Topic 4

Total Lagrangian Formulation for Incremental General Nonlinear Analysis

Contents:

- **Review of basic principle of virtual work equation, objective in incremental solution**
- **Incremental stress and strain decompositions in the total Lagrangian form of the principle of virtual work**
- **Linear and nonlinear strain increments**
- **Initial displacement effect**
- **Considerations for finite element discretization with continuum elements (isoparametric solids with translational degrees of freedom only) and structural elements (with translational and rotational degrees of freedom)**
- **Consistent linearization of terms in the principle of virtual work for the incremental solution**
- **The "out-of-balance" virtual work term**
- **Derivation of iterative equations**
- **The modified Newton-Raphson iteration, flow chart of complete solution**

Textbook:

Sections 6.2.3, 8.6, 8.6.1

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TOTAL LAGRANGIAN FORMULATION

We have so far established that

$$\int_{^0V} {}^{t+\Delta t}_{0}S_{ij}\, \delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij}\, {}^{0}dV = {}^{t+\Delta t}\mathcal{R}$$

is totally equivalent to

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

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Recall:

▸ $$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

is an expression of

- Equilibrium
- Compatibility
- The stress-strain law

all at time $t+\Delta t$.

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▸ We employ an incremental solution procedure:

Given the solution at time t, we seek the displacement increments u_i to obtain the displacements at time $t+\Delta t$

$$^{t+\Delta t}u_i = {}^{t}u_i + u_i$$

We can then evaluate, from the total displacements, the Cauchy stresses at time $t+\Delta t$. These stresses will satisfy the principle of virtual work at time $t+\Delta t$.

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▸ Our goal is, for the finite element solution, to linearize the equation of the principle of virtual work, so as to finally obtain

$$\underbrace{{}^{t}\underline{\underline{K}}}_{\text{tangent stiffness matrix}}\;\underbrace{\Delta\underline{U}^{(1)}}_{\text{nodal point displacement increments}} = \underbrace{{}^{t+\Delta t}\underline{R}}_{\text{externally applied loads at time } t+\Delta t} - \underbrace{{}^{t}\underline{F}}_{\text{vector of nodal point forces corresponding to the element internal stresses at time t}}$$

The vector $\Delta\underline{U}^{(1)}$ now gives an approximation to the displacement increment $\underline{U} = {}^{t+\Delta t}\underline{U} - {}^{t}\underline{U}$.

The equation

$$\begin{matrix} {}^{t}\underline{K} & \Delta\underline{U}^{(1)} & = & {}^{t+\Delta t}\underline{R} & - & {}^{t}\underline{F} \\ \begin{bmatrix} & & \\ & & \end{bmatrix} & \begin{bmatrix} \\ \end{bmatrix} & = & \begin{bmatrix} \\ \end{bmatrix} & - & \begin{bmatrix} \\ \end{bmatrix} \\ n \times n & n \times 1 & & n \times 1 & & n \times 1 \end{matrix}$$

is valid

- for a single finite element (n = number of element degrees of freedom)
- for an assemblage of elements (n = total number of degrees of freedom)

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▸ We cannot "simply" linearize the principle of virtual work when it is written in the form

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

- We cannot integrate over an unknown volume.
- We cannot directly increment the Cauchy stresses.

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- To linearize, we choose a known reference configuration and use 2nd Piola-Kirchhoff stresses and Green-Lagrange strains as described below.

  Two practical choices for the reference configuration:
  - time = 0 → total Lagrangian formulation
  - time = t → updated Lagrangian formulation

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TOTAL LAGRANGIAN FORMULATION

Because ${}^{t+\Delta t}_{0}S_{ij}$ and ${}^{t+\Delta t}_{0}\varepsilon_{ij}$ are energetically conjugate,

the principle of virtual work

$$\int_{{}^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

can be written as

$$\int_{{}^{0}V} {}^{t+\Delta t}_{0}S_{ij}\, \delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij}\, {}^{0}dV = {}^{t+\Delta t}\mathcal{R}$$

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We already know the solution at time t (${}^{t}_{0}S_{ij}$, ${}^{t}_{0}u_{i,j}$, etc.). Therefore we decompose the unknown stresses and strains as

$$ {}^{t+\Delta t}_{0}S_{ij} = \underbrace{{}^{t}_{0}S_{ij}}_{\text{known}} + \underbrace{{}_{0}S_{ij}}_{\text{unknown increments}} $$

$$ {}^{t+\Delta t}_{0}\varepsilon_{ij} = \overbrace{{}^{t}_{0}\varepsilon_{ij}} + \overbrace{{}_{0}\varepsilon_{ij}} $$

