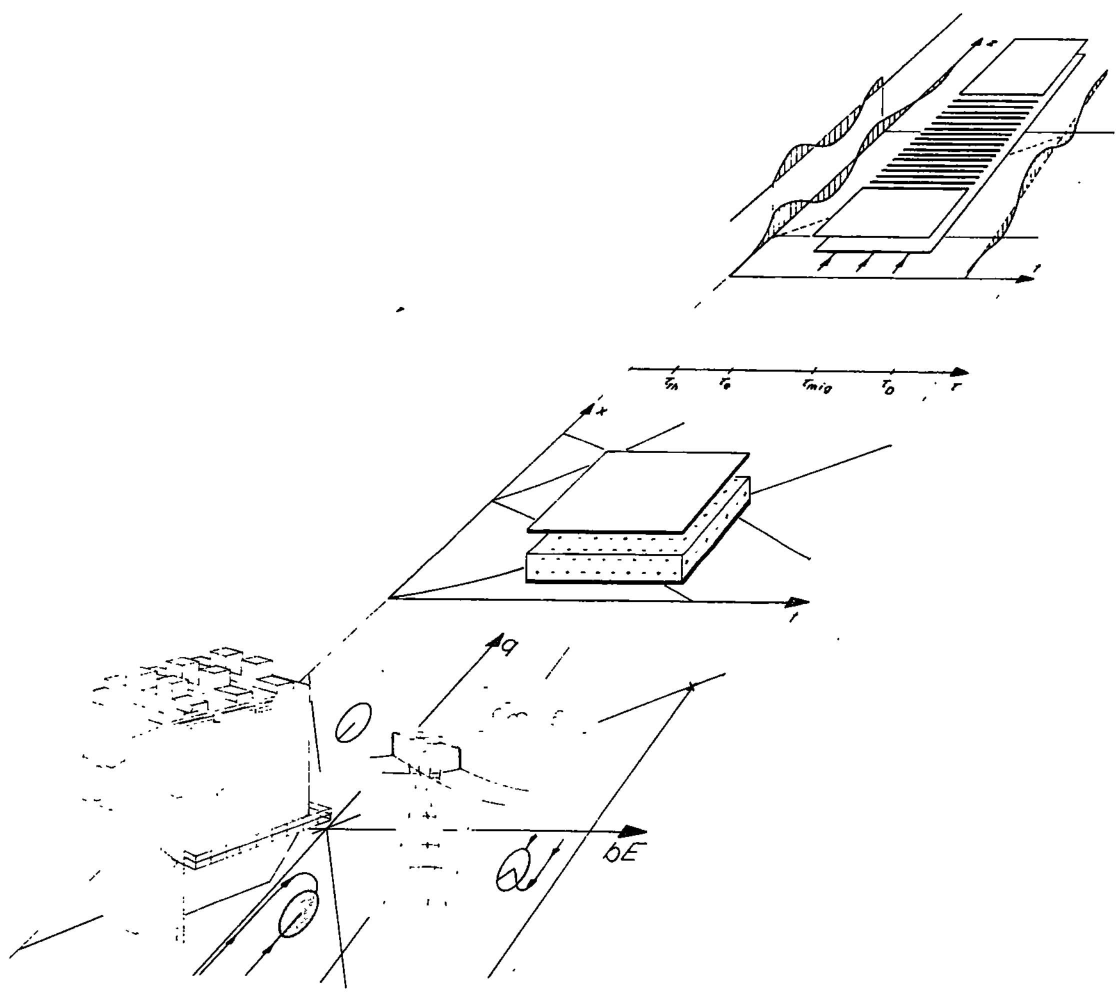
5

Charge Migration, Convection and Relaxation



Prob. 5.3.1 In cartesian coordinates (x,y)

$$\begin{bmatrix} \bar{E} \\ \bar{v} \end{bmatrix} = \left[\bar{i}_x \frac{\partial}{\partial y} - \bar{i}_y \frac{\partial}{\partial x} \right] \begin{bmatrix} A_E \\ A_v \end{bmatrix} \tag{1}$$

Thus, the characteristic equation, Eq. 5.3.4, becomes

$$\frac{dx}{dt} = \frac{\partial}{\partial y}\left(A_v \pm b_i A_E\right) \tag{2}$$

$$\frac{dy}{dt} = -\frac{\partial}{\partial x}\left(A_v \pm b_i A_E\right) \tag{3}$$

The ratio of these expressions is

$$\frac{dx}{dy} = -\frac{\frac{\partial}{\partial y}\left(A_v \pm b_i A_E\right)}{\frac{\partial}{\partial x}\left(A_v \pm b_i A_E\right)} \tag{4}$$

which, multiplied out, becomes

$$\frac{\partial}{\partial x}\left(A_v \pm b_i A_E\right)dx + \frac{\partial}{\partial y}\left(A_v \pm b_i A_E\right) = 0 \tag{5}$$

If A_v and A_E are independent of time, the quantity $A_v \pm b_i A_E$ is a perfect differential. That is,

$$A_v \pm b_i A_E = \text{constant} \tag{6}$$

is a solution to Eq. 5.3.4. Along these lines ρ_i = constant.

<u>Prob. 5.3.2</u> In axisymmetric cylindrical coordinates (r,z), Eq. (h) of Table 2.18.1 can be used to represent the solenoidal $\bar{E}$ and $\bar{v}$.

$$\begin{bmatrix} \bar{E} \\ \bar{v} \end{bmatrix} = \left[-\bar{i}_r \frac{1}{r}\frac{\partial}{\partial z} + \bar{i}_z \frac{1}{r}\frac{\partial}{\partial r}\right]\begin{bmatrix} \Lambda_E \\ \Lambda_v \end{bmatrix} \tag{1}$$

In terms of Λ_E and Λ_v, Eq. 5.3.4 becomes

$$\frac{dr}{dt} = -\frac{1}{r}\frac{\partial}{\partial z}(\Lambda_v \pm b_i \Lambda_E) \tag{2}$$

$$\frac{dz}{dt} = \frac{1}{r}\frac{\partial}{\partial r}(\Lambda_v \pm b_i \Lambda_E) \tag{3}$$

The ratio of these two expressions gives

$$\frac{dr}{dz} = \frac{-\frac{\partial}{\partial z}(\Lambda_v \pm b_i \Lambda_E)}{\frac{\partial}{\partial r}(\Lambda_v \pm b_i \Lambda_E)} \tag{4}$$

and hence

$$\frac{\partial}{\partial r}(\Lambda_v \pm b_i \Lambda_E)dr + \frac{\partial}{\partial z}(\Lambda_v \pm b_i \Lambda_E)dz = 0 \tag{5}$$

Provided Λ_v and Λ_E are independent of time, this is a perfect differential. Hence

$$\Lambda_v \pm b_i \Lambda_E = \text{constant} \tag{6}$$

represents the characteristic lines along which ρ_i is a constant.

