Topic 6

Formulation of Finite Element Matrices

Contents:

- **Summary of principle of virtual work equations in total and updated Lagrangian formulations**
- **Deformation-independent and deformation-dependent loading**
- **Materially-nonlinear-only analysis**
- **Dynamic analysis, implicit and explicit time integration**
- **Derivations of finite element matrices for total and updated Lagrangian formulations, materially-nonlinear-only analysis**
- **Displacement and strain-displacement interpolation matrices**
- **Stress matrices**
- **Numerical integration and application of Gauss and Newton-Cotes formulas**
- **Example analysis: Elasto-plastic beam in bending**
- **Example analysis: A numerical experiment to test for correct element rigid body behavior**

Textbook:

Sections 6.3, 6.5.4

- WE HAVE DEVELOPED THE GENERAL INCRE-MENTAL CONTINUUM MECHANICS EQUATIONS IN THE PREVIOUS LEC-TURES
- IN THIS LECTURE
    - WE DISCUSS THE F.E. MATRICES USED IN STATIC AND DYNA-MIC ANALYSIS, IN GENERAL MATRIX TERMS
- THE F.E. MATRICES ARE FORMULATED, AND WE DISCUSS THEIR EVALUATION BY NUMERICAL INTE-GRATION

Markerboard 6-1

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DERIVATION OF ELEMENT MATRICES

The governing continuum mechanics equation for the total Lagrangian (T.L.) formulation is

$$\int_{^0V} {}_0C_{ijrs}\,{}_0e_{rs}\,\delta_0e_{ij}\,{}^0dV + \int_{^0V} {}_0^tS_{ij}\,\delta_0\eta_{ij}\,{}^0dV = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\,\delta_0e_{ij}\,{}^0dV$$

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The governing continuum mechanics equation for the updated Lagrangian (U.L.) formulation is

$$\int_{^tV} {}_tC_{ijrs}\,{}_te_{rs}\,\delta_te_{ij}\,{}^tdV + \int_{^tV} {}^t\tau_{ij}\,\delta_t\eta_{ij}\,{}^tdV = {}^{t+\Delta t}\mathcal{R} - \int_{^tV} {}^t\tau_{ij}\,\delta_te_{ij}\,{}^tdV$$

For the T.L. formulation, the modified Newton iteration procedure is
(for k = 1, 2, 3, ...)

$$\int_{^0V} {}_0C_{ijrs}\,\Delta_0 e_{rs}^{(k)}\,\delta_0 e_{ij}\,{}^0dV + \int_{^0V} {}_0^tS_{ij}\,\delta\Delta_0\eta_{ij}^{(k)}\,{}^0dV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}^{(k-1)}\,\delta\,{}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}^{(k-1)}\,{}^0dV$$

where we use

$${}^{t+\Delta t}u_i^{(k)} = {}^{t+\Delta t}u_i^{(k-1)} + \Delta u_i^{(k)}$$

with initial conditions

$${}^{t+\Delta t}u_i^{(0)} = {}^tu_i, \quad {}^{t+\Delta t}_{\ \ 0}S_{ij}^{(0)} = {}_0^tS_{ij}, \quad {}^{t+\Delta t}_{\ \ 0}\varepsilon_{ij}^{(0)} = {}_0^t\varepsilon_{ij}$$

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For the U. L. formulation, the modified Newton iteration procedure is
(for k = 1, 2, 3, ...)

$$\int_{^tV} {}_tC_{ijrs}\,\Delta_t e_{rs}^{(k)}\,\delta_t e_{ij}\,{}^tdV + \int_{^tV} {}^t\tau_{ij}\,\delta\Delta_t\eta_{ij}^{(k)}\,{}^tdV$$

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^{t+\Delta t}V^{(k-1)}} {}^{t+\Delta t}\tau_{ij}^{(k-1)}\,\delta_{t+\Delta t}e_{ij}^{(k-1)}\,{}^{t+\Delta t}dV$$

where we use

$${}^{t+\Delta t}u_i^{(k)} = {}^{t+\Delta t}u_i^{(k-1)} + \Delta u_i^{(k)}$$

with initial conditions

$${}^{t+\Delta t}u_i^{(0)} = {}^tu_i, \quad {}^{t+\Delta t}\tau_{ij}^{(0)} = {}^t\tau_{ij}, \quad {}_{t+\Delta t}e_{ij}^{(0)} = {}_te_{ij}$$