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In terms of displacements, using

$$ {}^{t}_{0}\varepsilon_{ij} = \frac{1}{2}\left({}^{t}_{0}u_{i,j} + {}^{t}_{0}u_{j,i} + {}^{t}_{0}u_{k,i}\,{}^{t}_{0}u_{k,j}\right) $$

and

$$ {}^{t+\Delta t}_{0}\varepsilon_{ij} = \frac{1}{2}\left({}^{t+\Delta t}_{0}u_{i,j} + {}^{t+\Delta t}_{0}u_{j,i} + {}^{t+\Delta t}_{0}u_{k,i}\,{}^{t+\Delta t}_{0}u_{k,j}\right) $$

we find

$$ {}_{0}\varepsilon_{ij} = \underbrace{\frac{1}{2}\left({}_{0}u_{i,j} + {}_{0}u_{j,i} + \overbrace{{}^{t}_{0}u_{k,i}\,{}_{0}u_{k,j} + {}_{0}u_{k,i}\,{}^{t}_{0}u_{k,j}}^{\text{initial displacement effect}}\right)}_{\text{linear in } u_i} + \underbrace{\frac{1}{2}\,{}_{0}u_{k,i}\,{}_{0}u_{k,j}}_{\text{nonlinear in } u_i} $$

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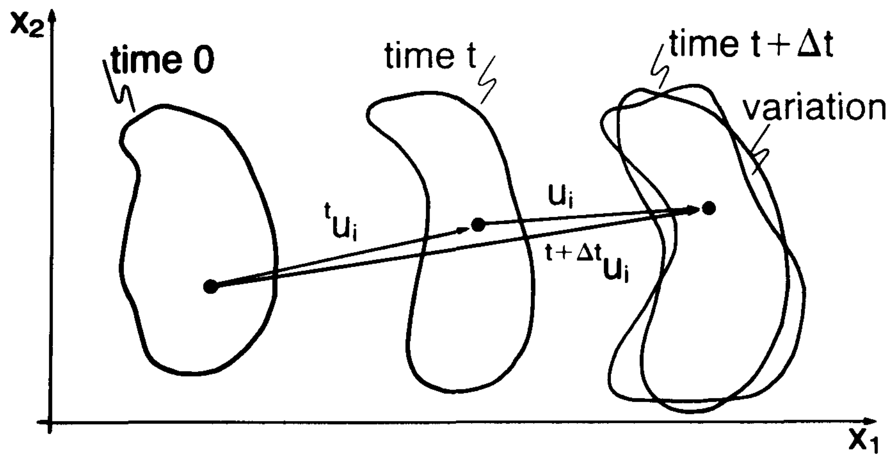
We note $\delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij} = \delta_0\varepsilon_{ij}$

- **Makes sense physically, because each variation is taken on the displacements at time $t+\Delta t$, with ${}^{t}u_i$ fixed.**



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We define

$${}_0e_{ij} = \underbrace{\frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + {}^{t}_{0}u_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}^{t}_{0}u_{k,j}\right)}_{\text{LINEAR STRAIN INCREMENT}}$$

$${}_0\eta_{ij} = \underbrace{\frac{1}{2}\,{}_0u_{k,i}\,{}_0u_{k,j}}_{\text{NONLINEAR STRAIN INCREMENT}}$$

Hence

$${}_0\varepsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij}\,, \quad \delta_0\varepsilon_{ij} = \delta_0 e_{ij} + \delta_0\eta_{ij}$$

An interesting observation:

- We have identified above, from continuum mechanics considerations, incremental strain terms

  ${}_0e_{ij}$ — linear in the displacement increments u_i
  ${}_0\eta_{ij}$ — nonlinear (second=order) in the displacement increments u_i

- In finite element analysis, the displacements are interpolated in terms of nodal point variables.

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- In isoparametric finite element analysis of solids, the finite element internal displacements depend linearly on the nodal point displacements.

$$ {}^t u_i = \sum_{k=1}^{N} h_k \, {}^t u_i^k $$

Hence, the exact linear strain increment and nonlinear strain increment are given by ${}_0e_{ij}$ and ${}_0\eta_{ij}$.

