Topic 5

Updated Lagrangian Formulation for Incremental General Nonlinear Analysis

Contents:

- **Principle of virtual work in terms of 2nd Piola-Kirchhoff stresses and Green-Lagrange strains referred to the configuration at time *t***
- **Incremental stress and strain decompositions in the updated Lagrangian form of the principle of virtual work**
- **Linear and nonlinear strain increments**
- **Consistent linearization of terms in the principle of virtual work**
- **The "out-of-balance" virtual work term**
- **Iterative equations for modified Newton-Raphson solution**
- **Flow chart of complete solution**
- **Comparison to total Lagrangian formulation**

Textbook: Section 6.2.3

A SUMMARY OF THE T.L.F.

• THE BASIC EQN. WE USE IS

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}R$$

• WE INTRODUCE

$${}^{t+\Delta t}_{0}S_{ij} \qquad {}^{t+\Delta t}_{t}S_{ij}$$

$${}^{t+\Delta t}_{0}\varepsilon_{ij} \qquad {}^{t+\Delta t}_{t}\varepsilon_{ij}$$

• WE DECOMPOSE

$${}^{t+\Delta t}_{0}S_{ij} = {}^{t}_{0}S_{ij} + {}_{0}S_{ij}$$

$${}^{t+\Delta t}_{0}\varepsilon_{ij} = {}^{t}_{0}\varepsilon_{ij} + {}_{0}\varepsilon_{ij}$$

• WE NOTE

$${}_0\varepsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij}$$

LINEAR / NONLINEAR IN u_i ← particle displ.

• WE OBTAIN

$$\int_{^0V} {}^{t+\Delta t}_{0}S_{ij}\; \delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij}\; {}^0dV = {}^{t+\Delta t}R$$

• SUBSTITUTION AND LINEARIZATION GIVES

$$\int_{^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}^t_0S_{ij}\, \delta_0\eta_{ij}\, {}^0dV = {}^{t+\Delta t}R - \int_{^0V} {}^t_0S_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

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• IN THE ITERATION WE HAVE

$$\ldots = {}^{t+\Delta t}R - \int_{^0V} {}^{t+\Delta t}_0S_{ij}^{(k-1)}\; \delta\, {}^{t+\Delta t}_0\varepsilon_{ij}^{(k-1)}\; {}^0dV$$

$k = 1, 2, 3, \ldots$

• THE F.E. DISCRETIZATION GIVES

$${}^t_0\underline{K}\, \Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_0\underline{F}^{(i-1)}$$

$i = 1, 2, 3, \ldots$

• AT CONVERGENCE

$${}^{t+\Delta t}\underline{R} = {}^{t+\Delta t}_0\underline{F}$$

WE SATISFY:

- COMPATIBILITY
- STRESS-STRAIN LAW
- EQUILIBRIUM
  - NODAL POINT EQUILIBRIUM
  - LOCAL EQUILIBRIUM IF MESH IS FINE ENOUGH

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UPDATED LAGRANGIAN FORMULATION

Because ${}^{t+\Delta t}_{\ t}S_{ij}$ and ${}^{t+\Delta t}_{\ t}\varepsilon_{ij}$ are energetically conjugate,

the principle of virtual work

$$\int_{t+\Delta t_V} {}^{t+\Delta t}\tau_{ij}\, \delta_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}dV = {}^{t+\Delta t}\mathcal{R}$$

can be written as

$$\int_{t_V} {}^{t+\Delta t}_{\ t}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ t}\varepsilon_{ij}\, {}^{t}dV = {}^{t+\Delta t}\mathcal{R}$$

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We already know the solution at time t (${}^{t}_{t}S_{ij}$, ${}^{t}_{t}u_{i,j}$, etc.). Therefore we decompose the unknown stresses and strains as

$$\underbrace{{}^{t+\Delta t}_{\ t}S_{ij} = {}^{t}_{t}S_{ij}}_{\text{known}} + \underbrace{{}_{t}S_{ij}}_{\text{unknown increments}} = {}^{t}\tau_{ij} + {}_{t}S_{ij}$$

$${}^{t+\Delta t}_{\ t}\varepsilon_{ij} = \overbrace{{}^{t}_{t}\varepsilon_{ij}}^{\to 0} + {}_{t}\varepsilon_{ij} = {}_{t}\varepsilon_{ij}$$