Prob. 5.4.1 Integration of the given electric field and flow velocity result in $A_E = Vy/d$ and $A_v = -(4U/d)[(x^2/2) - (x^3/3d)]$. It follows from the result of Prob. 5.3.1 that the characteristic lines are $A_v + bA_E$=constant, or the relation given in the problem statement. The characteristic originating at x=0 reaches the upper electrode at $y=y_1$ where y_1 is obtained from the characteristics by first evaluating the constant by setting x=0 and y=0 (constant = 0) and then evaluating the characteristic at x=d and $y=y_1$.

$$y_1 = \frac{2}{3}Ud/(bV/d) \tag{1}$$

Because the current density to the upper electrode is $nqbE_x$ and all characteristics reaching the electrode to the right of $y=y_1$ carry a uniform charge density, nq, the current per unit length is simply the product of the uniform current density and the length $(a-y_1)$. This is the given result.

Prob. 5.4.2 From the given distributions of electric potential and velocity potential, it follows that

$$\bar{E} = -VR^2\left[-\frac{2}{r^3}\cos\theta\,\bar{i}_r - \frac{1}{r^3}\sin\theta\,\bar{i}_\theta\right] \tag{2}$$

$$\bar{v} = UR\left[\left(\frac{1}{R} - \frac{R^2}{r^3}\right)\cos\theta\,\bar{i}_r - \frac{1}{r}\left(\frac{r}{R} + \frac{1}{2}\frac{R^2}{r^2}\right)\sin\theta\,\bar{i}_\theta\right] \tag{3}$$

From the spherical coordinate relations, Eqs. 5.3.8, it in turn is deduced that

$$\Lambda_E = \frac{VR^2\sin^2\theta}{r} \tag{4}$$

$$\Lambda_v = \frac{UR^2}{2}\left(\frac{r^2}{R^2} - \frac{R}{r}\right)\sin^2\theta \tag{5}$$

so the characteristic lines are (Eq. 5.3.13b)

$$\Lambda_v + b\Lambda_E = \frac{UR^2}{2}\left(\frac{r^2}{R^2} - \frac{R}{r}\right)\sin^2\theta + \frac{bVR^2}{r}\sin^2\theta = \text{constant} \tag{6}$$

Normalization makes it evident that the trajectories depend on only one parameter.

$$\left[\left(\frac{r}{R}\right)^2 - \frac{R}{r}\left(1 - \frac{2Vb}{UR}\right)\right]\sin^2\theta = C \tag{7}$$

The critical points are determined by the requirement that both the r and θ components of the force vanish.

Prob. 5.4.2(cont.)

$$b\frac{2VR^2}{r^3}\cos\theta + U\left(1-\frac{R^3}{r^3}\right)\cos\theta = 0 \qquad (8)$$

$$b\frac{VR^2}{r^3}\sin\theta - \frac{UR}{r}\left(\frac{r}{R}+\frac{1}{2}\frac{R^2}{r^2}\right)\sin\theta = 0 \qquad (9)$$

From the first expression,

$$\text{either} \quad \theta = \pi/2 \qquad \text{or} \quad \left(\frac{r}{R}\right)^3 = 1 - b\frac{2V}{RU} \qquad (10)$$

while from the second expression,

$$\text{either} \quad 0, \pi \qquad \text{or} \quad \left(\frac{r}{R}\right)^3 = -\frac{1}{2}\left(1-\frac{2bV}{RU}\right) \qquad (11)$$

For V > 0 and positive particles, the root of Eq. 10b is not physical. The roots of physical interest are given by Eqs. 10a and 11b. Because r/R > 1, the singular line (point) is physical only if bV/RU > 3/2.

Because there is no normal fluid velocity on the sphere surface, the characteristic lines have a direction there determined by $\bar{E}$ alone. Hence, the sphere can only accept charge over some part of its southern hemisphere. Just how much of this hemisphere is determined by the origins of the incident lines. Do they originate at infinity where the charge density enters, or do they come from some other part of the spherical surface? The critical point determines the answer to this question.

Characteristic lines typical of having no critical point in the volume and of having one are shown in the figure. For the lines on the right, bV/RU=1 so there is no critical point. For those on the left, bV/RU = 3 > 3/2.

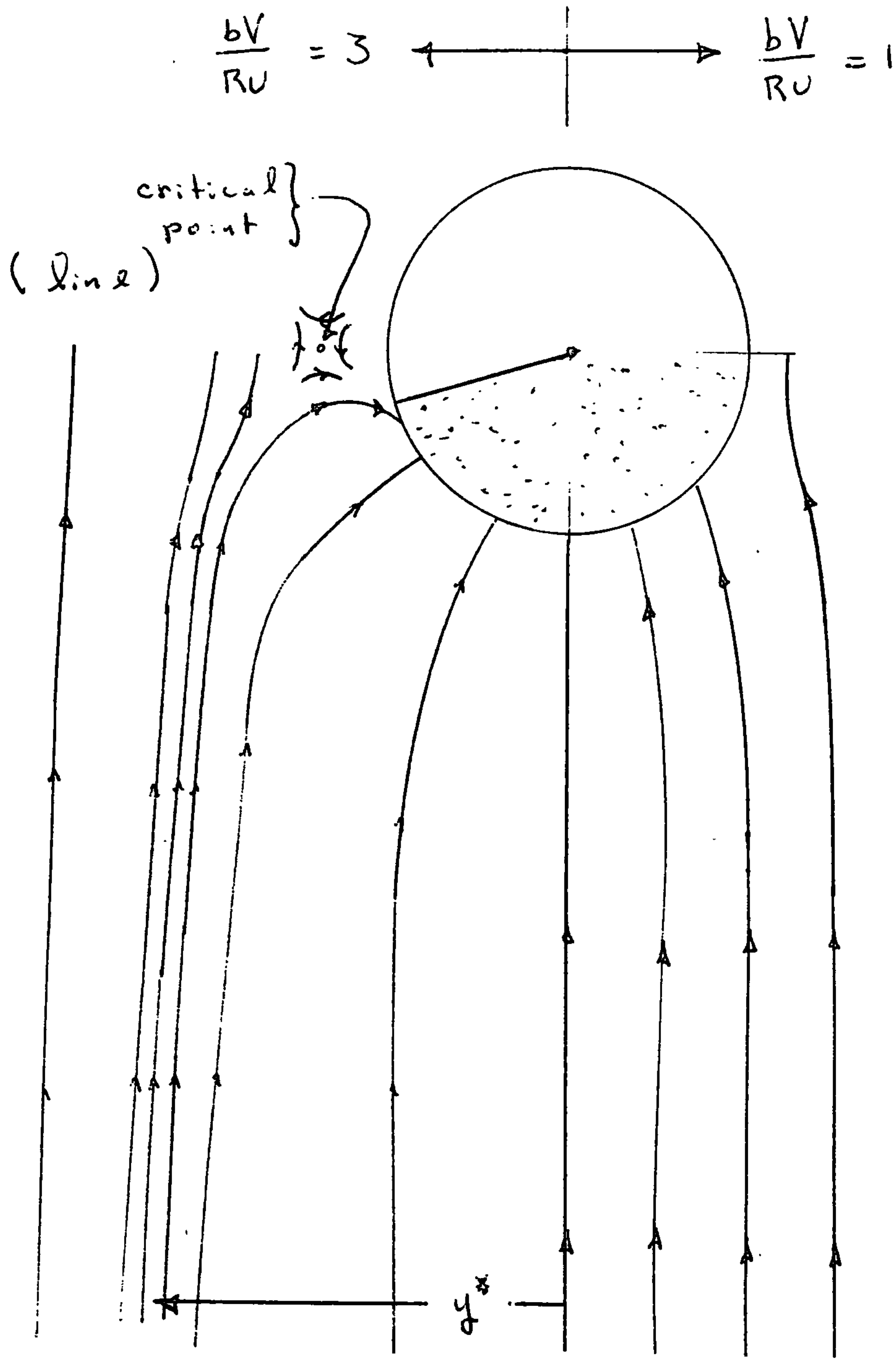
If the critical point is outside the sphere (bV/RU > 3/2) then the "window" having area $\pi(y^*)^2$ through which particles enter and ultimately impact the sphere is determined by the characteristic line passing through the critical point