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Transparency 6-5

Assuming that the loading is deformation-independent,

$$^{t+\Delta t}\mathcal{R} = \int_{^0V} {}^{t+\Delta t}_{\ 0}f_i^B \, \delta u_i \, {}^0dV + \int_{^0S} {}^{t+\Delta t}_{\ 0}f_i^S \, \delta u_i^S \, {}^0dS$$

For a dynamic analysis, the inertia force loading term is

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\rho \, {}^{t+\Delta t}\ddot{u}_i \, \delta u_i \, {}^{t+\Delta t}dV = \underbrace{\int_{^0V} {}^0\rho \, {}^{t+\Delta t}\ddot{u}_i \, \delta u_i \, {}^0dV}_{\text{may be evaluated at time 0}}$$

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If the external loads are deformation-dependent,

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}f_i^B \, \delta u_i \, {}^{t+\Delta t}dV \doteq \int_{^{t+\Delta t}V^{(k-1)}} {}^{t+\Delta t}f_i^{B(k-1)} \, \delta u_i \, {}^{t+\Delta t}dV$$

and

$$\int_{^{t+\Delta t}S} {}^{t+\Delta t}f_i^S \, \delta u_i^S \, {}^{t+\Delta t}dS \doteq \int_{^{t+\Delta t}S^{(k-1)}} {}^{t+\Delta t}f_i^{S(k-1)} \, \delta u_i^S \, {}^{t+\Delta t}dS$$

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Materially-nonlinear-only analysis:

$$\int_V C_{ijrs}\, \Delta e_{rs}^{(k)}\, \delta e_{ij}\, dV = {}^{t+\Delta t}\mathcal{R} - \int_V {}^{t+\Delta t}\sigma_{ij}^{(k-1)}\, \delta e_{ij}\, dV$$

This equation is obtained from the governing T.L. and U.L. equations by realizing that, neglecting geometric nonlinearities,

$$ {}^{t+\Delta t}_{\ 0}S_{ij} \equiv {}^{t+\Delta t}\tau_{ij} \equiv \underbrace{{}^{t+\Delta t}\sigma_{ij}}_{\text{physical stress}}$$

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Dynamic analysis:

Implicit time integration:

$$ {}^{t+\Delta t}\mathcal{R} = {}^{t+\Delta t}\mathcal{R}_{\substack{\text{external}\\ \text{loads}}} - \int_{{}^0V} {}^0\rho\, {}^{t+\Delta t}\ddot{u}_i\, \delta u_i\, {}^0dV$$

Explicit time integration:

T.L. $\int_{{}^0V} {}^t_0S_{ij}\, \delta\, {}^t_0\varepsilon_{ij}\, {}^0dV = {}^t\mathcal{R}$

U.L. $\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^tdV = {}^t\mathcal{R}$

M.N.O. $\int_V {}^t\sigma_{ij}\, \delta e_{ij}\, dV = {}^t\mathcal{R}$

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The finite element equations corresponding to the continuum mechanics equations are

Materially-nonlinear-only analysis:

Static analysis:

$$ {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \qquad (6.55)$$

Dynamic analysis, implicit time integration:

$$\underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i)} + {}^{t}\underline{K}\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)} \qquad (6.56)$$

Dynamic analysis, explicit time integration:

$$\underline{M}\,{}^{t}\ddot{\underline{U}} = {}^{t}\underline{R} - {}^{t}\underline{F} \qquad (6.57)$$

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Total Lagrangian formulation:

Static analysis:

$$({}^{t}_{0}\underline{K}_{L} + {}^{t}_{0}\underline{K}_{NL})\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{0}\underline{F}^{(i-1)}$$

