PROFESSOR: Ladies and gentlemen, welcome to this lecture on nonlinear finite element analysis of solids and structures. In the previous lectures, we discussed the general continuum mechanics formulations that we use in nonlinear finite element analysis. And we also derived some element matrices.

In this lecture, I'd like to discuss this you the truss element. The truss element is a very interesting element and a very important element. It is important because it's used in the analysis of many truss structures and also in the analysis of cable structures. It's an interesting element to study from a theoretical point of view because we can use the general continuum mechanics equations analytically and derive directly the finite element equations and the finite matrices corresponding to truss element. These are derived in closed form. We can study these matrices and get some insight, some physical insight, into what really these individual terms in the continuum mechanics equations mean.

We want to study in this lecture the updated Lagrangian formulation. And in the next lecture, the total Lagrangian formulation. I mentioned earlier in a lecture that these two formulations really reduce to exactly the same matrices provided certain transformations routes are followed. And for the truss element, we can actually very nicely analytically demonstrate that all is true.

So let us study now the updated Lagrangian formulation and later on the total Lagrangian formulation of the element. The assumptions that we are using for the truss element are that the stresses are transmitted only in the direction normal to the cross section. The stress is constant over the cross section. And the cross sectional area remains constant doing deformations.

Of course, these first two assumptions here are those that we are also using in linear analysis. This we add now as an assumption, meaning that we really only consider small strain problems. We will consider large rotation, large displacement, but only small strain problems.

Here we have a typical element. We align that to another truss element with the x1 in its original configuration. And the elemental goes through large deformations as you can see and the large rotation, theta. This is node 1, originally here, moves there. This node 2, originally here, moves there.

Notice that the length of the element is L in its original configuration and it is also L in the configuration at time t. We use Young's modulus, an elastic material, in other words, we assume for the element. The cross sectional area is A. And once again, we also use the fact that the element lies in the x1, x2 space.

The same kind of deviation that I'm now following through could, of course, also be generalized to the three dimensional case, in other words were you have three axes, x3 being included. But all of the relevant information, particularly the physical insight that we want to get from this deviation, is very well obtained by just considering element lying in the x1, x2 plane and moving in that plane.

If we now look at the deformations of the element, we can see that at times 0 it was here once again. The displacement off node 2 is into the x1 direction signified by this symbol here, time t, u, into the 1 direction at node 2. Into the 2 direction, we have u2. The lower 2 means 2 direction, 2 of course being upwards. The upper 2 means node 2. The t, once again, means at time t.

If you look at the left node here, node 1, that one moves this way and that way. We use, of course, a similar symbolism to denote this movement. So this is the deformation from time 0 to times t given by these nodal point displacements.

Now from time t to time t plus delta t, we obtain am addition deformation. And that additional deformation is given by these displacements, at node 2 here and at node 1 here. Notice that, in other words, from time t to times t plus delta t, the truss has moved from the red configuration to the blue configuration.

Of course, note that these displacements are measured in the stationary coordinate frame. I made a big point in the earlier lectures out of the fact that the coordinate frame remains stationary and the elements move through the coordinate frame.

Of course, what we're interested in doing is to calculate, in the finite element solution, these incremental displacements. We assume that the configuration at time t is known. We want to calculate the new configuration. That means we want to calculate these incremental displacements. And we achieve that by setting up the appropriate matrices for the element.

Well to develop the appropriate matrices for the updated Lagrangian formulation which I would like to discuss in this lecture, it is best to introduce an auxiliary coordinate frame. And that auxiliary coordinate frame is one that is aligned with one axis along the element. We introduce x1 curl and x2 curl. The curl always meaning body aligned coordinate frame.

Notice that this is now the rotation here that the element has undergone. And of course, we really want to get the stiffness matrix in the x1, x2 frame. x2, of course, being perpendicular to x1, not shown on this viewgraph, but shown on the earlier viewgraphs. We know that if we have calculated the matrices in this curl coordinate frame, the body attached coordinate frame, we very easily can transform to the stationary coordinate frame.

So let us now concentrate on finding the K-matrix, the F-vector, corresponding to this body attached coordinate frame. To do so, we look back to our continuum mechanics equations that we developed in earlier lectures. Here we have the basic continuum mechanics equation, the principal of virtual work written for the update Lagrangian formulation in the curled coordinate frame, see that curl there.

And this was the starting point for the development off all of the equations that we're using in the update Lagrangian formulation. The linearization resided in this equation. And we talked about this equation quite a bit. Except, of course, we did not have to curls there because we were talking about the x1, x2, x3 uncurled coordinate frame. Now we're talking about this curled coordinate frame, so I simply put a curl on there. But otherwise, the terms are identical.