$$\frac{r}{R} = \left[\frac{1}{2}\left(\frac{2Vb}{RU}-1\right)\right]^{1/3}, \quad \theta = \pm\frac{\pi}{2} \qquad (12)$$

Thus, in Eq. 7,

$$C = \frac{3}{2}(2)^{1/3}\left(\frac{2bV}{RU}-1\right)^{2/3} \qquad (13)$$

Prob. 5.4.2(cont.)



In the limit $r \to \infty$, $\theta \to \pi/2$

$$C \to \left(\frac{r}{R}\right)^2 \sin^2\theta = (y^*)^2/R^2 \tag{14}$$

so, for $bV/RU > 3/2$,

$$i = \rho U (y^*)^2 \pi = \frac{3\pi R^2}{2} \rho U (2)^{1/3} \left(\frac{2bV}{RU} - 1\right)^{2/3} \tag{15}$$

Prob. 5.4.2(cont.)

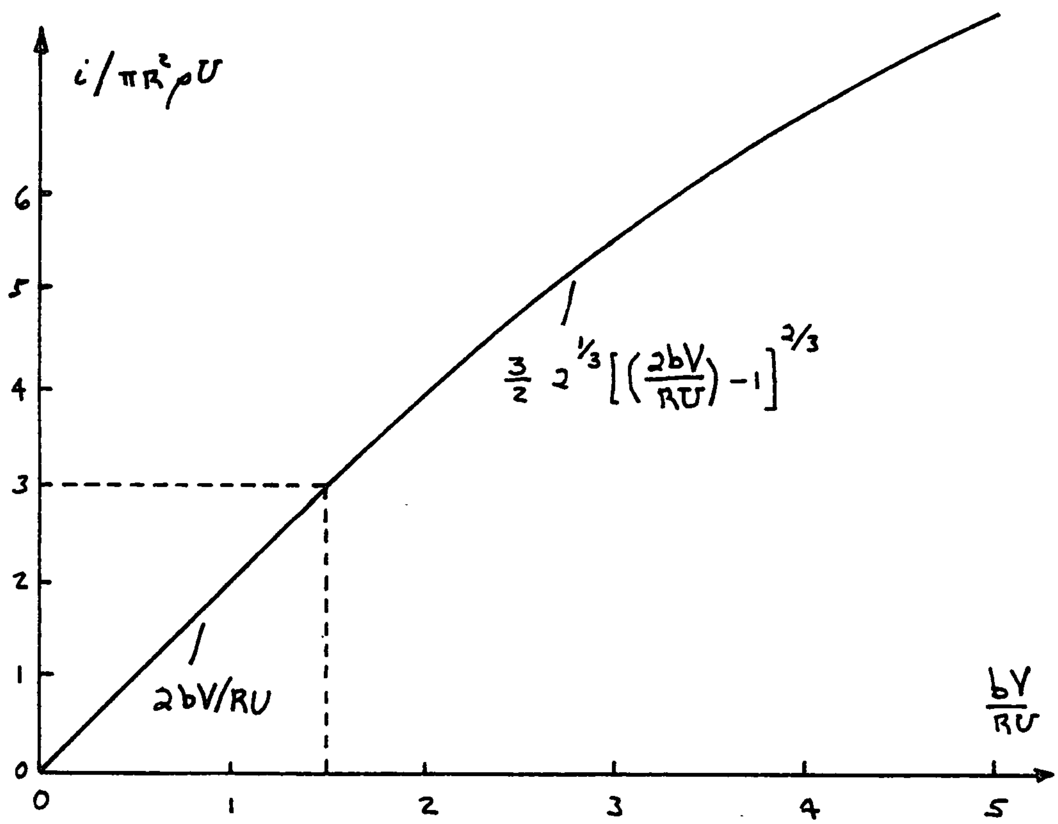
For bV/RU < 3/2, the entire southern hemisphere collects, and the window for collection is defined (not by the singular point, which no longer exists in the volume) by the line passing through the equator, $\theta = \pi/2, r/R = 1$

$$(\psi^*/R)^2 = 2bV/RU \tag{16}$$

Thus, in this range the current is

$$i = \frac{2bV}{RU}\pi R^2 \rho U \tag{17}$$

In terms of normalized variables, the current therefore has the voltage dependence summarized in the figure.



Prob. 5.4.3 (a) The critical points form lines in three dimensions. They occur where the net force is zero. Thus, they occur where the θ component balances

$$U\left(1+\frac{a^2}{r^2}\right)\sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$$

and where the r component is zero

$$-U\left(1-\frac{a^2}{r^2}\right)\cos\theta + bV\frac{1}{r\ln\left(\frac{R_0}{a}\right)} = 0$$

Because the first of these fixes the angle, the second can be evaluated to give the radius

$$\frac{r}{a} = \frac{\underline{V}}{2\cos\theta} + \sqrt{\left(\frac{\underline{V}}{2}\right)^2 + 1} \quad ; \quad \underline{V} \equiv \frac{bV}{aU\ln\left(\frac{R_0}{a}\right)} ; \cos\theta = \pm 1$$

Note that this critical point exists if charge and conductor have the same polarity $(\underline{V} > 0)$ at $\theta = 0$ and if $(\underline{V} < 0)$ at $\theta = \pi$.

(b) It follows from the given field and flow that

$$A_E = \frac{V\theta}{\ln(R_0/a)} \quad ; \quad A_v = -U\left(r - \frac{a^2}{r}\right)\sin\theta$$

and hence the characteristic lines are

$$A_v + bA_E = -U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{bV\theta}{\ln(R_0/a)} = \text{const.}$$

These are sketched for the two cases in the figure.

