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PROFESSOR: Ladies and gentlemen, welcome to this lecture on nonlinear finite element analysis of solids and structures. In the previous lectures, we considered the general continuum mechanics formulations that we use for nonlinear finite element analysis. And we also introduced briefly the finite element matrices. In the coming lectures, I would like now to discuss with you these finite element matrices in more detail.

Finite element matrices can generally be categorized as continuum elements. We call them sometimes also solid elements. The truss element, for example, would be a continuum element. The 2D element, the 3D elements would be continuum elements. The 2D and 3D elements we may also call solid elements. And as another category, we have the structural elements. Structure elements are beam elements, plate elements, shell elements.

Of course, a distinguishing feature between the structural elements and the continuum elements is that continuum elements carry only nodal point displacements as degrees of freedom, whereas the structural elements have also rotational degrees of freedom at their nodes.

In this lecture, I like to talk about the 2D continuum elements, the 2D plane stress, plane strain, and axisymmetric elements. These elements are used very, very widely in the engineering professions for all sorts of analyses-- plane stress analyses of plates, plane strain analysis all dams, axisymmetric analysis of shells, and so on and so on.

The elements are very general, and can be used for geometric and material nonlinear analyses. I also like to then, at the end of the lecture, talk briefly about the 3D elements that are also very widely used, and that are really formulated in the same way as the 2D elements. Therefore, once you understand to 2D elements very well, it is fairly easy to generalize these concepts or use these concepts also to construct and formulate 3D elements.

Let me now go over to my view graph and discuss it with you the information that I have on these view graphs. Once again, I like to talk about plane stress, plane strain elements, and axisymmetric elements in this lecture. And these derivations that we will be discussing, as I said already, are directly applicable also, or can directly be extended to three dimensional elements.

Let's look at a typical 2D element, two dimensional element. This is a nine-node element in the stationary coordinate frame, x1, x2. x times 0, we would see this element here. Notice there are nine nodes, 1 to 9. Notice that we will be talking about the isoparametric elements. And these have the R and F auxiliary coordinate system, the natural coordinate system, just like in linear analysis.

X times 0, to the element is here. And at time t, the element is here. Notice that the element has undergone large displacements, large rotations. You don't see very large rotations here, but the rotations could be very large. And certainly also large strains. You can see directly that the element here has grown from its size, so certainly it must have been subjected to large strains.

So we consider really a very general motion. But remember, once again, that the coordinate frame, x1, x2, the Cartesian coordinate frame, remains stationary, as we have discussed in the previous lectures.

Because the elements are isoparametric elements, we can directly write these expressions. That 0x1's, the coordinates of the material points in the elements at time 0, are given via this interpolation. The hk are the interpolation functions that we also use in linear analysis.

The 0x1k are the nodal point coordinates. k refers to the nodal point. 1 refers to the coordinate direction. 0 refers to the fact that they're looking at the configuration at time 0. This 0 is, of course, the same 0 that we see here. This one is the same one that we see there. We are summing, of course, of all the nodes. And for the element that I just had on the previous view graph, it would be nine nodes, so n is equal to 9.

We have a similar expression for the x2 coordinate direction. In other words, 0x2 is given like that. This is the x2 coordinate of a material particle, and it's expressed in terms of the nodal point coordinates of the elements expressed in terms of the nodal point coordinates of the element.

The same expression is also applicable at time t. Notice all we have exchanged is the 0 to a t, 0 to a t. And similarly here for the x2 coordinate. 0 to t, and similar here, 0 to t.

Let us look at an example. Here we have depicted a four-node element. The original element lies here. It's black. The R and F system, of course, is the natural coordinate system. This is a configuration of the element at time 0. It moves into that configuration to time t, or it is at, in this configuration, at time t.

Notice that these are the interpolations that I just introduced you to. The hk functions are, of course, interpolation functions. Once again, these are here, the nodal point coordinates. k is the nodal point, t is the time that we're looking at, xi means the i's direction. i, of course, in this particular case, 1 or 2. Similarly, for the original geometry of the element.

The hk's are listed out here on the right-hand side. Notice these are the same interpolation functions that we are using in linear analysis. No difference there. For example, h1 is given right here. And this is, of course, the interpolation function corresponding to nodal point 1, as shown right there. You are probably very familiar with these interpolation functions, and I don't need to go into details there.

