognize that the exact linear strain increment, the way we have defined it earlier linear in the incremental displacements, is still properly given as a linear strain increment in the nodal point variables.

However, the term that we defined earlier as a nonlinear strain increment, this one, is not the full story of all the nonlinear strain increments. Because the rotations will put additional nonlinear strain increments into this term here. The reason being, that for large rotations, of course, we have cosine sine terms in these rotations and when we evaluate the total nonlinear strain term, we find that there are additional terms coming up here.

Well, what is the effect of these two statements? It means that the right hand side force vector, or rather the vector t 0 f, the vector of nodal point forces corresponding to the internal elements stresses, is always correctly calculated. However, the stiffness matrix, t 0 k, for the formulation is only approximated because some of the nonlinear terms here have been dropped in the case of structural elements.

Of course, you can also include the nonlinear terms. Then you would have to add these terms to this expression here, and then you would get a true tangent stiffness matrix. This means, of course, for an actual analysis, that when you form the equations, we have on the left hand side-- let me just put all of it in here-- for the first iteration, we talked about the fact that we have a stiffness matrix times an increment in displacements. I already put a 1 there because it will be only an approximation to the total incremental displacement vector.

Well, we notice then that this matrix here is approximated if you were to drop these additional terms. But the right hand side vector f here is properly calculated. Therefore if you iterate, you always get the exact solution, the correct solution. But the convergence in the iteration might be a bit slower than what you could have obtained if you had included these additional terms here that I referred to.

Let us now continue with the continuum mechanics formulation. The equation of the principle of virtual work becomes, if we substitute the quantities that I defined earlier, directly this equation. Notice here we have now the increment in stress times the variation of the incremental Green-Lagrange strain, from time t to t plus delta t. Of course, it's integrated over the original volume plus this integral here.

Here we have the total stress, second Piola-Kirchhoff stress, corresponding to time t operating on the nonlinear strain increment of the total Green-Lagrange strain increment. And once again, we are integrating over the total volume corresponding to time zero. And on the right hand side, we have the external virtual work minus the stress at time t, operating on the linear strain increment corresponding to the total Green-Lagrange strain increment.

We have so far made no approximations, and this is a relationship that has to hold for any arbitrary variation in displacements which go in here, or any arbitrary virtual displacement that go in here. And of course, corresponding strains that appear here. In essence, what we have been doing is that we have established a variation on the configuration at time t plus delta t. We have looked at the principle of virtual work corresponding to time t plus delta t, but we have introduced this increment in displacement.

Notice that this increment in displacement is measured from the displacement at time t. All we have done therefore, is to rewrite the principle of virtual work in terms of tu and u instead of dealing with t plus delta tu. No approximation being done yet so far. Of course, we really haven't been talking about finite elements either yet, when we just look at this basic equation of principle of virtual work.

It is this principle or the equation that we are dealing with, is in general, a very complicated function of the unknown displacement increment. We obtain an approximate relationship in finite element analysis, by linearizing the governing equation, and, of course, then in the finite element sense cast it into this set of linear equations. The process of linearizing is a very important process, interesting process for us to study, and that's what I want to do now.

We begin to look, in this linearization at all the individual terms that appear in the integral. And let us look first at this term here. This term is actually already linear in ui. So there is nothing much to linearize. t 0 sij does not contain ui. It's a known quantity. It's a known stress value. And notice that when we take the variation on this nonlinear strain increment, we directly get this expression, where this part here is a variation on the displacement increment. Similarly here. And of course, these are going to be constants.

Therefore, this term here is linear in the incremental displacements. Linear in ui. So nothing to linearize there. The term on this viewgraph here, contains an incremental stress and an incremental strain. Notice this incremental strain contains the linear and the nonlinear term. And this, of course, this product is a linear functional off ui, sorry it contains linear terms, nonlinear terms as well. This stress here is, in general, a nonlinear function of 0 epsilon ij.

The variation that we are taking here means we are taking a variation on the linear part and the nonlinear part. We will see actually that this one here is constant. But this one carries also terms of ui, so this total expression is linear in ui. And if we multiply these terms here, we want to end up in just one expression that has neglected all higher-order terms in ui, but just contains ui. Well, let us do so.