(c) There are two ways to compute the current to the conductor when the voltage is negative. First, the entire surface of the conductor collects with a current density $-\rho bE_r$ that is uniform over its surface. Hence, because the charge density is uniform along a characteristic line, and all striking the conductor surface carry this density,

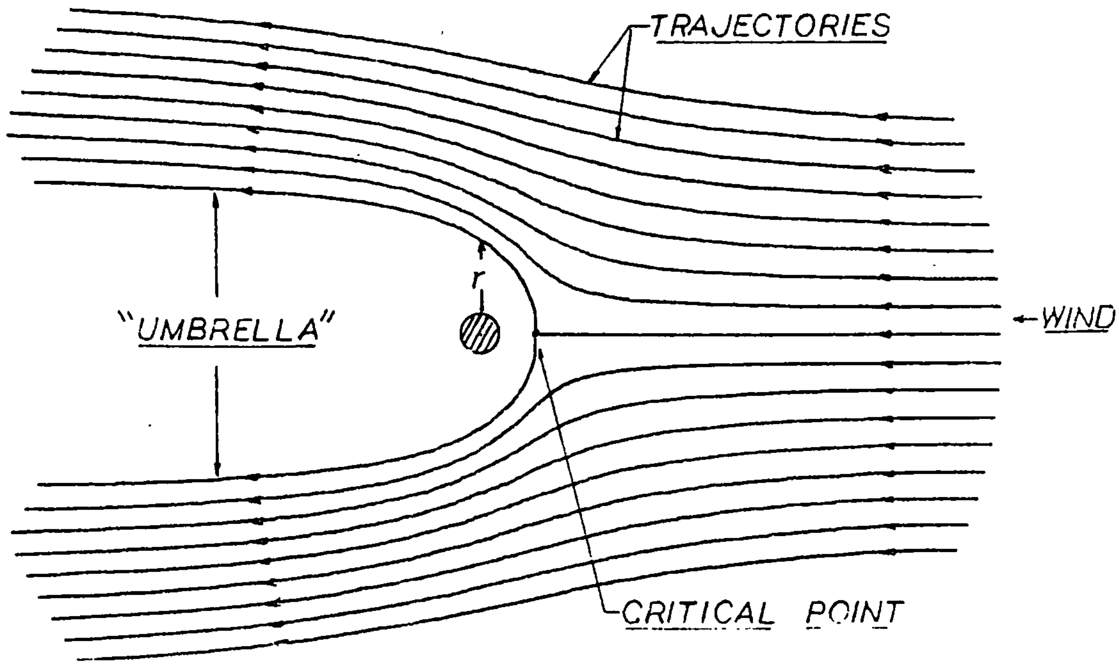
$$i = (2\pi a w)\rho b E_r = 2\pi a w \rho b\left[V/a\ln(R_0/a)\right] \quad ; \quad V < 0$$

and i is zero for V > 0. Second, the window at infinity, y*, can be found by evaluating (const.) for the line passing through the critical point. This must be the same constant as found for $r \to \infty$ to the right.

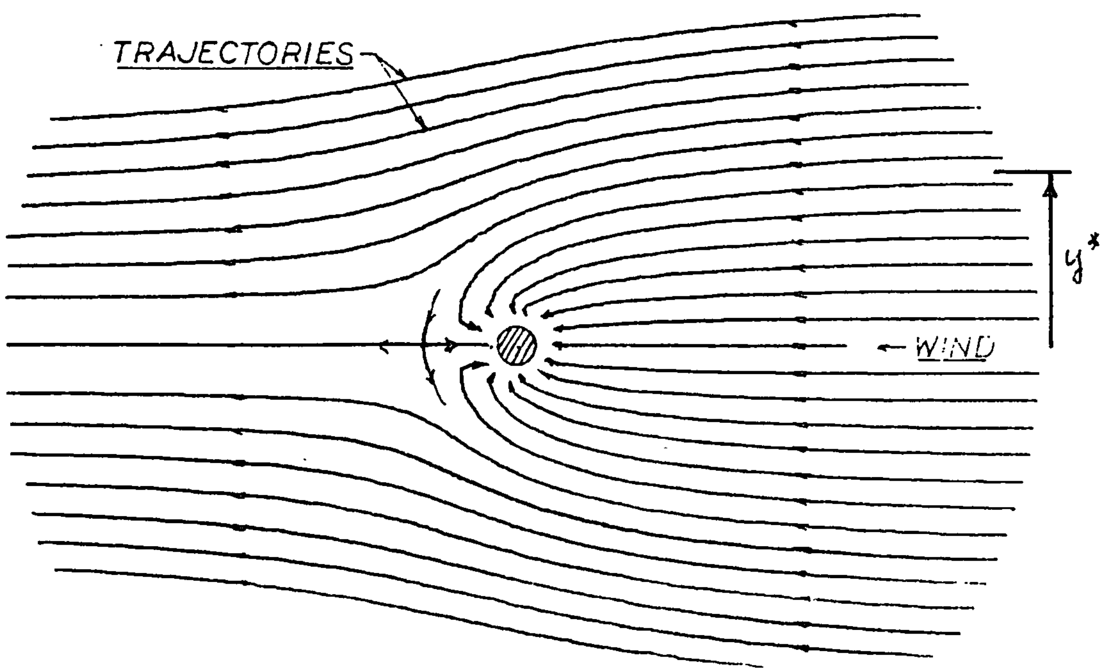
$$\text{const.} = -Uy^* = bV\pi/\ln(R_0/a)$$

It follows that $i = (2y^* w)\rho U$, which is the same current as given above.

Prob. 5.4.3 (cont.)



Positive Particle Trajectories for a Positive Conductor in the Stationary Flow Case (Repelled Particles)



Negative Particle Trajectories for a Positive Conductor in the Stationary Flow Case (Attracted Particles)

Prob. 5.4.4 In terms of the stream function from Table 2.18.1, the velocity is represented by 2Cxy. The volume rate of flow is equal to ℓ times the difference between the stream function evaluated on the electrodes to left and right, so it follows that $-4Ca^2\ell = Q_v$. Thus, the desired stream function is

$$A_v = -\frac{Q_v}{2a^2\ell}\,xy \tag{1}$$

The electric potential is $\Phi = V_o xy/a^2$. Thus, $\bar{E} = -V_o(y\bar{i}_x + x\bar{i}_y)/a^2$ and it follows that the electric stream function is

$$A_E = V_o(x^2-y^2)/2a^2 \tag{2}$$

(b) The critical lines (points) are given by

$$\bar{v} + b\bar{E} = -\frac{Q_v}{2a^2\ell}(x\bar{i}_x - y\bar{i}_y) - \frac{bV_o}{a^2}(y\bar{i}_x + x\bar{i}_y) = 0 \tag{3}$$

Thus, elimination between these two equations gives

$$\frac{-Q_v^2}{4\ell^2(bV_o)^2}\,y = y \tag{4}$$

so that the only lines are at the origin where both the velocity and the electric field vanish.