But let us look now at the following, namely what had happened in the motion to a nodal point. A typical nodal point would be the second nodal point here. The original coordinates are as shown. And these original coordinates have grown, or have become larger as shown here, because this node here has moved to that new position. Notice this is now here, the coordinate of node 2, of course, x1 coordinate.

This superscript 2 means node 2, this 1 means 1 direction. t means time t. Here we have 1 and 2 and time 0. Notice here, 2, 2 times 0. This 2 here, the bottom 2, means coordinate direction 2. This top 2 means nodal point 2. This is a convention that we want to use. It is a bit heavy, but we have to somehow use a convention to label our coordinates, and this is the one that I chose some time ago.

Similar here for the 2 coordinate, x2 coordinate, at time t. tx2 at nodal point 2. That upper 2, once again, be the nodal point 2. And of course, this would also apply for all the other nodal points.

If we look at the motion of a material particle that is in the element, we would obtain that motion from the motion of the nodal points. Here we have now, in a nine-node element that is originality here, and moves into this position. At time t it is in this position.

Let's look at one particular particle within the element. Here we have one particle right there. Notice that this particle here is given via this relationship here. r and s are both 0.5-- you see r positive means this direction, s positive means that direction, r and s 0.5 is there. We would use r and s equal to 0.5, substitute into hk. And then, of course, we have 9 such hk's, substitute r and s equal to 0.5 into each of these hk's. And sum out, this right-hand side, to get the coordinate, the coordinates-- there are two, of course-- of this point here, this material particle, at time 0. That's how we would obtain the coordinates of that material particle.

Now, at time t, we proceed much in the same way. Here we have the equation. We would, again, take hk at r equal to 0.5, s equal to 0.5, for all case, and multiply these hk values by these values here, which of course, are given because we must know where these nodal points have arrived at.

So we can evaluate the right-hand side to directly get these two values, there are two such values, tx1 and tx2, which gives then the position of this material particle at time t. Notice that the isoparametric coordinates of a material particle never change. You put that here in red because that is very important to keep in mind. Of course, the actual coordinates of the particle change, because that particle moves through space in a stationary coordinate frame, x1 and x2. But we are in s coordinates. The natural coordinates do not change.

Well a major advantage of the isoparametric finite element analysis is that we can directly write, of course, at the displacements, are given as shown here via the nodal point displacements. tU1 is the displacement of the material particle into the one direction. The hk's are the isoparametric interpolation functions. And these are the nodal point displacements. k being the nodal point displacements of nodal point k. Similarly, for the 2 direction. And this holds at time t, and it also holds for the incremental displacements from time t to time t plus delta t, as written right here.

That this is, in fact, 2 can easily be shown from the coordinate interpolations. You see we had already these two interpolations, and all we need to do now is subtract on the left-hand side, and on the right-hand side, to obtain this equation here. And what are left with here, of course, must be the displacement of the material particle that we're looking at-- tUi. And here, we must have the displacement of the nodal points. And these are denoted as tUik. And that is exactly the relationships that we were just dealing with, tUi is equal to hk tUik.

There is one very important point that I like to point out to you. Namely, that these equations show directly that if we use a finite element mesh that is originally compatible, in other words, compatible in a linear analysis, than this finite element mesh will remain compatible throughout the motion, throughout the large deformation motion. And that is a very important point. That we can say that the mesh which originally is compatible will remain compatible throughout the analysis. That follows directly from these equations.

The element matrices that we need, of course, require derivatives, and these are obtained much in the same way as in linear analysis. We need this derivative here, partial tUi, with respect to the original coordinates. This is the actual derivative. This is the abbreviation that we used in the earlier lectures. And we obtained this derivative by taking the differentiation of the hk's with respect to the original coordinates. Of course, these are numbers. These are the nodal point displacements. So it's this one that we really need to evaluate this derivative.

Similarly, for the incremental displacements. We want to take the differentiation of the incremental displacement with respect to the original coordinates. It's achieved this way. Once again, here we have an expression that we need to evaluate, which also goes in here, of course. And as we evaluate, as we will just now see, much in the same way as in linear analysis.