Dynamic analysis, implicit time integration:

$$\underline{M}\,{}^{t+\Delta t}\ddot{\underline{U}}^{(i)} + ({}^{t}_{0}\underline{K}_{L} + {}^{t}_{0}\underline{K}_{NL})\,\Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}_{0}\underline{F}^{(i-1)}$$

Dynamic analysis, explicit time integration:

$$\underline{M}\,{}^{t}\ddot{\underline{U}} = {}^{t}\underline{R} - {}^{t}_{0}\underline{F}$$

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Updated Lagrangian formulation:

Static analysis:

$$({}_t^t\underline{K}_L + {}_t^t\underline{K}_{NL})\, \Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}_{t+\Delta t}^{t+\Delta t}\underline{F}^{(i-1)}$$

Dynamic analysis, implicit time integration:

$$\underline{M}\, {}^{t+\Delta t}\ddot{\underline{U}}^{(i)} + ({}_t^t\underline{K}_L + {}_t^t\underline{K}_{NL})\, \Delta\underline{U}^{(i)} = {}^{t+\Delta t}\underline{R} - {}_{t+\Delta t}^{t+\Delta t}\underline{F}^{(i-1)}$$

Dynamic analysis, explicit time integration:

$$\underline{M}\, {}^t\ddot{\underline{U}} = {}^t\underline{R} - {}_t^t\underline{F}$$

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The above expressions are valid for

- a single finite element ($\underline{U}$ contains the element nodal point displacements)
- an assemblage of elements ($\underline{U}$ contains all nodal point displacements)

In practice, element matrices are calculated and then assembled into the global matrices using the direct stiffness method.

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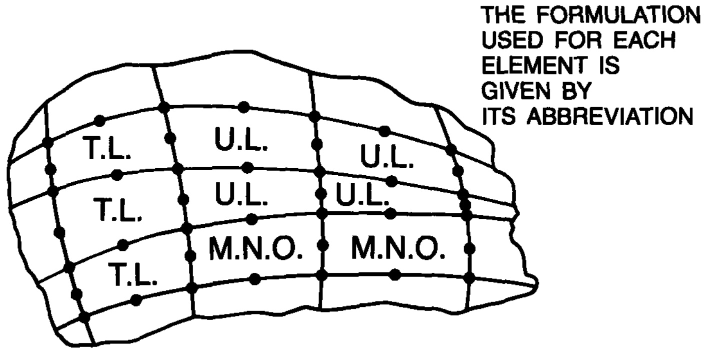
Considering an assemblage of elements, we will see that different formulations may be used in the same analysis:



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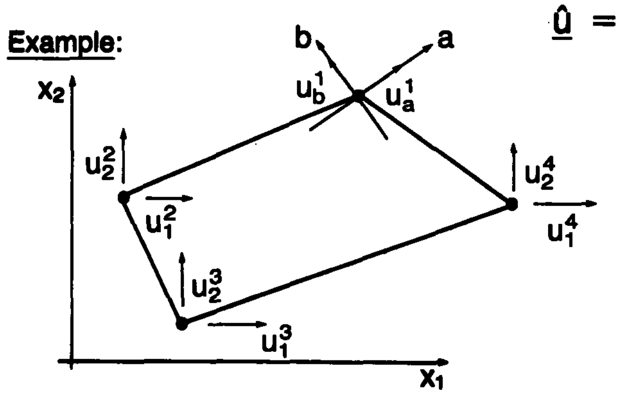
We now concentrate on a single element.

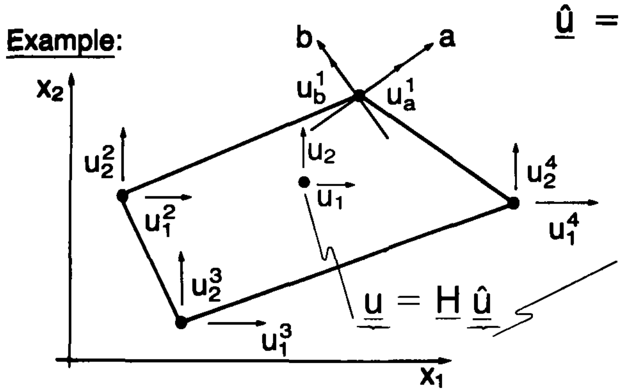
The vector $\hat{\underline{u}}$ contains the element incremental nodal point displacements



$$\hat{\underline{u}} = \begin{bmatrix} u_a^1 \\ u_b^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \\ u_1^4 \\ u_2^4 \end{bmatrix}$$