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In terms of displacements, using

$$ {}^{t+\Delta t}_{\ t}\varepsilon_{ij} = \frac{1}{2}\left({}^{t+\Delta t}_{\ t}u_{i,j} + {}^{t+\Delta t}_{\ t}u_{j,i} + {}^{t+\Delta t}_{\ t}u_{k,i}\ {}^{t+\Delta t}_{\ t}u_{k,j} \right) $$

we find

$$ {}_{t}\varepsilon_{ij} = \underbrace{\frac{1}{2}\left({}_{t}u_{i,j} + {}_{t}u_{j,i}\right)}_{\text{linear in } u_i} + \underbrace{\frac{1}{2}\,{}_{t}u_{k,i}\ {}_{t}u_{k,j}}_{\text{nonlinear in } u_i} $$

(No initial displacement effect)

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We define

$$ {}_{t}e_{ij} = \frac{1}{2}\left({}_{t}u_{i,j} + {}_{t}u_{j,i}\right) \quad \text{linear strain increment} $$

$$ {}_{t}\eta_{ij} = \frac{1}{2}\,{}_{t}u_{k,i}\ {}_{t}u_{k,j} \quad \text{nonlinear strain increment} $$

Hence

$$ {}_{t}\varepsilon_{ij} = {}_{t}e_{ij} + {}_{t}\eta_{ij} $$

$$ \delta_{t}\varepsilon_{ij} = \delta_{t}e_{ij} + \delta_{t}\eta_{ij} $$

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The equation of the principle of virtual work becomes

$$\int_{^tV} {}_tS_{ij}\, \delta_t\varepsilon_{ij}\, {}^tdV + \int_{^tV} {}^t\tau_{ij}\, \delta_t\eta_{ij}\, {}^tdV = {}^{t+\Delta t}\mathcal{R} - \int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

Given a variation δu_i, the right-hand-side is known. The left-hand-side contains unknown displacement increments.

Important: So far, no approximations have been made.

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Just as in the total Lagrangian formulation,

- The equation of the principle of virtual work is in general a complicated nonlinear function in the unknown displacement increment.
- Therefore we linearize this equation to obtain the approximate equation

$$^t_t\underline{K}\, \Delta\underline{U} = {}^{t+\Delta t}\underline{R} - {}^t_t\underline{F}$$

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We begin to linearize the terms containing the unknown displacement increments.

1) The term $\int_{{}^tV} {}^t\tau_{ij}\, \delta_t\eta_{ij}\, {}^tdV$

is linear in u_i.

- ${}^t\tau_{ij}$ does not contain u_i.
- $\delta_t\eta_{ij} = \frac{1}{2}\, {}_tu_{k,i}\, \delta_tu_{k,j} + \frac{1}{2}\, \delta_tu_{k,i}\, {}_tu_{k,j}$

  is linear in u_i.

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2) The term $\int_{{}^tV} {}_tS_{ij}\, \delta_t\varepsilon_{ij}\, {}^tdV$ contains linear and higher-order terms in u_i.

- ${}_tS_{ij}$ is a nonlinear function (in general) of ${}_t\varepsilon_{ij}$.
- $\delta_t\varepsilon_{ij} = \delta_te_{ij} + \delta_t\eta_{ij}$ is a linear function of u_i.

We need to neglect all higher-order terms in u_i.

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${}_tS_{ij}$ can be written as a Taylor series in ${}_t\varepsilon_{ij}$:

$$ {}_tS_{ij} = \underbrace{\left.\frac{\partial_t S_{ij}}{\partial_t \varepsilon_{rs}}\right|_t}_{\text{known}} \; \underbrace{{}_t\varepsilon_{rs}}_{\text{linear and quadratic in } u_i} + \text{higher-order terms} $$

$$ \doteq \left.\frac{\partial_t S_{ij}}{\partial_t \varepsilon_{rs}}\right|_t \big(\underbrace{{}_te_{rs}}_{\substack{\text{linear}\\ \text{in } u_i}} + \underbrace{{}_t\eta_{rs}}_{\substack{\text{quadratic}\\ \text{in } u_i}}\big) \doteq \underbrace{{}_tC_{ijrs}\,{}_te_{rs}}_{\text{linearized term}} $$

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Hence we obtain

$$ \underbrace{{}_tS_{ij}}\,\delta_t\varepsilon_{ij} \doteq \underbrace{{}_tC_{ijrs}\,{}_te_{rs}}\,(\delta_te_{ij} + \delta_t\eta_{ij}) $$

$$ = \underbrace{{}_tC_{ijrs}\,{}_te_{rs}\,\underbrace{\delta_te_{ij}}_{\substack{\text{does not}\\ \text{contain } u_i}}}_{\text{linear in } u_i} + \underbrace{{}_tC_{ijrs}\,{}_te_{rs}\,\underbrace{\delta_t\eta_{ij}}_{\text{linear in } u_i}}_{\text{quadratic in } u_i} $$

$$ \doteq \underbrace{{}_tC_{ijrs}\;{}_te_{rs}\;\delta_te_{ij}}_{\text{linearized result}} $$