Notice, here we have written down the partial of Ui with respect to the current coordinates obtained as given here. Of course, these are the nodal point displacement increments, and here we have the differentiation of the hk's with respect to the current coordinates now. So once we have these evaluated, these expressions evaluated, we can obtain all of the derivatives that go into the strain displacement matrices that we want to have for the element.

The derivatives are evaluated using the chain rule, just as in linear analysis. We are using that partial hk with respect to r, is given as partial hk with respect to x1 times partial x1 with respect to r, et cetera. Of course, these are here the derivatives that we want to calculate. Notice, this is what we need to calculate in order to get these. And these here, just like in linear analysis, go into the Jacobian matrix written down here. Here we have the Jacobian matrix.

And this is what we want. Therefore, we need to invert the Jacobian matrix, as in linear analysis. And we obtain via this relationship here, the required derivatives. The entries in this matrix involve derivatives of this form. Partial x1, 0x1 with respect to r, with respect to s and so on.

And those are obtained here as shown on the right-hand side. Notice here, of course, we only need differentiations with respect to r, with respect to the natural coordinates. And since the functions hk are a function of r and f, we can directly evaluate these kinds of expressions that go in the Jacobian matrix, and the inverse, of course.

If we want to take derivatives with respect to the current coordinates, we proceed much in the same way. This is the relationship that we arrive at by simply substituting instead of the 0xi, the txi in to the Jacobian matrix, and of course, into the expressions that are in here.

So here we have a Jacobian matrix that is giving the derivatives of the current coordinates at time t with respect to the natural coordinates. Such an element is obtained as shown here. It involves again only the differentiation of the interpolation functions with respect to the natural coordinates. Here, of course with respect to s, because we want to differentiate with respect to s. Of course, these are the nodal point coordinate at time t, which are known. We invert this relationship here to obtain the differentiation that we need to have.

We are now ready to compute the required element matrices for the total Lagrangian formulation. And the element matrices that we want to compute, of course, our those established in the early lectures. The matrices that go into evaluating these matrices are listed here. Here, of course, the tangent constitutive relationship, which we will talk about much later. Not in this lecture. Here we have the linear strain displacement matrix, the nonlinear strain displacement matrix. Here again, the linear strain displacement matrix, a stress vector, and a stress matrix. Let's see now how these matrices look for the two dimensional case.

The constitutive relation, just very briefly-- again, much more detail will be given in a later lecture-- would be relating the increment in stress to the increments in strain. Notice there's a 2 here, and this is the matrix which is established from this relationship here. In the early lectures, we used this relationship, the tensor notation. Well we have now a matrix notation, and this is the relationship with that matrix notation.

C for a linear elastic material would look as shown here, and you are familiar with this relationship from linear analysis. It's the same matrix that we encounter in linear analysis. Of course, e being Young's modulus, nu being Poisson's ratio.

We also derived in the early lectures the expression for the incremental strain terms. And here they are listed out. We have derived these right-hand side expressions. We notice that what is here underscored by a blue line was the initial displacement effect. And is this initial displacement effect, of course, is a particular in gradient off the total Lagrangian formulation, as we discussed in the early lectures.

The nonlinear strain terms that we also derived in the earlier lectures, are listed here for the two dimensional case. We have seen all these expressions on the earlier view graph, except that we use then additional notation, in other words, a notation that involved k and j's and i's, and we have to sum over k. Well if you do so, you directly arrive at these relationships here.

It's an interesting point to derive this expression and the linear strain part for the hoop strain-- this is called the hoop strain-- in the axisymmetric case. And let us look at that derivation a bit more in detail. Here we have an axisymmetric element in its original configuration. At time 0 it has moved into this configuration up to time t plus delta t. Since we want to get the incremental strain from time t to time t plus delta t, we're looking at the configuration of time t plus delta t in this derivation.

Notice that this here is the axis of revolution, which we denote as x2, x1 is this axis. And if we look as a plan view onto this element, we would see this x1, x2 coming out of the view graph, and x3 going down like this. Notice this here we label as 0ds, the initial length of a hoop fiber, so to say. In fact, this hoop fiber starts right there at 0x1, which is, and I have to go now upwards to the upper graph again, which is nothing else than the starting point of that fiber.