(c) Force lines follow from the stream functions as

$$\frac{-Q_v}{2a^2\ell}\,xy + \frac{bV_o}{2a^2}(x^2-y^2) = \text{constant} \tag{5}$$

The line entering at the right edge of the throat is given by

$$-\frac{Q_v}{\ell}\,xy + bV_o(x^2-y^2) = -\frac{Q_v}{\ell}a^2 + \frac{bV_o}{c^2}(c^4-a^4) \tag{6}$$

and it reaches the plane x=0 at

$$y^2 = \frac{Q_v}{\ell bV_o}a^2 - \frac{(c^4-a^4)}{c^2} \tag{7}$$

Clearly, force lines do not terminate on the left side of the collection electrode, so the desired current is given by

$$i = -\int_0^{y_1} \ell\rho bE_x(0,y)\,dy = \frac{\ell\rho bV_o}{2a^2}\,y_1^2 \tag{8}$$

where y_1 is equal to a unless the line from $(c,a^2/c)$ strikes to the left of a, in which case y_1 follows from evaluation of Eq. 7, provided that it

Prob. 5.4.4(cont.)

is positive. For still larger values of bV_o, $i=0$.

Thus, at low voltage, where the full width is collecting, $i = \ell\rho bV_o/2$.

This current gives way to a new relation as the force line from the right edge of the throat just reaches (0,a).

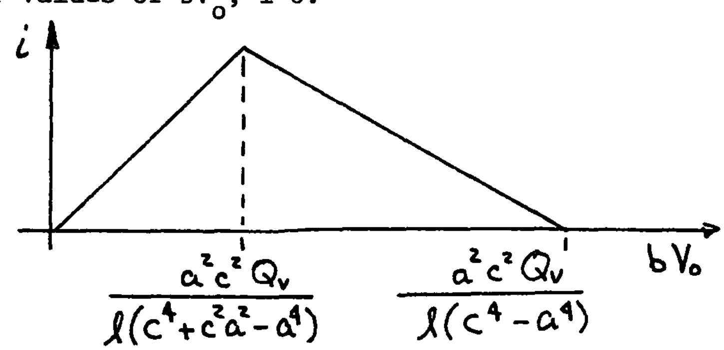
$$bV_o = \frac{Q_v a^2 c^2}{\ell(c^4 + c^2a^2 - a^4)} \tag{9}$$

$$i = \frac{\ell\rho bV_o}{2a^2}\left[\frac{Q_v}{\ell bV_o}a^2 - \frac{(c^4 - a^4)}{c^2}\right] = \frac{\rho Q_v}{2} - \frac{\ell\rho bV_o}{2a^2c^2}(c^4 - a^4) \tag{10}$$

Thus, as bV_o is raised, the current diminishes until $y_1=0$, which occurs at

$$bV_o = \frac{\rho Q_v a^2 c^2}{\ell\rho(c^4 - a^4)} \tag{11}$$

For greater values of bV_o, $i=0$.



Prob. 5.5.1 With both positive and negative ions, the charging current is, in general, the sum of the respective positive and negative ion currents. These two contributions act against each other, and final particle charges other than zero and $\pm q_c$ result. These final charges are those at which the two contributions are equal. The diagram is divided into 12 charging regimes by the coordinate axes q and E_o and the four lines

$$E_o = U_o/b_+ \tag{1}$$

$$E_o = -U_o/b_- \tag{2}$$

$$q = \pm q_c = \pm 12\pi\epsilon_o R^2 E_o \tag{3}$$

In each regime, the charging rate is given by the sum of the four possible current components

$$i_1^{\pm} = \pm 3|I_\pm|\left(1 \mp \frac{q}{|q_c|}\right)^2 \tag{4}$$

$$i_2^{\pm} = -12\frac{|I_\pm|}{|q_c|}q \tag{5}$$

where $I_\pm \equiv \pi R^2 b_\pm \rho_\pm E_o$, as in the unipolar cases.

In regimes (a), (b), (c) and (d), only i_2^- is acting, driving the particle charge down to the $\pm q_c$ lines. Similarly, in regimes (m), (n), (o) and (p), only i_2^+ is charging the particle, driving q up to the lower $\pm q_c$ lines.

In regimes (e), (i), (h) and (l), the current is $i_1^+ + i_1^-$; the equilibrium charge, defined by

$$i_1^+(q_1) + i_1^-(q_1) = 0 \tag{6}$$

is

$$q_1 = |q_c|\left\{\frac{\frac{|I_+|}{|I_-|}+1}{\frac{|I_+|}{|I_-|}-1} \mp \left[\frac{\left(\frac{|I_+|}{|I_-|}+1\right)^2}{\left(\frac{|I_+|}{|I_-|}-1\right)^2} - 1\right]^{1/2}\right\} \tag{7}$$

where the upper sign holds for $|I_+|>|I_-|$ while the lower one holds for $|I_+|<|I_-|$. In other words, the root of the quadratic which gives $|q_1|<|q_c|$ is taken. Note that q_1 depends linearly on $|E_o|$; the sign of q_1 is that of $|I_+|-|I_-|$. This is seen clearly in the limit $|I_+|\to 0$ or $|I_-|\to 0$.

Prob. 5.5.1 (cont.)

In regime (j), i_1^+ is the only current; in regime (g), i_1^- is the only contribution. In both cases, the particle charge is brought to zero and respectively into regime (f) (where the current is $i_2^- + i_1^+$) or into regime (k) (where the current is $i_2^+ + i_1^-$). The final charge in these regimes is q_2, given by

