Topic 9

The Two-Noded Truss Element—Total Lagrangian Formulation

Contents:

- **Derivation of total Lagrangian truss element displacement and strain-displacement matrices from continuum mechanics equations**
- **Mathematical and physical explanation that only one component (${}_0^tS_{11}$) of the 2nd Piola-Kirchhoff stress tensor is nonzero**
- **Physical explanation of the matrices obtained directly by application of the principle of virtual work**
- **Discussion of initial displacement effect**
- **Comparison of updated and total Lagrangian formulations**
- **Example analysis: Collapse of a truss structure**
- **Example analysis: Large displacements of a cable**

Textbook: Section 6.3.1

Examples: 6.15, 6.16

TOTAL LAGRANGIAN FORMULATION OF TRUSS ELEMENT

We directly derive all required matrices in the stationary global coordinate system.

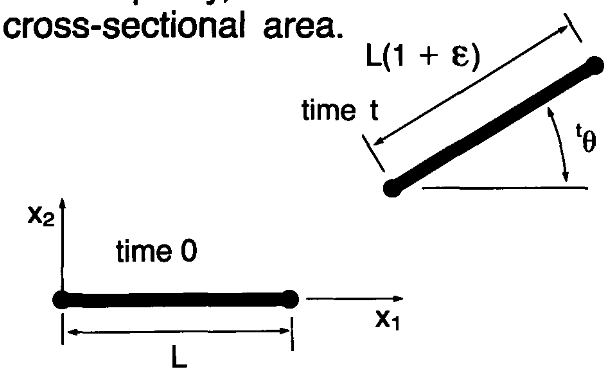
Recall that the linearized equation of the principle of virtual work is

$$\int_{^0V} {}_0C_{ijrs}\, {}_0e_{rs}\, \delta_0 e_{ij}\, {}^0dV + \int_{^0V} {}_0^tS_{ij}\, \delta_0\eta_{ij}\, {}^0dV = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}_0^tS_{ij}\, \delta_0 e_{ij}\, {}^0dV$$

Transparency 9-1

We will now show that the only non-zero stress component is ${}_0^tS_{11}$.

1) Mathematical explanation:
For simplicity, we assume constant cross-sectional area.



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Transparency 9-3

We may show that for the fibers of the truss element

$$ {}_0^t\underline{X} = \begin{bmatrix} (1+\varepsilon)\cos{}^t\theta & -\sin{}^t\theta \\ (1+\varepsilon)\sin{}^t\theta & \cos{}^t\theta \end{bmatrix} $$

Since the truss carries only axial stresses,

$$ {}^t\underline{\tau} = \frac{{}^tP}{A} \underbrace{\begin{bmatrix} (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) \\ (\cos{}^t\theta)(\sin{}^t\theta) & (\sin{}^t\theta)^2 \end{bmatrix}}_{\text{written in the stationary coordinate frame}} $$

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Hence using

$$ {}_0^t\underline{S} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0\underline{X}\, {}^t\underline{\tau}\, {}_t^0\underline{X}^T $$

we find

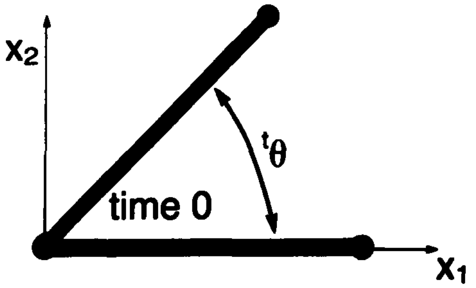
$$ {}_0^t\underline{S} = \begin{bmatrix} \frac{{}^tP}{A}\cancel{\left(\frac{1}{1+\varepsilon}\right)} & 0 \\ 0 & 0 \end{bmatrix} $$

1 for small ε

Physical explanation: we utilize an intermediate configuration t^*

time t^* (conceptual): Element is stretched by ${}^{t}P$.

time t: The element is moved as a rigid body.



$$ {}^{0}_{0}\underline{S} = {}^{0}\underline{\tau} = \begin{bmatrix} {}^{0}P/A & 0 \\ 0 & 0 \end{bmatrix} $$

$$ {}^{t^*}_{0}\underline{S} = {}^{t^*}\underline{\tau} = \begin{bmatrix} {}^{t}P/A & 0 \\ 0 & 0 \end{bmatrix} $$

$$ {}^{t}_{0}S = \begin{bmatrix} {}^{t}P/A & 0 \\ 0 & 0 \end{bmatrix} $$

(the components of the 2nd Piola-Kirchhoff stress tensor do not change during a rigid body motion)