In other words, this green dot, this green dot is nothing else than the start of this arrow. I could say let's line them up like that. So this arrow here goes really into the view graph up there. If you look further to the right here we see that we have also a blue arrow, of course, curved. We call it the hoop, it's a circle really, the radius around this origin that we're looking at, like that.

And notice that the start of this arrow here is this blue dot right there. A material fiber, a material particle, let's put it this way, a material particle that has moved from here to there causes, in axisymmetric analysis, this fiber here to take on this length.

And of course, the change in this length gives us the hoop strain. Let us look at this relationship here, because that gives us a relationship between the change in the length of the fiber, green fiber, blue fiber, so to say, in plan view. And that is nothing else than 0x1 dividing t plus delta x1. Why is that the case? Well you see it by geometry from this picture. And we will use this expression here now to evaluate the actual strain.

We find that the Green-Lagrangian strain can be written as in this form. We substitute the expression that we just obtained for this relationship. We substitute the displacements. Displacements to configuration t, and incremental displacement from t to t plus delta t divided by the 0x1, of course, still here. We then go through a number of steps of arithmetic, I'm sure you can easily do those, and you arrive at this result. If we look at this, we find that this here expression only involves the displacements to time t. And that must, therefore, be the Green-Lagrangian strain at time t.

Notice that this expression here involves incremental displacements, linear incremental displacement, no products of them, and the initial displacement at time t. This is the initial displacement effect, which we always have in the total Lagrangian formulation. This is a linear strain term involving only the incremental displacement. And this the total linear strain increment.

Notice this is a total nonlinear strain incremental. It involves the incremental displacement u1 squared, and that's why it is a nonlinear strain increment. Of course, this total here is the incremental strain. This total is the incremental strain. It's a nice derivation that gives some insight into how these expressions that I had already on the earlier view graph are arrived at.

Well we are now ready to construct the B matrix. On the left-hand side, we would have, of course, the strains. If you look at the linear strain displacement matrix, we have the linear strains here. Notice there's a 2 here because of the 0e12 being equal to 0e21. We simply put 2 times 0e12 in here. And this is a total linear strain increment. Here we have a sum of two matrices giving us a total linear strain displacement matrix. This is the one that does not include the initial displacement effect. That one includes initial displacement effect. And here we have the nodal point incremental displacement the way we defined them in an earlier lecture.

Well the entries in t0BL0 are shown here, involving of course, four-nodal point k derivatives of the interpolation functions corresponding to nodal point k. Notice that this last row is only included if we are dealing with an axisymmetric analysis. And notice that these are then exactly the terms that we also have in linear analysis.

So no surprises here. No new entries, as a matter of fact, when compared to linear analysis. Except that we see a 0 here, meaning, of course, that we're taking a differentiation with respect to the original coordinates. Generally, in linear analysis, if you look at the book, of course, you would not see this 0 here because it's not necessary to have that 0 there. We always take differentiations with respect to the original geometry.

Maybe a quick word also how we want to read this here. Notice these two entries here, of course, nothing else in these two blue entries. Because these two entries multiply these two columns, for readability I put this entry there, put this entry there, so that you directly see this column here corresponds to u1k, and this column corresponds to u2k. Of course, both columns correspond to node k.

If we look at the matrix t0BL1, which includes now the initial displacement effect, in fact, introduces the initial displacement effect to the total strain displacement matrix, it looks this way. Once again, this is a contribution coming corresponding to u1k This is the contribution corresponding to u2k. And notice here you have the initial displacement effect appearing right here. All initial displacement effects. And similarly here, initial displacement effect.

Once again, if you don't have an axisymmetric analysis, in other words, you have a plane stress, plane strain case, you would drop this last row. We construct the t0BNL and T0S matrix. Next for the geometric stiffness matrix, and we talked about this expression in the earlier lecture, notice that this here is giving up, of course, the matrix, the k matrix, that we're looking for, this part here. And the F matrix looks as shown here.

I pointed out very strongly in the earlier lecture that we construct the S and the BNL such that when this product is taken, we get directly this one. Because this is basic. This is obtained from the continuum mechanics formulation. And we want to evaluate this. Therefore, we construct the BNL and S such that this product, indeed, gives us this contribution. And as is constructed as shown here. The BNL is constructed as shown here. Notice the u2k contributions, the u1k contribution for node k.