The final linearized equation is

$$\underbrace{\int_{{}^tV} {}_tC_{ijrs}\, {}_te_{rs}\, \delta_t e_{ij}\, {}^tdV + \int_{{}^tV} {}^t\tau_{ij}\, \delta_t \eta_{ij}\, {}^tdV}_{\delta \underline{U}^T\, {}^t_t\underline{K}\, \Delta \underline{U}}$$

$$= \underbrace{{}^{t+\Delta t}\mathcal{R} - \int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV}_{\delta \underline{U}^T\, ({}^{t+\Delta t}\underline{R} - {}^t_t\underline{F})}$$

when discretized using the finite element method

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An important point is that

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

is the virtual work due to element internal stresses at time t. We interpret

$${}^{t+\Delta t}\mathcal{R} - \int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV$$

as an “out-of-balance” virtual work term.

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Solution using updated Lagrangian formulation

Displacement iteration:

$$^{t+\Delta t}u_i^{(k)} = {}^{t+\Delta t}u_i^{(k-1)} + \Delta u_i^{(k)}, \quad {}^{t+\Delta t}u_i^{(0)} = {}^{t}u_i$$

Modified Newton iteration:

$$\int_{^tV} {}_tC_{ijrs}\, \Delta_t e_{rs}^{(k)}\, \delta_t e_{ij}\, {}^t dV + \int_{^tV} {}^t\tau_{ij}\, \delta \Delta_t \eta_{ij}^{(k)}\, {}^t dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^{t+\Delta t}V^{(k-1)}} {}^{t+\Delta t}\tau_{ij}^{(k-1)}\, \delta_{t+\Delta t} e_{ij}^{(k-1)}\, {}^{t+\Delta t}dV$$

$k = 1, 2, \ldots$

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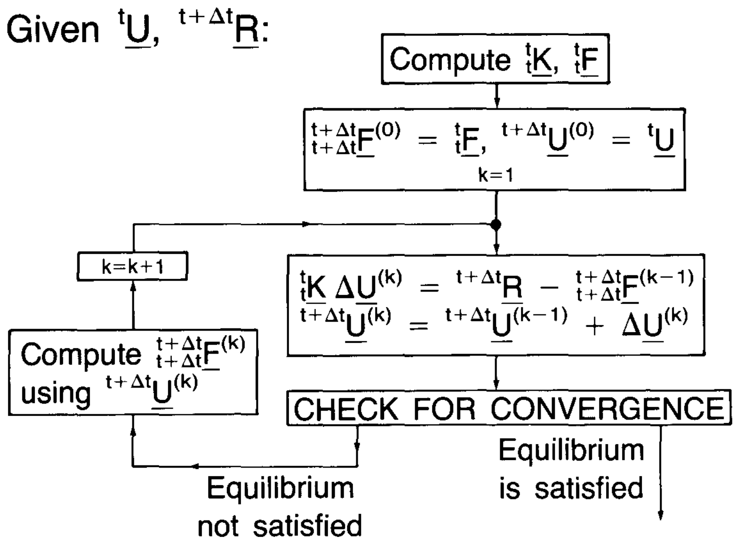
which, when discretized, gives

$$^t_t\underline{K}\, \Delta \underline{U}^{(k)} = {}^{t+\Delta t}\underline{R} - \underbrace{{}^{t+\Delta t}_{t+\Delta t}\underline{F}^{(k-1)}}_{\text{computed from } {}^{t+\Delta t}u_i^{(k-1)}}$$

(for $k = 1, 2, 3, \ldots$)

Note that $^{t+\Delta t}\underline{U}^{(k)} = {}^t\underline{U} + \sum_{j=1}^{k} \Delta \underline{U}^{(j)}$.



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Comparison of T.L. and U.L. formulations

- In the T.L. formulation, all derivatives are with respect to the initial coordinates whereas in the U.L. formulation, all derivatives are with respect to the current coordinates.

- In the U.L. formulation we work with the actual physical stresses (Cauchy stress).

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The same assumptions are made in the linearization and indeed the same finite element stiffness and force vectors are calculated (when certain transformation rules are followed).