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The linearized equation of motion simplifies to

$$ \int_{{}^{0}V} {}_{0}C_{1111}\, {}_{0}e_{11}\, \delta_{0}e_{11}\, {}^{0}dV + \int_{{}^{0}V} {}^{t}_{0}S_{11}\, \delta_{0}\eta_{11}\, {}^{0}dV $$

$$ = {}^{t+\Delta t}\mathcal{R} - \int_{{}^{0}V} {}^{t}_{0}S_{11}\, \delta_{0}e_{11}\, {}^{0}dV $$

Again, we need only consider one component of the strain tensor.

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Next we recognize:

$${}_0^tS_{11} = \frac{{}^tP}{A}$$

$${}_0C_{1111} = E \ , \ {}^0V = A\,L$$

The stress and strain states are constant along the truss.

Hence the equation of motion becomes

$$(EA)\ {}_0e_{11}\ \delta_0e_{11}\ L + {}^tP\ \delta_0\eta_{11}\ L = {}^{t+\Delta t}\mathcal{R} - {}^tP\ \delta_0e_{11}\ L$$

Transparency 9-8

To proceed, we must express the strain increments in terms of the displacement increments:

$${}_0e_{11} = {}_0^t\underline{B}_L\ \underline{\hat{u}}\ ,$$

$$\delta_0\eta_{11} = (\delta\underline{\hat{u}}^T\ {}_0^t\underline{B}_{NL}^T)\ ({}_0^t\underline{B}_{NL}\ \underline{\hat{u}})$$

where

$$\underline{\hat{u}} = \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix}$$

Since $\;{}_0\varepsilon_{11} = {}_0u_{1,1} + {}_0^tu_{1,1}\,{}_0u_{1,1} + {}_0^tu_{2,1}\,{}_0u_{2,1}$

$$+ \frac{1}{2}\left(({}_0u_{1,1})^2 + ({}_0u_{2,1})^2\right)$$

we recognize

$${}_0e_{11} = {}_0u_{1,1} + {}_0^tu_{1,1}\,{}_0u_{1,1} + {}_0^tu_{2,1}\,{}_0u_{2,1}$$

$$\begin{aligned}\delta_0\eta_{11} &= \delta_0u_{1,1}\,{}_0u_{1,1} + \delta_0u_{2,1}\,{}_0u_{2,1} \\ &= [\delta_0u_{1,1} \quad \delta_0u_{2,1}]\begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix}\end{aligned}$$

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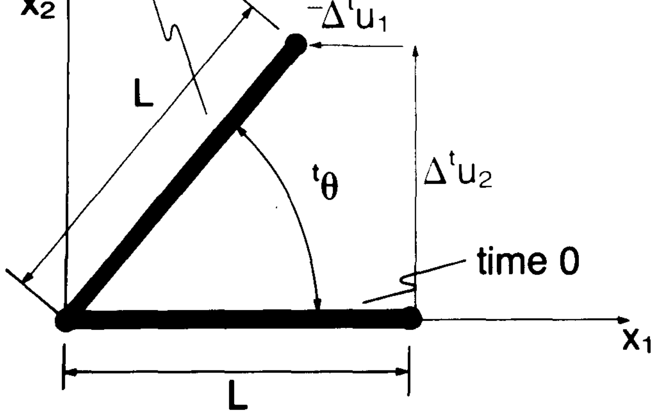
We notice the presence of ${}_0^tu_{1,1}$ and ${}_0^tu_{2,1}$ in ${}_0e_{11}$. These can be evaluated using kinematics:

$${}_0^tu_{1,1} = \frac{\Delta^tu_1}{L} \quad , \quad {}_0^tu_{2,1} = \frac{\Delta\,^tu_2}{L}$$

$$= \cos{}^t\theta - 1 \qquad\qquad = \sin{}^t\theta$$



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Transparency 9-11

We can now write the displacement derivatives in terms of the displacements (this is simple because all quantities are constant along the truss). For example,

$$ {}_0u_{1,1} = \frac{\partial u_1}{\partial {}^0x_1} = \frac{\Delta u_1}{\Delta {}^0x_1} = \frac{u_1^2 - u_1^1}{L} $$