Finally, we also need the t0S vector, hat vector. I pointed out in the earlier lecture that this vector and this matrix are constructed in such a way that this product here gives us exactly this expression here. This is basic coming from continuum mechanics, and this is what we have to capture. The entries in F hat are given here. And with this then, we would be ready to actually calculate all the matrices and vectors for the total Lagrangian formulation.

Let us now look at an example to reinforce our understanding of how all of these matrices are constructed, how they are evaluated. Here we have the following case. A four-node element originally in the configuration shown in black. Four nodes, as you can see, one, two, three, four. This element moves from time 0 to time t into this configuration. The RAD configuration. Node 0.1 move there, node 0.2 move there.

Notice that the element has stretched and sheared over, but it has only stretched into the vertical direction, not into the horizontal direction. Let us identify the lengths, values that we have to deal with. 0.2 here, 0.2 there. Notice that the displacement upward is 0.1. And the shearing over, so to say, is 0.1. We want to consider plane strain conditions. And the problem that we like to pose is calculate these two matrices, the linear strain, and the nonlinear strain displacement matrix for this particular case.

Let's first look now a bit at what's happening here to the material fiber. If we look at the horizontal material fiber, shown here in black in the original configuration, we notice that these material fibers are simply translated rigidly over horizontally. As shown here via the red arrows. So these black material fibers lying horizontally are simply sheared over rigidly, as shown by the red arrows. 2 time t, of course.

Let's look now at the vertical fibers in this problem. Let's see what has happened to them. Here we have the vertical fiber shown in black in the original configuration. Let's see how they end up in their final configuration. We see that these fibers have actually stretched, they've become longer, and they have also rotated over.

Well this is the kinematics that we are looking at. And let us now attack the problem of constructing, once again, the linear strain displacement matrix, and the nonlinear strain displacement matrix. We do so by first identifying which is the isoparametric coordinate system that we use. That coordinate system is shown here, r and s.

Let us then now start this problem. And we identify simply by inspection really because it's a fairly simple geometry that we're dealing with. That this differentiation is nothing else than 0.1, this differentiation is 0, that one is 0, and this one is 0.1. Of course, these are the elements that go into the Jacobian matrix, as I discussed earlier.

We put these elements into this 2 by 2 matrix, calculate the determinant, interesting to note what the value is. That one is needed, of course, later on in the numerical integration of the matrices. And we also recognize that to obtain this derivative of an interpolation function, we need to invert this matrix here.

This 0.1 inverted gives us 10, and that is this 10 right there. Therefore, we have now partial with respect to x1, 0x1 is 10 times partial with respect to R. Of course, in the actual expression, we would put hk's in here, and hk's in there. Similar we get the differentiation with respect to x2. We will use this expression here, putting the actual hk's in there and in there.

Well this is some preparatory work to actually complete the problem of constructing the linear strain and nonlinear strain displacement matrix. And I think this is actually a very good point where you might want to sit back and try to complete the whole problem.

Ladies and gentlemen, welcome again. I hope you've had a close look at the example, and surely I would be very interested in knowing how it went. But let us now look at the solution. We already discussed that for this example, the Jacobian is given by this matrix here. And therefore, these are the differentiations that we can directly use. Notice, of course, we need the differentiation with respect to the x1 coordinate, and with respect to the x2 coordinate of all the interpolation functions.

One way to proceed now is to make a little table where we have here in this column the nodes, we have in this column here the differentiations that we need, partial hk with respect to 0x1. Further differentiations here, partial hk with respect to x2. Notice that the Jacobian, the inverse of the Jacobian matrix being 10, makes a 1/4 equal to 2.-- 1/4 times 10 being equal to 2.5. So that's why you see the 2.5 here. These are the nodal point displacements, which, of course, are given for this particular case. And with these nodal point displacements given, and these differentiations calculated as listed here, we can now evaluate these products here. And that is done as shown in these columns.

Notice the sum of these gives us these differentiations here, which go into the initial displacement effect of the strain displacement matrix. Well like that, of course, you can also calculate the differentiations of the t0u21 and the t0u22. And both these terms are also required in the initial displacement effect of the strain displacement matrix.