$$i_2^{\mp}(q_2) + i_1^{\pm}(q_2) = 0 \tag{8}$$

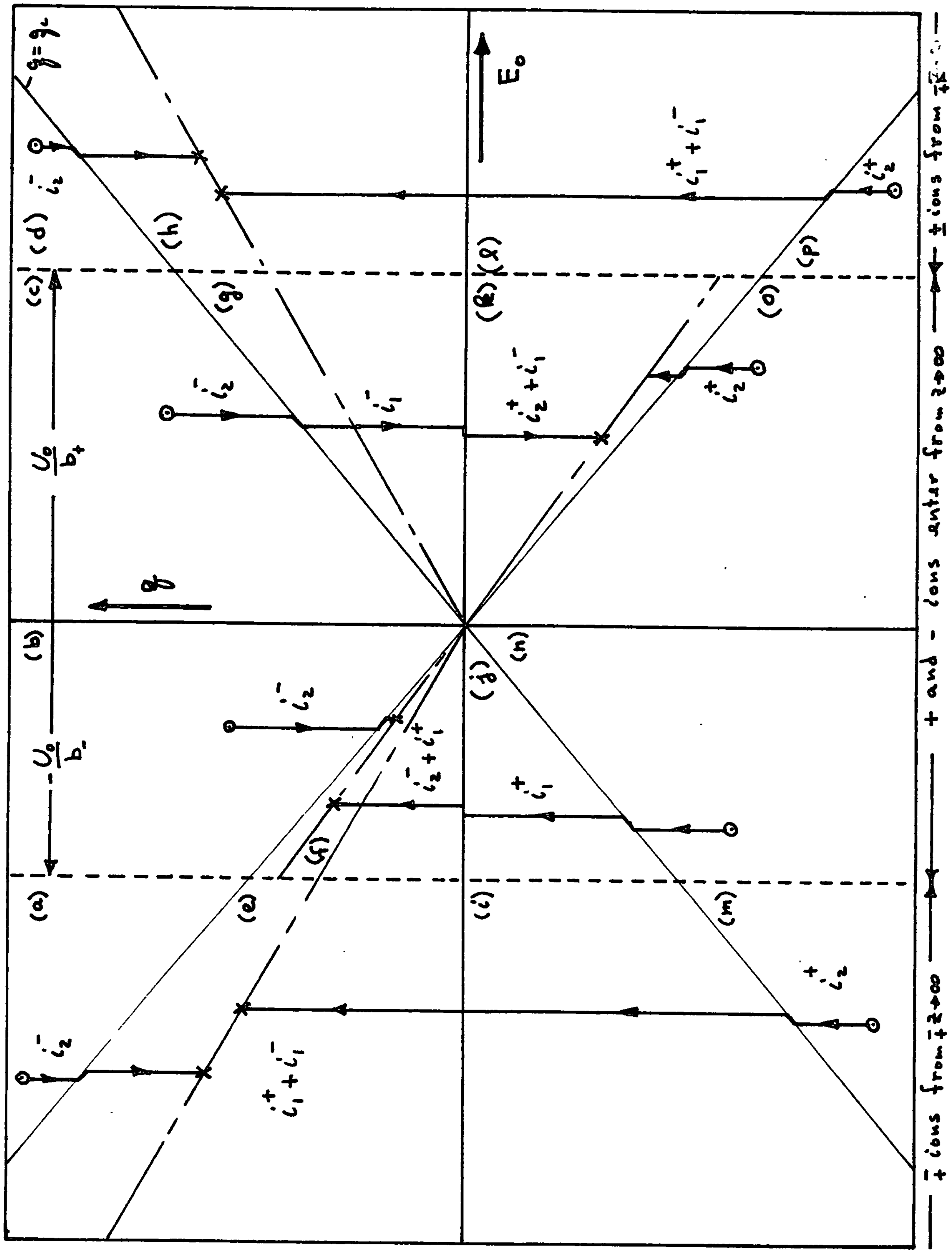
which can be used to find q_2.

$$q_2 = \mp |q_c| \left\{ \left(1 + 2\frac{|I_\pm|}{|I_\mp|}\right) - \left[\left(1 + 2\frac{|I_\pm|}{|I_\mp|}\right)^2 - 1\right]^{1/2} \right\} \tag{9}$$

Here, the upper and lower signs apply to regimes (k) and (f) respectively. Note that q_2 depends linearly on E_o and hence passes straight through the origin.

In summary, as a function of time the particle charge, q, goes to q_1 for $E_o < -U_o/b_-$ or $E_o > U_o/b_+$ and goes to q_2 for $-U_o/b_- < E_o < U_o/b_+$. In the diagram, a shift from the vertical at a regime boundary denotes a change in the functional form of the charging current. Of course, the current itself is continuous there.

Prob. 5.5.1 (cont.)



Prob. 5.5.2 (a) In view of Eq. (k) of Table 2.18.1

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Lambda_v}{\partial \theta} = -U\left(1 - \frac{R^3}{r^3}\right)\cos\theta \qquad (1)$$

$$v_\theta = \frac{-1}{r \sin\theta}\frac{\partial \Lambda_v}{\partial r} = U\left(1 + \frac{R^3}{2r^3}\right)\sin\theta \qquad (2)$$

and it follows by integration that

$$\Lambda_v = -\frac{U}{2}\left(r^2 - \frac{R^3}{r}\right)\sin^2\theta \qquad (3)$$

Thus, because Λ_E remains Eq. 5.5.4, it follows that the characteristic lines, Eq. 5.3.13b, take the normalized form

$$-\frac{1}{2}\left(\underline{r}^2 - \frac{1}{\underline{r}}\right)\sin^2\theta \pm \underline{E}\left(\frac{1}{\underline{r}} + \frac{1}{2}\underline{r}^2\right)\sin^2\theta \mp 3\underline{q}\cos\theta = \text{const.} \qquad (4)$$

where as in the text, $q_c \equiv 12\pi\epsilon_0 R^2 E$, and $\underline{r} = r/R$, $\underline{E} \equiv b_\pm E/U$ and $\underline{q} = \underline{E}\, q/q_c$.

Note that $\underline{E}/q_c$ is independent of E and, provided U > 0, is always positive. Without restricting the analysis, U can be taken as positive. Then, $\underline{E}$ can be taken as a normalized imposed field and $\underline{q}$ (which is actually independent of E because $\underline{E}/q_c$ is independent of E) can be taken as a normalized charge on the drop.

(b) Critical points occur where

$$\bar{v} \pm b_\pm \bar{E} = 0 \qquad (5)$$

The components of this equation, evaluated using Eq. 5.5.3 for $\bar{E}$ and Eqs. 1 and 2 for $\bar{v}$, are

$$-\left(1 - \frac{1}{\underline{r}^3}\right)\cos\theta \pm \underline{E}\left(\frac{2}{\underline{r}^3} + 1\right)\cos\theta \pm \frac{3\underline{q}}{\underline{r}^2} = 0 \qquad (6)$$

$$\left(1 + \frac{1}{2\underline{r}^3}\right)\sin\theta \pm \underline{E}\left(\frac{1}{\underline{r}^3} - 1\right)\sin\theta = 0 \qquad (7)$$

One set of solutions to these simultaneous equations for (r, θ) follows by recognizing that Eq. 7 is satisfied if

Prob. 5.5.2 (cont.)

$$\sin\theta = 0 \Rightarrow \theta = \begin{pmatrix}0\\ \pi\end{pmatrix} \Rightarrow \cos\theta = \pm 1 \equiv k \tag{8}$$

Then, Eq. (6) becomes an expression for r.

$$k\left[-(r^3-1) \pm E(2+r^3)\right] \pm 3qr = 0 \tag{9}$$

This cubic expression for r has up to three roots that are of interest. These roots must be real and greater than unity to be of physical interest. Rather than attempting to deal directly with the cubic, Eq. 9 is solved for the normalized charge, q,

$$q = \frac{k}{3}\left[(\pm 1 - E)r^2 - \frac{1}{r}(\pm 1 + 2E)\right] \tag{10}$$

The objective is to determine the charging current (and hence current of mass) to the drop when it has some location in the charge-imposed field plane (q, E). Sketches of the right-hand side of Eq. 10 as a function of r, fall in three categories, associated with the three regimes of this plane $E<-\frac{1}{2}$, $-\frac{1}{2}<E<1$, $1<E$ as shown in Fig. P5.5.2a.