Because for the truss as I state earlier the only non-zero stress is the stress along the length of the truss which acts normally to the cross section of the truss, we can simplify this general equation to this equation. The only non-zero stress is tau curl 1 1. The only strain that we call stress, only small strain that we call stress, is e curl 1 1. Of course a t on the left-hand side because it's referred to as the configuration at time t.

So this very general equation reduces to a much simpler equation already as shown here. What we are out now to do is to evaluate these quantities here and there using the finite element interpolation.

First let us now look at what are some of the terms that we easily can identify. We identify this tensile term here to be simply Young's modulus. This stress term to be simply the force in the truss divided by the cross sectional area. The volume of the truss is given here. Notice, once again, the length of the truss is assumed to be constant, therefore we only consider small strain conditions, I mentioned that earlier.

If we now use this information in the general equation, we directly arrive at this equation here. You simply substitute and you will see immediately that these are the terms. Of course, what we now have to evaluate are these curled terms here.

To proceed, we identify that the e curl 1 1, the linear strain term, is simply given by the linear strain displacement matrix with a curl on it times the nodal point displacement vector. These are the nodal point displacements. There are four such displacements because we have two nodes and two displacements per node.

Notice that I have a curl here signifying that we're talking about the curled coordinate frame. And there's a hat on top off that curl as you can see here. That hat means these are discrete nodal point displacements.

This term here is evaluated via that term. Notice that we construct the BNL in such a way that this right-hand side is equal to that left-hand side. I mentioned that also in the earlier lectures for the two dimensional and the three dimensional finite element cases.

Well the vector of nodal point displacements, this vector here, is listed out here. Notice it contains the four displacements that I referred to earlier. And these are the displacements, u curl 1 1, u curl 2 1. This upper 1 always denoting nodal point, the lower 1 and 2 meaning coordinate directions. Similarly for these two terms.

To evaluate now these terms, we recognize that the total strain, the total incremental Green-Lagrange strain I should say, is given by this relationship here from our general continuum mechanics equation. The linear strain term is given here. And the nonlinear strain term is the rest. The rest meaning taking this total, subtracting the linear one, you're left with that one.

And once again, these expressions are directly obtained by just taking the general continuum mechanics equations and varying the indices the way you want to see them varied. 1 1, for i and j, if you would refer back. And k, the k that used earlier goes over 1 and 2 in this particular case. In a three dimensional case, of course, you would have k also going to 3.

The variation on this term resides directly in this expression. And we notice that this expression can also be written in matrix form. One row vector times one column vector. It is convenient to work now with this matrix form because we want to bring it into a form BNL transposed BNL.

Before doing so, let us identify one important point, namely that the displacement derivatives are constant along the truss. The reason for it is that we have only two nodes and these two notes can only specify a linear variation and displacement between the two nodes, meaning a linear variation along the truss and the displacement derivatives are therefore constant. For example, this displacement derivative is simply obtained by taking the difference in the nodal point displacements, of course, in direction 1 because we're talking about 1 here, over the length L.

So this is directly obtained as u curl 1 comma 1. Similarly, we also evaluate u curl 2, 1 and you see the lower part here gives us that expression. Of course, the upper row here is nothing else than the rewriting of that equation. If you rewrite this equation in matrix form, you directly obtain this relationship. Or another check would be simply takes this vector, multiply this matrix by that vector, and the upper part of that multiplication will directly be this result.

If we now use this information, we directly extract the BL matrix, which of course links up the linear strain increment with the nodal point displacements. Here there is a small error. Let me just point it out. This bracket, of course, should go to there because the B matrix does not contain the nodal point displacement vector. It only goes up to there. Because the e curl 1 1 is equal to B matrix times the nodal point displacement vector.

Similarly we can write now the variation in eta 1 1 in this form. Notice that this part here captures this amount and this part here captures that amount. This one, of course, we had already derived earlier. We just plug it in now.

With this information we can not directly develop the k matrices. We notice that the linear strain stiffness matrix is obtained from this expression. And here you have it. You simply substitute for these terms and obtain this matrix, much in the same way as we are dealing in linear analysis.

The nonlinear strain stiffness matrix, that's the one we want to look at next, is obtained from this relationship. And we simply now substitute what we have derived for this term and directly obtain this expression here. And what's under this blue bracket, of course, is our nonlinear strain stiffness matrix.