Another way to proceed, to get this initial displacement effect, is to evaluate the deformation gradient. In an earlier lecture, we talked about the deformation gradient and it's listed right here. It's a 2 by 2 matrix because we're talking about a two dimensional motion. And here you have, for example, partial tx2 with respect to 0x1. Here you have partial tx1 with respect to 0x1. 0x1, et cetera. And if we take this deformation gradient or the elements of that information gradient, we can directly obtain these differentiations here.

In other words, t0x11 minus 1 must give us this expression here. t0x12, being that one here, gives us tu1 with respect to 0x2, in shorthand notation written as shown here, et cetera. So we can obtain, in other words, these elements also from the deformation gradient.

Let us now look at how the columns of the strain displacement matrices look. We simply substitute into the general expressions that I gave you earlier, and here we have the t0 be a 0 entry for node three. Of course there are eight columns altogether because we have a four-node element. We just showed two such columns here for-- namely, those corresponding to node three. Similarly, for t0BL1, we get these entries here. Once again, also for node three, of course.

Similarly, we can also construct the corresponding columns in the t0BNL matrix. And these columns are given right there. Once again, for node three. This is, of course, also a matrix that have altogether eight columns, because there are four nodes, eight degrees of freedom.

Let us now, as a next step, consider the updated Lagrangian formulation in the general plane stress, plane strain, axisymmetric case. In the updated Lagrangian formulation, we have identified earlier, in an earlier lecture, that we need these matrices and that vector. This is, of course, a linear strain stiffness matrix, that's a nonlinear strain stiffness matrix, and that is the force vector that corresponds to the internet element stresses.

The matrices that go into the calculation of these matrices are listed here. The tangent material relationship, the linear strain displacement matrix go into the calculation of this k matrix. This stress matrix, and that strain displacement matrix, the nonlinear strain displacement matrix, these two quantities go into the calculation of this k matrix. And this stress vector, and that linear strain displacement matrix goes into the calculation of that F vector. We already derived that all earlier in an earlier lecture.

The stress strain matrix is listed here. Notice that we have the incremental stresses. Of course, these are increments in the second Piola-Kirchhoff stresses from time t onwards. That's why you have the t here. That's why this t corresponds, of course, to the updated Lagrangian formulation. Here we have the material tensor, here we have the strain terms.

Once again, the 2 that I pointed out early already for the total Lagrangian case as well. Of course this matrix here, contains the elements of the material tensor, tCijrs, which we used earlier when we talked about the continuum mechanics formations.

Well for the linear elastic case, we would simply use the very familiar stress strain law that we are using also in linear analysis. Young's modulus, Poisson's ratio. So this is one typical case for this C matrix. Of course, we will discuss later on in later lectures how we construct the C matrix for all sorts of material conditions.

We need for the B matrix, for the linear strain displacement matrix, these entries here, meaning we need these differentiations. Notice partial u1 with respect to tx1 is in shorthand notation written as shown here. te22 is equal to that element, and similarly we go on. Notice this, of course, is again, the hoop strain.

For the nonlinear strain displacement matrix, we need to look at the nonlinear strain terms. And the nonlinear strain terms are listed out here now. We identified these strain terms earlier when we discussed the updated Lagrangian formulation in an earlier lecture. Of course, at that time, the strain terms were represented in terms of ijk indices. Now we have to simply put i and j equal to 1, for example, and k equal to 1, and equal to 2, and you will directly obtain from the earlier expression that I presented to you this term here. And similarly, you obtain all the other terms as well.

By the way, the hoop strain, the linear, and here we see the nonlinear hoop strain, expression would be derived in the same way as we have done it in this lecture for the total Lagrangian formulation. It would actually be a good exercise for you to do so one and see how that goes. We construct the ttBL matrix to capture the total strain, total linear strain, listed in this vector here as shown by this equation. Of course, here we have the nodal point displacements. The nodal point displacements are in this vector. Denote always that these are the discrete nodal point displacements.