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- However, in the formulation of degenerate isoparametric beam and shell elements, the finite element internal displacements are expressed in terms of nodal point displacements and rotations.

$^{t}u_i$ = f (linear in nodal point displacements but nonlinear in nodal point rotations)

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- For isoparametric beam and shell elements
  - — the exact linear strain increment is given by ${}_0e_{ij}$, linear in the incremental nodal point variables
  - — only an approximation to the second-order nonlinear strain increment is given by $\frac{1}{2}\,{}_0u_{k,i}\;{}_0u_{k,j}$, second-order in the incremental nodal point displacements and rotations

The equation of the principle of virtual work becomes

$$\int_{^0V} {}_0S_{ij}\,\delta_0\varepsilon_{ij}\,{}^0dV + \int_{^0V} {}_0^tS_{ij}\,\delta_0\eta_{ij}\,{}^0dV$$
$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\,\delta_0 e_{ij}\,{}^0dV$$

Given a variation δu_i, the right-hand-side is known. The left-hand-side contains unknown displacement increments.

Important: So far, no approximations have been made.

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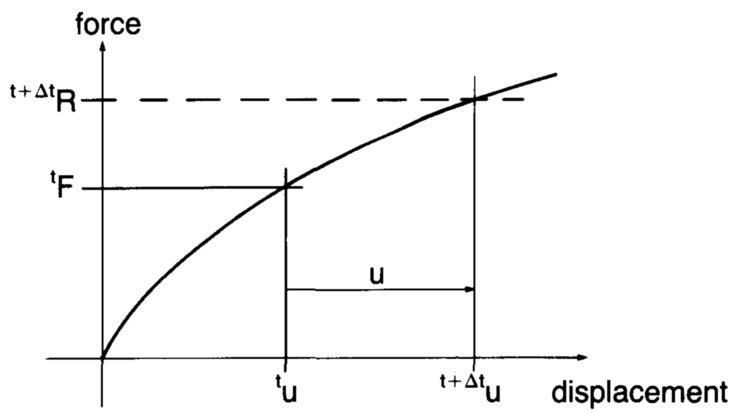

All we have done so far is to write the principle of virtual work in terms of ^tu_i and u_i.

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- The equation of the principle of virtual work is in general a complicated nonlinear function in the unknown displacement increment.
- We obtain an approximate equation by neglecting all higher-order terms in u_i (so that only linear terms in u_i remain). This leads to

$$ {}_0^t\underline{K}\, \Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}_0^t\underline{F} $$

The process of neglecting higher-order terms is called linearization.

Transparency 4-20

Now we begin to linearize the terms that contain the unknown displacement increments.

1) The term $\int_{{}^0V} {}_0^tS_{ij}\, \delta_0\eta_{ij}\, {}^0dV$

is linear in u_i:

- ${}_0^tS_{ij}$ does not contain u_i.
- $\delta_0\eta_{ij} = \frac{1}{2}\, {}_0u_{k,i}\, \delta_0u_{k,j} + \frac{1}{2}\, \delta_0u_{k,i}\, {}_0u_{k,j}$

  is linear in u_i.

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2) The term $\int_{^0V} {}_0S_{ij}\, \delta_0\varepsilon_{ij}\, {}^0dV$ contains linear and higher-order terms in u_i:

- ${}_0S_{ij}$ is a nonlinear function (in general) of ${}_0\varepsilon_{ij}$.
- $\delta_0\varepsilon_{ij} = \delta_0 e_{ij} + \delta_0\eta_{ij}$ is a linear function of u_i.

We need to neglect all higher-order terms in u_i.

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Linearization of ${}_0S_{ij}\, \delta_0\varepsilon_{ij}$:

Our objective is to express (by approximation) ${}_0S_{ij}$ as a linear function of u_i (noting that ${}_0S_{ij}$ equals zero if u_i equals zero).

We also recognize that $\delta_0\varepsilon_{ij}$ contains only constant and linear terms in u_i. We will see that only the constant term $\delta_0 e_{ij}$ should be included.