Finally, the force vector is obtained from this expression. And you, once again, simply plug in and obtain this expression here where what's under the blue is the force vector. The beauty is that we have started with a fairly complicated continuum mechanics equation. We have specialized it to the truss element, recognizing that only certain terms really need to be included or are included in this particular case. We have evaluated them in a fairly simple, straightforward manner, plugged in, and obtained now matrices that can of course be analytically evaluated the way that they have it done. And now we can in fact even think about these matrices and try to physically interpret their meaning. We will do that just now.

However, before getting into that, we want to get back to one point, namely that we have sued so far a curled coordinate system. And as we mentioned earlier, we will have to transform from that curled coordinate system to the stationary global coordinate system. That is achieved by this transformation. Notice that the curled displacements are nothing else than transformation matrix times the uncurled displacement. This transformation matrix is very well known from linear analysis. It simply transforms displacement from one coordinate system into another.

This is the one that holds anywhere along the element for the continuous displacement. But it also holds at the nodes. So if we apply that relationship at the nodes, be directly obtain this transformation here. The curled displacements at the nodes are given in terms of the transformation matrix times the uncurled displacement at the nodes. Of course, we want to get the matrix expressions related to this uncurled displacement vector.

Well having obtained this relationship, we go back to the basic equations, recognizing that this part here is coming from the linear strain stiffness matrix. We substitute for u curl, the uncurled vectors with the T's in front. There is a transpose appearing now here. This capital transpose has to be there because there's a transpose there. And what's in here underlined in red is the linear strain stiffness matrix of the element corresponding to the stationery global coordinate system.

We proceed much the same way for the nonlinear strain stiffness matrix. And what's in here now is what we want to get, the nonlinear strain stiffness matrix corresponding to the global stationary coordinate frame. Same thing holds of course for the F-vector. And here we have the F-vector now in the uncurled global stationary coordinate frame. So it's really this part, that part, and this part that we're using in our finite element analysis when we assemble truss elements into a global assemblage of truss elements.

Performing the indicated matrix multiplications gives this expression here. In fact, this is the same stiffness matrix that we use in linear analysis when we have an element that is oriented at an angle theta to the global x-axis, the horizontal x-axis. So you might have very well seen this matrix before.

The nonlinear strain stiffness matrix looks this way. Notice there is a tP over L and then these terms. Of course, it's also symmetric. And this is the matrix corresponding to the global coordinate frame now.

We notice immediately that this matrix is in fact the same matrix that we already had evaluated in the curved coordinate frame. We will get back to that just now. The F-vector comes out to be this.

Let's now look at these terms physically. Here we have a single truss element pinned at the left-hand side. And a load R is applied to that truss at the other end. Of course, this direction of the load must be along the truss element. Or rather the truss element aligns its direction so that it balances this load that is applied. The internal force is tP. tP balances tR. And tP, of course, acts along the truss element.

Now if you look at this node here, node 2, we notice immediately that this tR vector can be decomposed. Or this tR can be decomposed into tR cosine theta along this axis and tR sin theta along that axis. P is acting here and P can also be decomposed as shown.

Now if we look here, we find therefore that the tR vector corresponding to the global coordinate system contains these two entries shown here. And the tF from our earlier viewgraph at node 2 was indeed simply this part. We notice therefore that tR is equal to tF as show here or tR minus ttF equal to 0. And that is, of course, what has to hold when we satisfy equilibrium at that node. So we have a nice interpretation of this F-vector. And in fact, we can see that our arithmetic of getting that F-vector has been correct.

Let's look now at the KNL matrix, the nonlinear strain stiffness matrix. We already pointed out earlier that the KNL in the uncurled frame was equal to the KNL in the curled frame. And we can ask ourself, of course, why is that so? Well the reason is the following.

Let's look again at this truss element pinned here and we look at node 2, then we see that this vector here, which is by Pythagoras the component or the resultant vector of these two displacements, has this length. Now this length can be evaluated as shown here. Because the derivatives are constants, we discussed that earlier, we can write this derivative squared plus that derivative squared, square root out of it, times L, being equal to that.

But then we recognize that what we're seeing here is nothing else than that term there, in other words, our nonlinear strain increment. Which in this particular case we can see, for the given displacement is constant and independent of the coordinate frame used. And that is really the reason why the KNL matrix is constant for any coordinate frame that we are using.

Let us now try to understand what the elements in the KNL matrix physically mean. They, in fact, give the change in force, the required change in the externally applied force, when the element is rotated. Here we have an element at time t, once again fixed at the left end, and subjected to a force tR. Of course, this force tR is balanced by the internal force tP.

Let us now impose a displacement such that the element rotates about this point. We reach the configuration at time t plus delta t by imposing this displacement here, which we denote u curl 2 2. Now in this configuration, a new force has to act on this element. And this new force, denoted as t plus delta tR, is shown here.