Hence we obtain

$$ \begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix} $$

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Therefore

$$ {}_0e_{11} = {}_0u_{1,1} + [{}_0^tu_{1,1} \quad {}_0^tu_{2,1}] \begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix} $$

$$ = \underbrace{\frac{1}{L}[-1 \quad 0 \quad 1 \quad 0]}_{{}_0^t\underline{B}_{L0}} \hat{\underline{u}} $$

$$ + \underbrace{[\cos{}^t\theta - 1 \quad \sin{}^t\theta]\left(\frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right)}_{\text{initial displacement effect } {}_0^t\underline{B}_{L1}} \hat{\underline{u}} $$

Transparency 9-13

$$
{}_0e_{11} = \underbrace{\frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}}_{{}^t_0\underline{B}_{L0}} \hat{\underline{u}}
$$

$$
+ \underbrace{\frac{1}{L}\left[-(\cos{}^t\theta - 1) \;\vdots\; -\sin{}^t\theta \;\vdots\; \cos{}^t\theta - 1 \;\vdots\; \sin{}^t\theta \right]}_{{}^t_0\underline{B}_{L1}} \hat{\underline{u}}
$$

$$
= \underbrace{\frac{1}{L}\begin{bmatrix} -\cos{}^t\theta & -\sin{}^t\theta & \cos{}^t\theta & \sin{}^t\theta \end{bmatrix}}_{{}^t_0\underline{B}_{L}} \hat{\underline{u}}
$$

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Also

$$
\delta_0\eta_{11} = \delta\hat{\underline{u}}^T \underbrace{\overbrace{\left(\frac{1}{L}\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)}^{{}^t_0\underline{B}_{NL}^T}}_{[\delta_0u_{1,1} \quad \delta_0u_{2,1}]} \underbrace{\overbrace{\left(\frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \right) \hat{\underline{u}}}^{{}^t_0\underline{B}_{NL}}}_{\begin{bmatrix} {}_0u_{1,1} \\ {}_0u_{2,1} \end{bmatrix}}
$$

Transparency 9-15

Using these expressions,

$$\underbrace{(EA)\ {}_0e_{11}\ \delta_0 e_{11}\ L}$$

$$\delta\underline{\hat{u}}^T \left(\underbrace{\frac{EA}{L} \begin{bmatrix} (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) & -(\cos{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) \\ & (\sin{}^t\theta)^2 & -(\cos{}^t\theta)(\sin{}^t\theta) & -(\sin{}^t\theta)^2 \\ & & (\cos{}^t\theta)^2 & (\cos{}^t\theta)(\sin{}^t\theta) \\ \text{symmetric} & & & (\sin{}^t\theta)^2 \end{bmatrix}}_{{}_0^t\underline{K}_L} \right) \underline{\hat{u}}$$

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$${}^tP\ \delta_0\eta_{11}\ L$$

$$\delta\underline{\hat{u}}^T \left(\underbrace{\frac{{}^tP}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{{}_0^t\underline{K}_{NL}} \right) \underline{\hat{u}}$$

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and

$$\delta \underline{\hat{u}}^T \underbrace{\left({}^tP \begin{bmatrix} -\cos{}^t\theta \\ -\sin{}^t\theta \\ \cos{}^t\theta \\ \sin{}^t\theta \end{bmatrix} \right)}_{{}^t_0\underline{F}} \quad \longleftarrow \quad {}^tP \, \delta_0 e_{11} \, L$$

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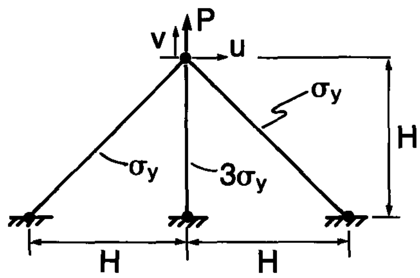
We notice that the element matrices corresponding to the T.L. and U.L. formulations are identical:

- The coordinate transformation used in the U.L. formulation is contained in the "initial displacement effect" matrix used in the T.L. formulation.
- The same can also be shown in detail analytically for a beam element, see K. J. Bathe and S. Bolourchi, Int. J. Num. Meth. in Eng., Vol. 14, pp. 961–986, 1979.

Transparency 9-19

Example: Collapse analysis of a truss structure

$H = 5$
$A = 1$
$E = 200{,}000$
$E_T = 0$
$\sigma_y = 100$



- Perform collapse analysis using U.L. formulation.
- Test model response when using M.N.O. formulation.

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For this structure, we may analytically calculate the elastic limit load and the ultimate limit load. We assume for now that the deflections are infinitesimal.