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${}_0S_{ij}$ can be written as a Taylor series in ${}_0\varepsilon_{ij}$:

$$ {}_0S_{ij} = \underbrace{\left.\frac{\partial\, {}_0^tS_{ij}}{\partial\, {}_0^t\varepsilon_{rs}}\right|_t}_{\text{known}} \underbrace{{}_0\varepsilon_{rs}}_{\text{linear and quadratic in } u_i} + \text{higher-order terms} $$

$$ \doteq \left.\frac{\partial\, {}_0^tS_{ij}}{\partial\, {}_0^t\varepsilon_{rs}}\right|_t (\underbrace{{}_0e_{rs}}_{\text{linear in } u_i} + \underbrace{{}_0\eta_{rs}}_{\text{quadratic in } u_i}) \doteq \underbrace{{}_0C_{ijrs}\, {}_0e_{rs}}_{\text{linearized term}} $$

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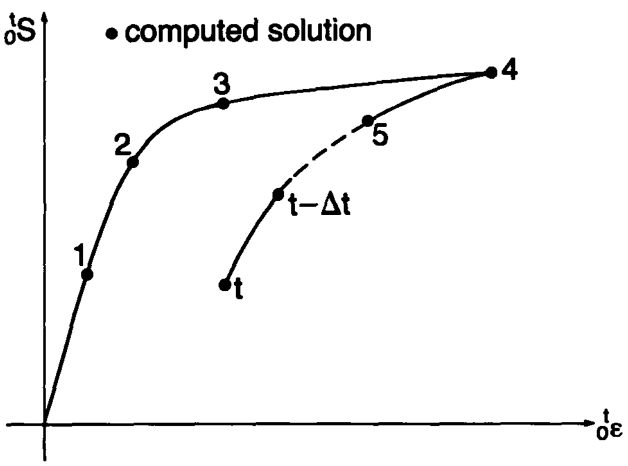
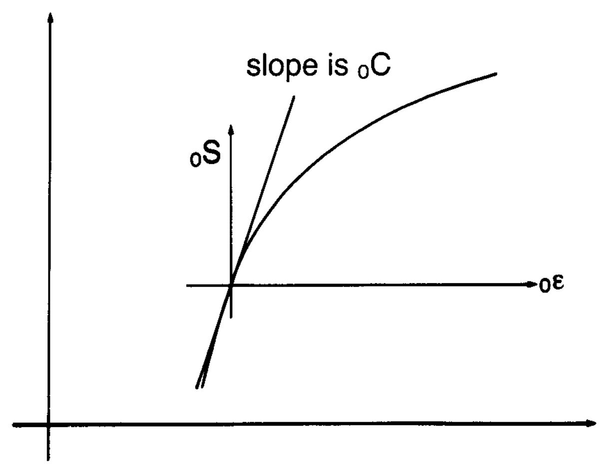
Example: A one-dimensional stress- strain law



At time t,



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Hence we obtain

$$\underbrace{{}_0S_{ij}}\,\delta_0\varepsilon_{ij} \doteq \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}}\,(\delta_0 e_{ij} + \delta_0\eta_{ij})$$

$$= \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}\,\underbrace{\delta_0 e_{ij}}_{\text{does not contain } u_i}}_{\text{linear in } u_i} + \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}\,\underbrace{\delta_0\eta_{ij}}_{\text{linear in } u_i}}_{\text{quadratic in } u_i}$$

$$\doteq \underbrace{{}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0 e_{ij}}_{\text{linearized result}}$$

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The final linearized equation is

$$\underbrace{\int_{^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}_0^tS_{ij}\, \delta_0\eta_{ij}\, {}^0dV}_{\delta\underline{U}^T\, {}_0^t\underline{K}\, \Delta\underline{U}}$$

$$= \underbrace{{}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV}_{\delta\underline{U}^T\, ({}^{t+\Delta t}\underline{R} - {}_0^t\underline{F})}$$

when discretized using the finite element method

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- An important point is that

$$\int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV = \underbrace{\int_{^0V} {}_0^tS_{ij}\, \delta_0^t\varepsilon_{ij}\, {}^0dV}_{\text{the virtual work due to the element internal stresses at time t}}$$

because

$$\delta_0 e_{ij} = \delta_0^t\varepsilon_{ij}$$

- We interpret

$${}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

as an "out-of-balance" virtual work term.

Mathematical explanation that $\delta_0 e_{ij} = \delta\,{}_0^t\varepsilon_{ij}$:

We had $\delta\,{}^{t+\Delta t}_{0}\varepsilon_{ij} = \delta_0 e_{ij} + \delta_0 \eta_{ij}$.