Here we have the force t plus delta tR. Notice it is aligned with the red element at time t plus delta t now. The change in the force from configuration t to t plus delta t is given by this vector. And the element in that vector can be calculated by taking moment equilibrium about this point.

We do just that right here. This is the equation of moment equilibrium about the fixed point of the truss. Notice that, of course, this displacement is small. And from this, we directly obtain that delta R is given as shown here on the right-hand side.

And this is actually the entry 4, 4 of the KNL matrix.

Now the same information is also expressed here on the right-hand side where we have written the delta R as a matrix product of KNL times u hat being equal to t plus delta tR minus tR.

So this completes our discussion of the finite element matrices of the truss element corresponding to the updated Lagrangian formulation. In the next lecture, we will actually discuss the total Lagrangian formulation of the truss element. And we will find that identically the same matrices are obtained as in the updated Lagrangian formulation. That will be a very interesting theoretical point, as well as practical point.

Let us now look at an example of using the truss element. And the example is a simple one, and yet quite practical one. We will analyze a prestressed cable.

The cable has a length 2L. At the midpoint, a load of 2 tR is supplied. The Young's modulus of the cable is E, the cross sectional area of the cable is A.

We can model one half of the cable because of symmetry conditions as shown down here. Notice that, of course, we assume the transverse displacement to be small, the angle therefore to be small, because we assume that the length of the cable element remains the same. In other words, it does not change from time zero to time t to time t plus delta t and so on.

Using the U.L. Formation, we obtain directly from the general matrices that we discussed us now, this equation here. Which gives the tangent stiffness matrix times the displacement increment equal to the externally applied load minus the nodal point force corresponding to the internal element stress, the internal element force tP, of course, in this case.

Notice that this is the equilibrium equation that carries us from time t to time t plus delta t. We would iterate on this, of course, introducing an iteration counter here on the right-hand side. We have discussed all that in earlier lectures to obtain an accurate solution for time t plus delta t, but these are some details that we don't need to look at now. For the moment, let's just look at this equation here, in which we imply basically a simple step by step forward incrementation of load without iteration. In practice once again, we would actually iterate to make sure that we are satisfying equilibrium at the end of each time step.

Of particular interest is the configuration at time 0. And there we recognized that the linear strain stiffness matrix up here is 0. And that's all we have in terms of stiffness is the nonlinear strain stiffness. And that is expressed in this equation here.

Notice that therefore if P0 is equal to 0, in other words, there's no prestress in the cable, then initially it has no stiffness. Of course, this element here, let's look at it here at time t, this element here will increase ad the deformations increase because tP will increase. And this element here will also increase as the deformations increase.

And in fact, this is expressed once more here. This total matrix will in fact be increasing. We say the cable stiffens and the load is applied.

If theta becomes very large, in fact if as theta goes to 90 degrees, the stiffness becomes quite large and the stiffness approaches as theta goes to 90 degrees EA over L. Now this, of course, is rather theoretical because we assume in our formulation that the length remains constant and therefore that the deformations are small, as I pointed out earlier.

Let us now look at the actual load displacement curve that is calculated for this particular truss structure, which is really a cable structure, we could look also at it as a trust structure because it is made up of two truss elements. Here we have the deflection plotted. And here we have the applied force plotted for a particular case.

Notice that L is 120 inches, A is 1 square inch. Notice that the maximum displacement we're looking at here is about 2 inches. So 2 inches over the length, 120 inches, means really small deformations.

But notice the stiffening effect shown by the increase in the slope of this curve. We can actually show the stiffness matrix components as a function of the applied load.

And this is a very interesting viewgraph. Here we see that at 0 load, the only stiffness that is there is given by KNL. We pointed that out already earlier. KL is equal to 0.

As the load increases, KNL increases as shown by the blue curve. As a load increases, also KL increases as shown here. And the sum of these two curves, of course, gives us the red curve. And the red curve is the total tangent stiffness matrix. At any particular applied force, we can directly read off the stiffness corresponding to that applied force.

Or, more naturally, we would of course look for the corresponding displacement. In other words, at a particular applied force, we have a corresponding displacement. And for that corresponding displacement, we would have a stiffness. Of course, these curves are directly obtained from the expression that I just showed you earlier.

Well, this brings us to the end of this lecture. In the next lecture then, as I pointed out already, I'd like to look with you at the total Lagrangian formulation of the truss element. And as I mentioned earlier, we will find something very interesting, namely that reducing all of these complex continuum mechanics equations down to what we really need for the truss element, we will find that the same matrices in a total Lagrangian formulation are obtained as in the updated Lagrangian formulation. Thank you very much for your attention.