We may write the displacements at any point in the element in terms of the element nodal displacements:

Example:



$$\underline{\hat{u}} = \begin{bmatrix} u_a^1 \\ u_b^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \\ u_1^4 \\ u_2^4 \end{bmatrix}$$

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Finite element discretization of governing continuum mechanics equations:

For all analysis types:

$$\int_V {}^0\rho\ {}^{t+\Delta t}\ddot{u}_i\ \delta u_i\ {}^0dV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left({}^0\rho \int_{{}^0V} \underline{H}^T\,\underline{H}\ {}^0dV\right)}_{\underline{M}} {}^{t+\Delta t}\ddot{\underline{\hat{u}}}$$

where we used

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\text{displacements at a point within the element}} = \underline{H}\,\underline{\hat{u}}$$

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and

$$ {}^{t+\Delta t}\mathcal{R} = \int_{^0V} {}^{t+\Delta t}_{\ 0}f_i^B\, \delta u_i\, {}^0dV + \int_{^0S} {}^{t+\Delta t}_{\ 0}f_i^S\, \delta u_i^S\, {}^0dS $$

$$ \downarrow $$

$$ \delta \underline{\hat{u}}^T \underbrace{\left(\int_{^0V} \underline{H}^T\, {}^{t+\Delta t}_{\ 0}\underline{f}^B\, {}^0dV + \int_{^0S} \underline{H}^{S\,T}\, {}^{t+\Delta t}_{\ 0}\underline{f}^S\, {}^0dS \right)}_{{}^{t+\Delta t}\underline{R}} $$

where

$$ \left. \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right|_{\text{on } S} = \underline{H}^S\, \underline{\hat{u}} $$

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Materially-nonlinear-only analysis:

Considering an incremental displacement u_i,

$$ \int_V C_{ijrs}\, e_{rs}\, \delta e_{ij}\, dV \rightarrow \delta \underline{\hat{u}}^T \underbrace{\left(\int_V \underline{B}_L^T\, \underline{C}\, \underline{B}_L\, dV \right)}_{{}^t\underline{K}} \underline{\hat{u}} $$

where

$$ \underline{e} = \underline{B}_L \underline{\hat{u}} $$

a vector containing components of e_{ij}

Example: Two-dimensional plane stress element:

$$ \underline{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ 2\,e_{12} \end{bmatrix} $$

and

$$\int_V {}^t\sigma_{ij}\, \delta e_{ij}\, dV \rightarrow \delta \hat{\underline{u}}^T \underbrace{\left(\int_V \underline{B}_L^T \, {}^t\hat{\underline{\Sigma}} \; dV \right)}_{{}^t\underline{F}}$$

where ${}^t\hat{\underline{\Sigma}}$ is a vector containing components of ${}^t\sigma_{ij}$.

Example: Two-dimensional plane stress element:

$${}^t\hat{\underline{\Sigma}} = \begin{bmatrix} {}^t\sigma_{11} \\ {}^t\sigma_{22} \\ {}^t\sigma_{12} \end{bmatrix}$$

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Total Lagrangian formulation:

Considering an incremental displacement u_i,

$$\int_{{}^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV \rightarrow \delta \hat{\underline{u}}^T \underbrace{\left(\int_{{}^0V} {}_0^t\underline{B}_L^T \, {}_0\underline{C} \, {}_0^t\underline{B}_L \, {}^0dV \right)}_{{}_0^t\underline{K}_L} \hat{\underline{u}}$$

where

$$\underbrace{{}_0\underline{e}}_{} = {}_0^t\underline{B}_L \, \hat{\underline{u}}$$

a vector containing components of ${}_0e_{ij}$

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$$\int_{{}^0V} {}_0^tS_{ij}\, \delta_0\eta_{ij}\, {}^0dV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left(\int_{{}^0V} {}_0^t\underline{B}_{NL}^T\, {}_0^t\underline{S}\, {}_0^t\underline{B}_{NL}\, {}^0dV\right)}_{{}_0^t\underline{K}_{NL}} \underline{\hat{u}}$$

where

${}_0^t\underline{S}$ is a matrix containing components of ${}_0^tS_{ij}$

${}_0^t\underline{B}_{NL}\, \underline{\hat{u}}$ contains components of ${}_0u_{i,j}$

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and

$$\int_{{}^0V} {}_0^tS_{ij}\, \delta_0e_{ij}\, {}^0dV \rightarrow \delta\underline{\hat{u}}^T \underbrace{\left(\int_{{}^0V} {}_0^t\underline{B}_L^T\, {}_0^t\underline{\hat{S}}\, {}^0dV\right)}_{{}_0^t\underline{F}}$$

where ${}_0^t\underline{\hat{S}}$ is a vector containing components of ${}_0^tS_{ij}$.

Updated Lagrangian formulation:

Considering an incremental displacement u_i,

$$\int_{^tV} {}_tC_{ijrs}\ {}_te_{rs}\ \delta_t e_{ij}\ {}^t dV \rightarrow \delta \underline{\hat{u}}^T \underbrace{\left(\int_{^tV} {}^t_t\underline{B}_L^T\ {}_t\underline{C}\ {}^t_t\underline{B}_L\ {}^t dV\right)}_{{}^t_t\underline{K}_L} \underline{\hat{u}}$$

where

$$\underbrace{{}_t\underline{e}}_{\text{a vector containing components of } {}_te_{ij}} = {}^t_t\underline{B}_L\ \underline{\hat{u}}$$

a vector containing
components of ${}_te_{ij}$

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$$\int_{^tV} {}^t\tau_{ij}\ \delta_t\eta_{ij}\ {}^t dV \rightarrow \delta \underline{\hat{u}}^T \underbrace{\left(\int_{^tV} {}^t_t\underline{B}_{NL}^T\ {}^t\underline{\tau}\ {}^t_t\underline{B}_{NL}\ {}^t dV\right)}_{{}^t_t\underline{K}_{NL}} \underline{\hat{u}}$$

where

${}^t\underline{\tau}$ is a matrix containing components of ${}^t\tau_{ij}$

${}^t_t\underline{B}_{NL}\ \underline{\hat{u}}$ contains components of ${}_tu_{i,j}$

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Transparency 6-25

and

$$\int_{{}^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, {}^t dV \rightarrow \delta \hat{\underline{u}}^T \underbrace{\left(\int_{{}^tV} {}^t_t\underline{B}_L^T\, {}^t\hat{\underline{\tau}}\, {}^t dV \right)}_{{}^t_t\underline{F}}$$

where ${}^t\hat{\underline{\tau}}$ is a vector containing components of ${}^t\tau_{ij}$

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- The finite element stiffness and mass matrices and force vectors are evaluated using numerical integration (as in linear analysis).
- In isoparametric finite element analysis we have, schematically, in 2-D analysis

$$\underline{K} = \int_{-1}^{+1} \int_{-1}^{+1} \underbrace{\underline{B}^T\, \underline{C}\, \underline{B}\, \det \underline{J}}_{\underline{G}}\, dr\, ds$$

$$\underline{K} \doteq \sum_i \sum_j \alpha_{ij}\, \underline{G}_{ij}$$

And similarly

$$\underline{F} = \int_{-1}^{+1} \int_{-1}^{+1} \underbrace{\underline{B}^T \hat{\underline{\tau}} \det \underline{J}}_{\underline{G}} \, dr \, ds$$

$$\underline{F} \doteq \sum_i \sum_j \alpha_{ij} \underline{G}_{ij}$$

$$\underline{M} = \int_{-1}^{+1} \int_{-1}^{+1} \underbrace{\rho \underline{H}^T \underline{H} \det \underline{J}}_{G} \, dr \, ds$$