The objective is expressed once more here, and we recognize that, of course, the variation on 0 epsilon ij contains only constant and linear terms in ui, the way I just expressed it. 0 sij can be written as a Taylor series expansion in 0 epsilon ij. This is the general expression. Here we take the partial of the stress at time t, with respect to the Green-Lagrange strain at time t, here we are having this bar with the t that denotes that we are actually doing this evaluation at time t. And here we have an increment in this strain, and of course, there are higher-order terms.

These higher-order terms we will neglect, and therefore we get directly for this stress here, this expression. Notice we already have substituted for 0 epsilon rs, these two quantities. This one here is quadratic in ui, this one is linear in ui. And if we linearize this term, we get directly this one here.

Notice that this is the appropriate stress-strain law that has to be used corresponding to this stress measure. And we will later on talk more about how we evaluate this stress-strain law. As an example, we may look here at this schematic solution, computed solution, over a number of time steps. We're going time 1, 2, 3, 4, 5, up to t minus delta t and then to t.

The stress-strain law relates basically the stress increment here to the strain increment. And notice it represents a tangent to this this curve. This green line here is a tangent to the red or the blue curve, the blue one now overlapping the red curve right there. So we have here the tangent, or the slope of this tangent is giving us the material tensor 0c. Of course, here we're talking just about one element. In general we have many elements corresponding to the strain and stress measures that we are dealing with.

If we now substitute the result that we just looked at into the general equation-- sorry to take this one down-- into the general equation here, we see directly that this part here, of course, is the stress. And this part here gives us that part. We simply multiply out and look at the individual terms. We notice that this one does not contain ui. We notice that this one is linear in ui.

So since this one is already linear in ui, this term will be quadratic in ui. We will have to drop it. This term here is linear in ui, but this one is constant, so this total term is linear in ui. And this is the one we keep, and which is then the linearized result. In other words, in summary once more, this total term here linearized is obtained via this expression here.

The final linearized equation that we are then dealing with, now that we're substituting all these results into the original equation that we have developed, is as follows. Here we have one term that comes from the incremental stress times the variation on the total incremental Green-Lagrange strain. That all has reduced to this term. Here we did not have to linearize at all. We kept what we already had. And on the right hand side, did not linearize either, because this was a term that we had already there, and of course this is the external virtual work.

Now it's interesting to note that these two terms here result into this expression, where this is the tangent stiffness matrix. Notice that tangent stiffness matrix contains the material tensor, as well as the current stresses. This incremental displacement here, vector, comes from this strain part and that strain part. Notice that this virtual displacement vector here comes from that strain part and also from that one here.

Below here we have, of course, that the external virtual work results into a vector of nodal point forces times these virtual displacements. And this one here results into the force vector corresponding to the internal element stresses.

And this is, of course, a very important quantity that we have to calculate accurately, as I pointed out earlier. Because we want ultimately in the iteration that this vector is equilibriating that vector, and if we do make a mistake in calculating this vector, then we might have converged, but we have converged to the wrong solution. So it's very important to recognize that this vector must be accurately calculated, by all means.

However, the k matrix here, if we go back once more to the discussion of this matrix, the k matrix here is a tangent stiffness matrix. And this tangent stiffness matrix is selected such as to obtain, of course, in the incremental solution, an appropriate incremental displacement. We will see later on that this matrix is updated in the iteration. It depends on what kind of intuitive scheme we are using, depending on that scheme we are updating this matrix differently.

In any case, it's a matrix that we will in quotes, "play around with", in order to accelerate the convergence. Therefore there is no unique matrix that can be used here, that should be used here. There are different possibilities that we will discuss in later lectures. But on the right hand side once again, this F vector is the F vector obtained from the current stresses, and that one has to be calculated uniquely in the correct way.

Well, an important point is that this relationship here on the left hand side, is equal to the relationship given here on the right hand side. And this holds because this term here is equal to that term. We interpret, in fact, this right hand side as an out of balance virtual work term.