This last term, of course, we only introduced in axisymmetric analysis. The entries in this ttBL matrix are shown here for a typical node k. We use the same kind of picture as for the total Lagrangian formulation. Notice here are the actual displacements, the way they would appear in a vector. But since this element here hits this column so to say, we have written it here once more in blue. This element here hits this column, and we have therefore written it here once more in blue. So this is really the column corresponding to the case node and the u1, the 1 direction. This is the column corresponding to the case node and the 2 direction.

If you compare this matrix with the matrix that we use in linear analysis, you would see that the matrix looks very similar, except that in linear analysis you would not have this t, or you don't put that t there in general because the t would, of course, be actually the 0 configuration because all the differentiations are referred to 0 configuration anyway. And it is common to not have an index down here. If you look at this textbook, chapter five, you would, for example, see there expressions such as this without this t there.

So there's really not much of a surprise right here. You will surely see that this is the right relationship to use in the B matrix based on your knowledge of linear analysis.

The expression of the nonlinear strain displacement matrix and the stress matrix are these two expressions are constructed in such a way as to have that this product on the left-hand side is equal to that expression there. I pointed that out already earlier, that on the right-hand side this is the continuum mechanics variable, and this continuum mechanics variable is, of course, the basic quantity, the basic quantity that we actually want to capture. And we construct this matrix and that matrix such as to do so. I should just point out there should be no bar here. Of course, this is a tensor quantity, it's not a matrix, and so there should be no bar there. That was a little error on my part.

The entries in the t tau are given right here. The last row and column are, of course, only included if you deal with an axisymmetric analysis. ttBNL is shown here. Notice again, differentiations with respect to time t. And the last row is only included if you have an axisymmetric analysis.

Finally, we want to also construct our stress vector such that when it is entered here with the BL matrix that we already have defined, we capture via this product here exactly that term. This is the term that is basic, that we have derived from the continuum mechanics formulation. This is what we want to capture, and we do so by this expression. And t tau hat is given right here. It lists all the stresses really, including the hoop stress if you have an axisymmetric analysis.

This completes what I wanted to say about the two dimensional elements. We talked about the total Lagrangian formulation, and the updated Lagrangian formulation of these elements. In other words, we really presented the matrices corresponding to these formulations in quite some detail. These details that we discussed for the two dimensional elements are also almost directly applicable for the three dimensional elements.

And here we have a typical eight-node element in a stationary coordinate frame, x1, x2, x3, in its original configuration. The nodal point coordinates are listed here. Notice k, of course, stands again for the node k. 1, 2, 3 stands for the directions 1, 2, 3. And the left superscript here stands for the time 0, the configuration 0. The element moves from time 0 to time t up to this configuration here. And the nodal points now are tx1k-- nodal point coordinate I should have said. Now tx1k, tx2k, tx3k.

The notation is much the same as we have been using in two dimensional analysis. Of course, we now have a third coordinate, the x3 coordinate also in the formulation. We now use interpolations much the same way as for the two dimensional elements. For the original configuration, we have these interpolations.

Notice, the third direction enters now. Once again, these are the interpolation functions. These are the nodal point coordinates corresponding to the 1 direction. These are the nodal point coordinates corresponding to the 2 direction, and so on.

Of course, we now also have to introduce the nodal point coordinates corresponding to the 3 direction. And if the number of nodes, if you have an eight-node element, and of course is 8. These are the interpolations used for the original geometry of the element, and for the configuration at time t, we use these interpolations.

Notice these are now the nodal points coordinates corresponding to time t. We use still the same interpolation functions, of course. And N is the same number as for the 0 interpolation, or rather for the interpolation of the original geometry. In other words, for the eight-node element that we briefly considered, N is in each case equal to 8.

We use these expressions, we subtract from tx1 the 0x1 from tx2, the 0x2 expression from tx3, the 0x3 expression, and directly obtain the interpolations for the displacement of the elements. And once we have the displacement interpolation, we, of course, can directly find the derivative of these displacement interpolations to obtain the strain interpolations. And these expressions then, the derivatives of the displacement interpolation, enter into the construction of the strain displacement matrices.

So we see that basically, the same procedure that we discussed with two dimensional analysis is directly applicable to the three dimensional analysis, the only difference being that the third direction, x3, has to be interpolated, the displacements have to be carried along corresponding to the third direction, et cetera.

This then brings me to the end of what I wanted to say in this lecture. Thank you very much for your attention.