The sketches make it possible to establish the number of critical points and their relative positions. Note that the extremum of the curves comes at

$$r_m = \left[\frac{1+2E}{2(E-1)}\right]^{1/3} \gtrless 1 \quad ; \quad \begin{cases} 1 < E \\ E < -\frac{1}{2}\end{cases} \tag{11}$$

For example, in the range $1<E$ this root is greater than unity and it is clear that on the $\theta = 0$ axis

$$-q^* < q \Rightarrow \text{no roots}; -E < q < -q^* \Rightarrow 2\text{ roots}; \; q < -E \Rightarrow 1 \text{ root} \tag{12}$$

where

$$q^* \equiv \begin{cases} \frac{1}{2}(1+2E)^{2/3}\left[2(E-1)\right]^{1/3}; & 1 < E \\ \frac{1}{2}(-2E-1)^{2/3}\left[2(1-E)\right]^{1/3}; & E < -\frac{1}{2}\end{cases} \tag{13}$$

With the aid of these sketches, similar reasoning discloses critical points on the z axis, as shown in Fig. P5.5.2b. Note that $q = E \Rightarrow q = q_c$.

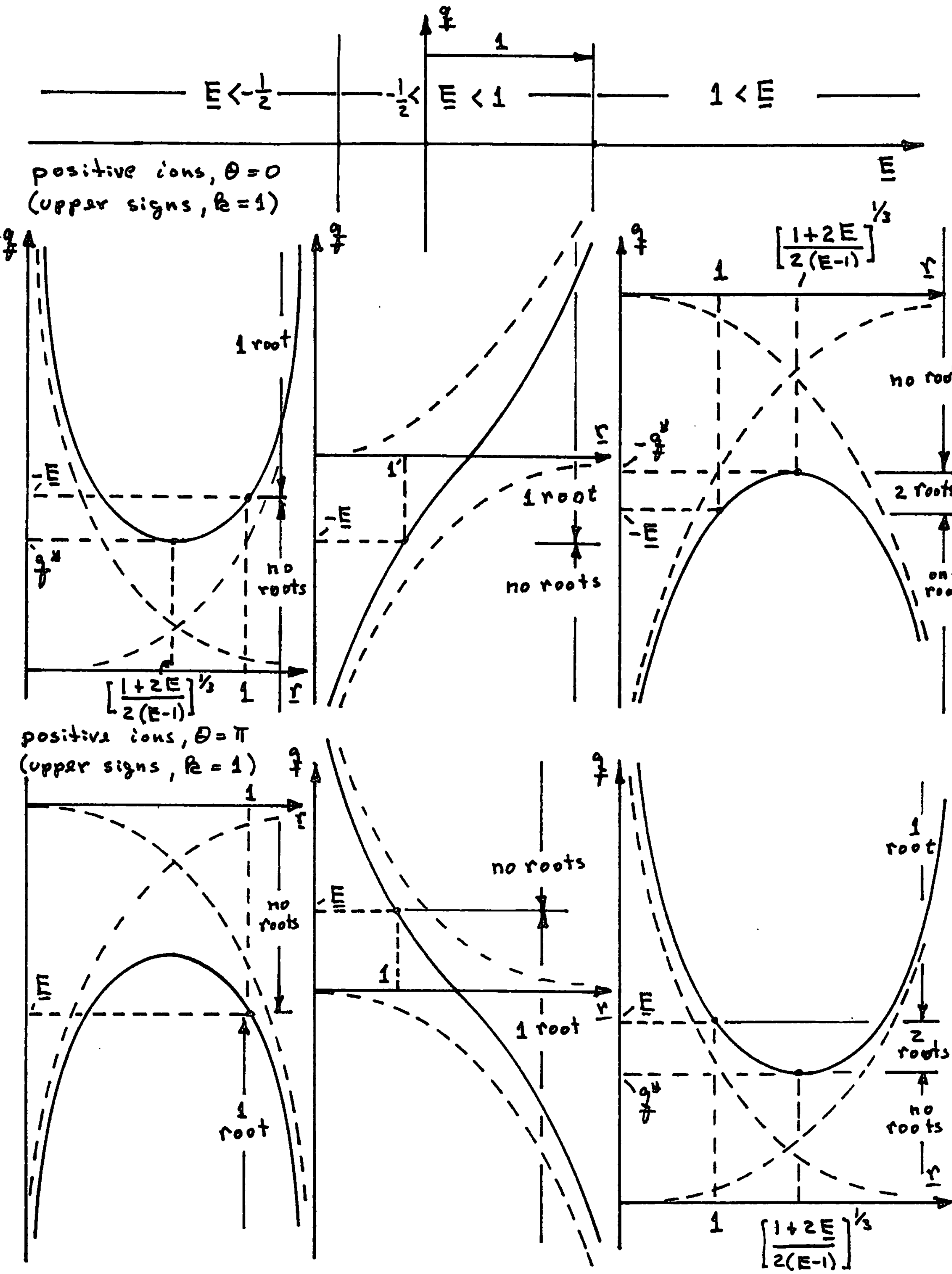

E < -1/2
-1/2 < E < 1
1 < E
positive ions, θ = 0
(upper signs, k = 1)
positive ions, θ = π
(upper signs, k = 1)
1 root
no roots
2 roots
one root
[(1+2E)/(2(E-1))]^1/3