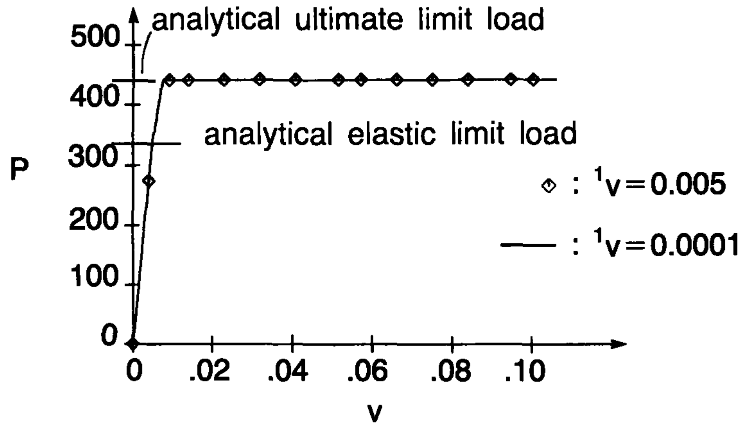
— Elastic limit load
(side trusses just become plastic)

$$P = 341.4$$

— Ultimate limit load
(center truss also becomes plastic)

$$P = 441.4$$

Using automatic load step incrementation and the U.L. formulation, we obtain the following results:



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We now consider an M.N.O. analysis.

We still use the automatic load step incrementation.

- If the stiffness matrix is not reformed, almost identical results are obtained (with reference to the U.L. results).

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- If the stiffness matrix is reformed for a load level larger than the elastic limit load, the structure is found to be unstable (a zero pivot is found in the stiffness matrix).

  Why?

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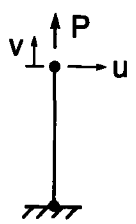
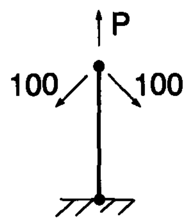
Explanation:

- In the M.N.O. analysis, once the side trusses have become plastic, they no longer contribute stiffness to the structure. Therefore the structure is unstable with respect to a rigid body rotation.



- In the U.L. analysis, once the side trusses have become plastic, they still contribute stiffness because they are transmitting forces (this effect is included in the ${}^{t}_{t}\underline{K}_{NL}$ matrix).

  Also, the internal force in the center truss provides stability through the ${}^{t}_{t}\underline{K}_{NL}$ matrix.



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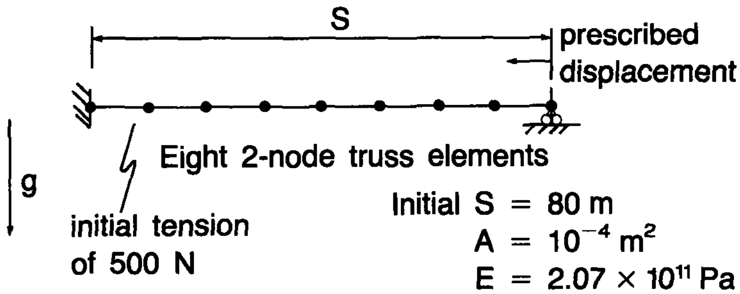
Example: Large displacements of a uniform cable



Initial S = 80 m
A = 10^{-4} m^2
E = 2.07×10^{11} Pa
ρ = 7750 kg/m^3

- Determine the deformed shape when S = 30 m.

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Transparency 9-27

This is a geometrically nonlinear problem (large displacements/large rotations but small strains).

The flexibility of the cable makes the analysis difficult.

- Small perturbations in the nodal coordinates lead to large changes in the out-of-balance loads.
- Use many load steps, with equilibrium iterations, so that the configuration of the cable is never far from an equilibrium configuration.

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Solution procedure employed to solve this problem:

- Full Newton iterations without line searches are employed.
- Convergence criteria:

$$\frac{\Delta \underline{U}^{(i)T}\,({}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)})}{\Delta \underline{U}^{(1)T}\,({}^{t+\Delta t}\underline{R} - {}^{t}\underline{F})} \le 0.001$$

$$\|{}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(i-1)}\|_2 \le 0.01\ \text{N}$$

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- The gravity loading and the prescribed displacement are applied as follows:

Time step	Comment	Number of equilibrium iterations required per time step
1	The gravity loading is applied.	14
2-1001	The prescribed displacement is applied in 1000 equal steps.	≤5

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Pictorially, the results are