Let's look once why this holds, because this might be a bit of a surprise. The mathematical explanation is given on this viewgraph. We had earlier already that the variation on the total Green-Lagrange strain is nothing else than the variation on the incremental total Green-Lagrange strain. Now if we look down here, we recognize that this variation here at ui equal to 0, is nothing else than the variation on the Green-Lagrange strain at time t. If ui is zero, then this term here is nothing else than the Green-Lagrange strain at time t.

So we're looking here at the variation of the Green-Lagrange at time, t if I put ui equal to 0. Now let's look at these expressions here. If I evaluate this term here, at ui equal to 0, we find that this term here, being a constant, is recovered. But notice that this term here, with ui equal to 0, simply turns out to be 0.

Therefore, this term is equal to that term. And this means, therefore, that the variation on t zero epsilon ij is equal to the variation on 0 eij, which we wanted to prove, of course. So this is the reason why this term here is equal to that term, as we have used it on the previous viewgraph.

This result also makes physical sense, because if we look at the governing equation, we find that this term here, replaced, is nothing else than tr. Should be tr. And of course, the script r the external virtual work at time t. And this, of course, is here the out of balance load term. Now if, for example, the material is elastic, and the external virtual work has not changed, then clearly the displacements would not change.

In other words, t plus delta tui would be equal to tui, and the incremental displacements would be 0. Therefore, equilibrium would be satisfied. Hence, this term, which we have here, must actually be equal to t 0 sij times variation t 0 epsilon ij, the way we discussed it just now.

We may rewrite the linearized governing equation as given on this viewgraph. And I am rewriting it in this form because we anticipate that we need an iteration. We simply introduce here, an iteration counter 1, with the delta in front of the incremental strain value. And similarly we introduce a delta here, and an iteration counter 1 there. And on the right hand side, we introduce also an iteration counter, but one lower.

In other words, 0 here, whereas we have a 1 there. Notice 0 here, a 1 here. And notice furthermore, that this term is nothing else than that term, and this term here is nothing else than the variation on t 0 epsilon ij.

This would be the equation that we have developed already, just having introduced now a different notation which leads us towards what we want to deal with in the iteration. The governing equations from a finite element point of view, then would be as shown here.

Notice tangent stiffness matrix, del u1, iteration counter, the external loads, which we still assume to be constant, by the way. We still assume deformation independent loading if we have deformation dependent loading, these would also change in the iteration. And here we have the vector corresponding to the internal element stresses at time t. We write that vector as t plus delta t 0, 0 there. And that means that we are really talking, at this point, about this vector.

We calculate this incremental displacement vector and add it to the previous displacements, which, of course, in this particular instance, in the first iteration, are nothing else than tu. We add those, and obtain our first estimates of displacements, namely the estimate at the end of iteration 1.

Having obtained now this estimate on displacements, we can repeat the process. And that is shown here on the next viewgraph. Notice that now I am talking about the right hand side, about the Green-Lagrange strains corresponding to time t plus delta t, and the first iteration. In other words, these are the Green-Lagrange strains corresponding to the end of the first iteration. They include now the incremental displacements that we just calculated.

Similarly, these are the stresses corresponding to the end of the first iteration. And this is an out of balance virtual work term. If that is non-zero, we will have further increment in displacements. And that increment results, of course, into an increment of strains. Namely, the second increment. Here is the 2 that denotes the increment. Here is the 2 that denotes the increment.

The finite element equations then would look as shown here. Same equation as before, but now with a 1 here and a 2 there. We calculate the incremental displacements, we add them to what we had already and get a better approximation to the displacements corresponding to time t plus delta t. This process, of course, can now be repeated, and on this viewgraph we show how it's being done for every iteration k.

In a particular iteration, we have now on the right hand side the external virtual work corresponding to time t plus delta t, the stresses corresponding to time t plus delta t and iteration k minus 1. At the end of iteration k minus 1, the strains corresponding to time t plus delta t, and at the end of iteration k minus 1. This is the out of balance virtual work term, and on the left hand side we have the same quantities as before, but now with the iteration counter k.