$$\underline{M} \doteq \sum_i \sum_j \alpha_{ij} \underline{G}_{ij}$$

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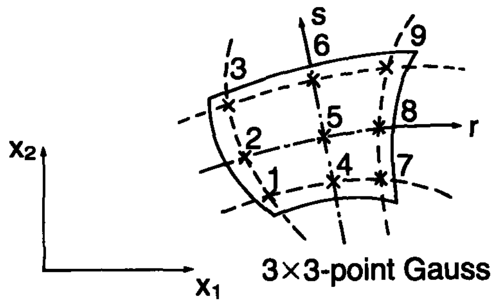
Frequently used is Gauss integration:

Example: 2-D analysis



r, s values:
±0.7745...
0.0

3×3-point Gauss integration

All integration points are in the interior of the element.

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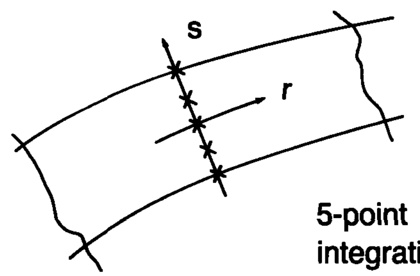
Also used is <u>Newton-Cotes integration:</u>

<u>Example</u>: shell element



5-point Newton-Cotes integration in s-direction

Integration points are on the boundary and the interior of the element.

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Gauss versus Newton-Cotes Integration:

- Use of n Gauss points integrates a polynomial of order 2n-1 exactly, whereas use of n Newton-Cotes points integrates only a polynomial of n-1 exactly.
  Hence, for analysis of solids we generally use Gauss integration.
- Newton-Cotes integration involves points on the boundaries.
  Hence, Newton-Cotes integration may be effective for structural elements.

In principle, the integration schemes are employed as in linear analysis:

- The integration order must be high enough not to have spurious zero energy modes in the elements.
- The appropriate integration order may, in nonlinear analysis, be higher than in linear analysis (for example, to model more accurately the spread of plasticity). On the other hand, too high an order of integration is also not effective; instead, more elements should be used.

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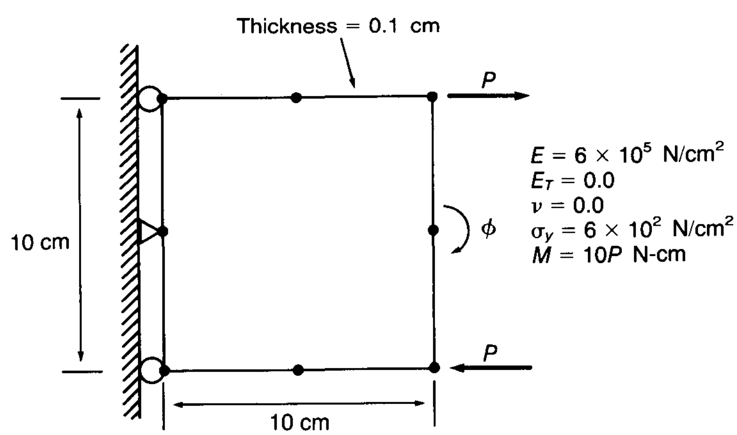
Example: Test of effect of integration order

Finite element model considered:



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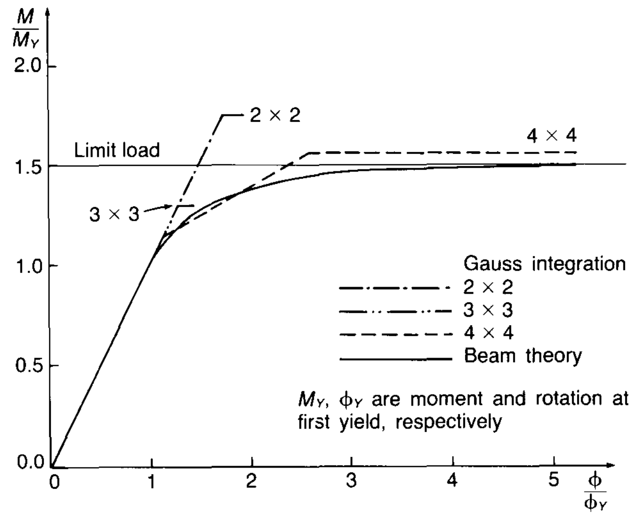
Calculated response:



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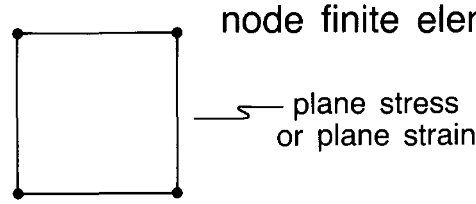
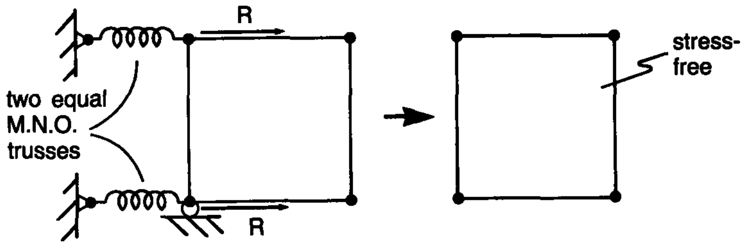
Problem: Design numerical experiments which test the ability of a finite element to correctly model large rigid body translations and large rigid body rotations.

— Consider a single two-dimensional square 4-node finite element:



Numerical experiment to test whether a 4-node element can model a large rigid body translation:



This result will be obtained if any of the finite element formulations discussed (T.L., U.L., M.N.O. or linear) is used.

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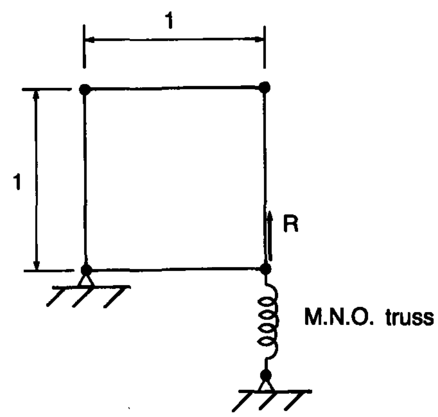
Numerical experiment to test whether a 4-node element can model a large rigid body rotation:



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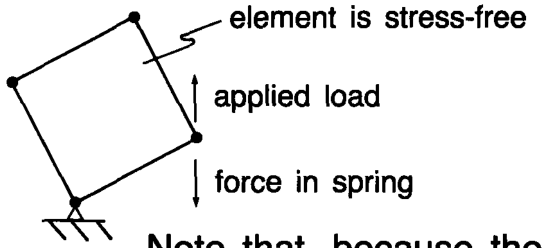
Transparency 6-37

When the load is applied, the element should rotate as a rigid body. The load should be transmitted entirely through the truss.



Note that, because the spring is modeled using an M.N.O. truss element, the force transmitted by the truss is always vertical.

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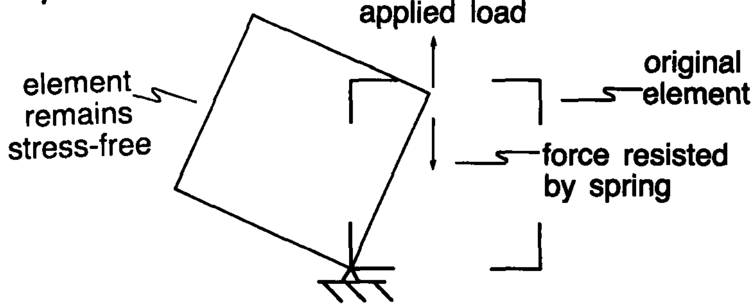
After the load is applied, the element should look as shown in the following picture.



This result will be obtained if the T.L. or U.L. formulations are used to model the 2-D element.

Resource: Finite Element Procedures for Solids and Structures
Klaus-Jürgen Bathe