If $u_i = 0$, then the configuration at time $t+\Delta t$ is identical to the configuration at time t. Hence $\delta\,{}^{t+\Delta t}_{0}\varepsilon_{ij}\Big|_{u_i=0} = \delta\,{}_0^t\varepsilon_{ij}$.

It follows that

$$\delta\,{}^{t+\Delta t}_{0}\varepsilon_{ij}\Big|_{u_i=0} = \overbrace{\delta_0 e_{ij}\Big|_{u_i=0}}^{\delta_0 e_{ij}} + \overbrace{\delta_0 \eta_{ij}\Big|_{u_i=0}}^{0} = \delta\,{}_0^t\varepsilon_{ij}$$

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This result makes physical sense because equilibrium was assumed to be satisfied at time t. Hence we can write

$$\int_{{}^0V} {}_0C_{ijrs}\;{}_0e_{rs}\;\delta_0 e_{ij}\;{}^0dV + \int_{{}^0V} {}_0^tS_{ij}\;\delta_0\eta_{ij}\;{}^0dV$$

$$= {}^{t+\Delta t}\mathfrak{R} - {}^t\mathfrak{R}$$

Check: Suppose that ${}^{t+\Delta t}\mathfrak{R} = {}^t\mathfrak{R}$ and that the material is elastic. Then ${}^{t+\Delta t}u_i$ must equal ${}^tu_i$, hence $u_i = 0$. This is satisfied by the above equation.

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We may rewrite the linearized governing equation as follows:

$$\int_{^0V} {}_0C_{ijrs}\, \Delta_0 e_{rs}^{(1)}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}_0^tS_{ij}\, \delta\Delta_0\eta_{ij}^{(1)}\, {}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} \underbrace{{}^{t+\Delta t}_{\ 0}S_{ij}^{(0)}}_{{}_0^tS_{ij}}\ \underbrace{\delta\, {}^{t+\Delta t}_{\ 0}\varepsilon_{ij}^{(0)}}_{\delta_0^t\varepsilon_{ij}}\, {}^0dV$$

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When the linearized governing equation is discretized, we obtain

$$ {}_0^t\underline{K}\, \Delta\underline{U}^{(1)} = {}^{t+\Delta t}\underline{R} - \underbrace{{}^{t+\Delta t}_{\ 0}\underline{F}^{(0)}}_{{}_0^t\underline{F}}$$

We then use

$$ {}^{t+\Delta t}\underline{U}^{(1)} = \underbrace{{}^{t+\Delta t}\underline{U}^{(0)}}_{{}^t\underline{U}} + \Delta\underline{U}^{(1)}$$

Having obtained an approximate solution ${}^{t+\Delta t}\underline{U}^{(1)}$, we can compute an improved solution:

$$\int_{{}^0V} {}_0C_{ijrs}\,\Delta_0 e_{rs}^{(2)}\,\delta_0 e_{ij}\,{}^0dV + \int_{{}^0V} {}_0^tS_{ij}\,\delta\Delta_0\eta_{ij}^{(2)}\,{}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}^{(1)}\,\delta\,{}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}^{(1)}\,{}^0dV$$

which, when discretized, gives

$${}_0^t\underline{K}\,\Delta\underline{U}^{(2)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{\ \ 0}\underline{F}^{(1)}$$

We then use

$${}^{t+\Delta t}\underline{U}^{(2)} = {}^{t+\Delta t}\underline{U}^{(1)} + \Delta\underline{U}^{(2)}$$

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In general,

$$\int_{{}^0V} {}_0C_{ijrs}\,\Delta_0 e_{rs}^{(k)}\,\delta_0 e_{ij}\,{}^0dV + \int_{{}^0V} {}_0^tS_{ij}\,\delta\Delta_0\eta_{ij}^{(k)}\,{}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}^{(k-1)}\,\delta\,{}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}^{(k-1)}\,{}^0dV$$

which, when discretized, gives

$${}_0^t\underline{K}\,\Delta\underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - \underbrace{{}^{t+\Delta t}_{\ \ 0}\underline{F}^{(k-1)}}_{\text{computed from } {}^{t+\Delta t}u_i^{(k-1)}}$$

(for k = 1, 2, 3, ...)

Note that ${}^{t+\Delta t}\underline{U}^{(k)} = {}^t\underline{U} + \sum_{j=1}^{k} \Delta\underline{U}^{(j)}$.

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