When discretized, we obtain these equations here. Delta uk, the increment in the iteration k of displacements, r minus f, computed, of course, from the current displacements. From the current displacements we calculate that vector here. Notice this vector is obtained from this integral here. We repeat this process for iteration k 1, 2, 3 et cetera until convergence. We will have to talk about how we measure convergence and so on.

But in this iteration we're adding up the total incremental displacements to obtain always a better estimate for the displacements corresponding to time t plus delta t, and here, of course, iteration k. Notice that the counter here goes from j equals 1 to k, that is this j and this k, of course, is nothing else than the k that we have on the left hand side here.

Notice once again that the first iteration, when k is equal to 1, it amounts to nothing else than what we discussed already at the very beginning a little earlier in the lecture. We simply generalized that discussion now to an equation in which we iterate and always try to calculate another increment in displacements until the right hand side is 0.

Notice when the right hand side is 0, the external virtual work is equilibrated by the internal virtual work, and of course, that is what we want to reach. That is what our criterion is for the analysis. The external virtual work must be equilibrated by the internal virtual work.

In the finite element discretization, the whole process is as summarized on this viewgraph. Initially, we are given tu, the displacements corresponding to time t, and the externally applied loads corresponding to time t plus delta t. We compute the stiffness matrix, the tangent stiffness matrix corresponding to time t. We calculate this vector, the nodal point forces corresponding to the stresses at time t.

Notice this 0 here refers always to the total Lagrangian formulation. Notice that this vector here is really the initial condition for the iteration. Similar, the displacements at time t are the initial conditions for the iteration for the displacements. Zero here. Zero there, meaning the initial conditions for our iteration. We set the iteration counter, k equal to 1, and we now go into the following loop.

Notice on the right hand side, an out of balance load vector is calculated. And that out of balance load vector gives us an incremental displacement vector. Of course, the tangent stiffness matrix is involved in that calculation. Notice that this incremental displacement vector is added to the displacements that we had previously calculated, to obtain a new estimate on the displacements. And like that, we have now obtained a displacement vector that, of course, hopefully is closer to satisfying certain convergence criteria.

We will have to talk about these convergence criteria later on. Basically, we check whether equilibrium is satisfied within a certain tolerance. If the equilibrium is not satisfied, we go here. We use these displacements that we just have calculated to compute a new vector of nodal point forces corresponding to the element stresses.

Notice this vector now contains the information that we have just calculated in solving this equation here. This was information that we obtained. And that information is contained in this vector, and that vector is used right here to calculate this nodal point force vector.

Having calculated that, we increase our iteration counter, come back here and like that we continue the iteration, looping around through here as I have just described. Notice at convergence once again, we want, of course, that r is equal to f, meaning that the external loads are equilibrated by the nodal point forces corresponding to the internal element stresses.

Well, this brings us to the end of this lecture. You might recognize that this last viewgraph really contains a lot of information that, of course, we discussed in this lecture, but that also I referred to already in the first lecture.

At that time, I tried to introduce you to this iteration process using rather physical concepts, not going through a lengthy mathematical derivation. We did not talk about continuum mechanics variables. We did not talk about total Lagrangian formulation, second Piola-Kirchhoff stresses, Green-Lagrange strains and so on.

I hoped at that time to give you just a physical feel of how we are iterating. The same kind of equations, the same equations we looked at at that time, now I hope with this lecture, I have given you the mathematical basis of these equations.

I hope you have learned how to derive these equations, how to work with these equations a little bit. And in some of the next lectures, we will look at how actually we do construct this k matrix for different elements. That is, of course, a very important consideration. How do we calculate this k matrix for 1D, 2D, 3D elements, shell elements, beam elements? How do we calculate this force vector for these different elements? We will have to discuss that as well.

Notice that in this k matrix goes the material law. Whether we are dealing with an elasto-plastic material, rubber type material, all that will affect the actual ingredients, the actual elements here in that k matrix. We we also have to discuss how we can possibly make this convergence that I'm talking about fast. In other words, how can we accelerate the convergence of these iterations?

So in essence, I like to just convey to you that what you see here you will see a number of times again. We will talk about the different parts that you have been looking at here already briefly once again in the upcoming lectures.

Thank you very much for your attention.