Fig. P5.5.2a

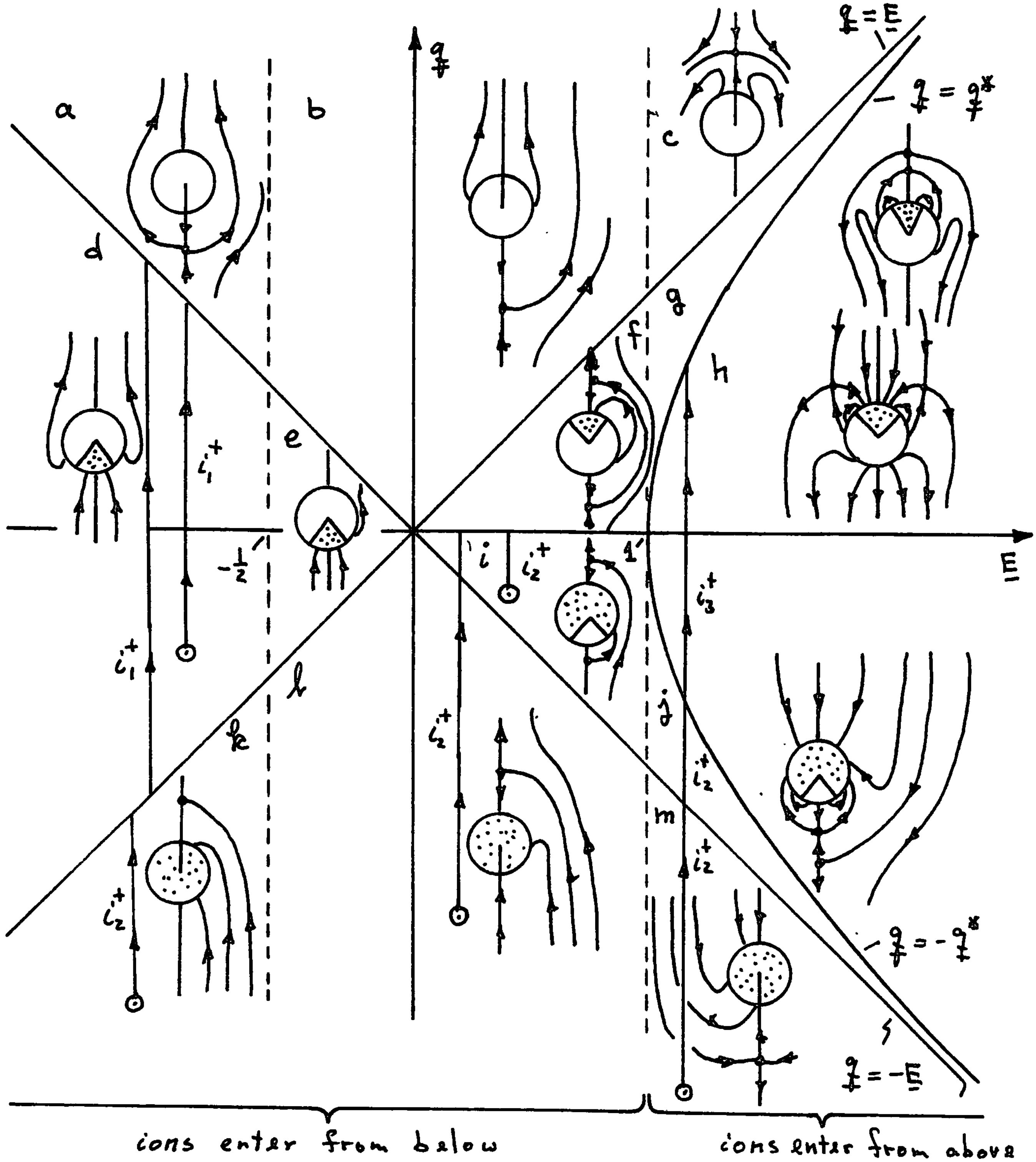


Fig. P5.5.2b Regimes of charging and critical points for positive ions.

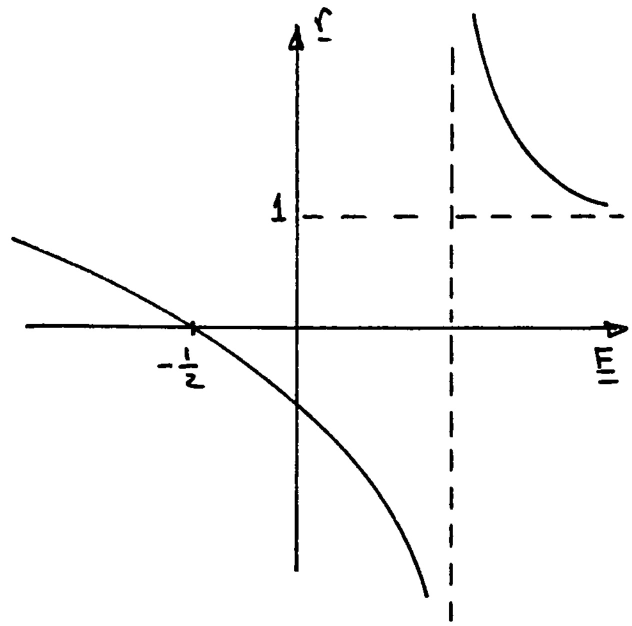
Prob. 5.5.2 (cont.)

Any possible off-axis roots of Eqs. 6 and 7 are found by first considering solutions to Eq. 7 for $\sin\theta \neq 0$. Solution for $\underline{r}$ then gives

$$\underline{r} = \frac{\left(\frac{1}{2} \pm \underline{E}\right)^{1/3}}{\left(\underline{E} \mp 1\right)^{1/3}} \tag{14}$$

This expression is then substituted into Eq. 6, which can then be solved for $\cos\theta$

$$\cos\theta = -\frac{\underline{q}}{\underline{q}^*} \tag{15}$$



A sketch if Eq. 14 as a function of $\underline{E}$ shows that the only possible roots that are greater than unity are in the regimes where $1 < \underline{E}$. Further, for there to be a solution to Eq. 15, it is clear that $|\underline{q}| < |\underline{q}^*|$. This means that off-axis critical points are limited to regime h in Fig. 5.5.2b.

Consider how the critical points evolve for the regimes where $1 < \underline{E}$ as $\underline{q}$ is lowered from a large positive value. First, there is an on-axis critical point in regime c. As $\underline{q}$ is lowered, this point approaches the drop from above. As regime g is entered, a second critical point comes out of the north pole of the drop. As regime h is reached, these points coalesce and split to form a ring in the northern hemisphere. As the charge passes to negative values, this ring moves into the southern hemisphere, where as regime i is reached, the ring collapses into a point, which then splits into two points. As regime ℓ is entered, one of these passes into the south pole while the other moves downward.

Prob. 5.5.2 (cont.)

There are two further clues to the ion trajectories. The part of the particle surface that can possibly accept ions is as in the case considered in the text, and indicated by shading in Fig. 5.5.2b. Over these parts of the surface, there is an inward directed electric field. In addition, if $1 < \underline{E}$, ions must enter the neighborhood of the drop from above, while if $\underline{E} < 1$ they enter from below.

Finally, the stage is set to sketch the ion trajectories and determine the charging currents. With the singularities already sketched, and with the direction of entry of the characteristic lines from infinity and from the surface of the drop determined, the lines shown in Fig. 5.5.2b follow.

In regions (a), (b) and (c), where there are no lines that reach the drop from the appropriate "infinity", the charging current is zero.

In regions (d) and (e) there are no critical points in the region of interest. The line of demarcation between ions collected by the drop as they come from below and those that pass by is the line reaching the drop where the radial field switches from "out" to "in". Thus, the constant in Eq. 4 is determined by evaluating the expression where $r = R$ and $\cos\theta = -q/q_c = -q/\underline{E}$ and hence $\sin^2\theta = 1 - \cos^2\theta = 1 - (q/\underline{E})^2$. Thus, the constant is

$$\text{const.} = \frac{3}{2}\underline{E}\left(1 + \frac{q^2}{\underline{E}^2}\right) \tag{16}$$

Now, following this line to $z \to \infty$, where $\cos\theta \to 1$ and $r\sin\theta \to y^*$ gives

$$y^{*2} = \frac{3bER^2}{U}\left(1 + \frac{q}{q_c}\right)^2 \Big/ \left(\frac{bE}{U} - 1\right) \tag{17}$$

Thus, the total current being collected is

$$i_1^+ = \pi y^{*2}\rho_+(U - bE) = -3\pi R^2 b|E|\rho_+\left(1 - \frac{q}{|q_c|}\right)^2 \tag{18}$$

Prob. 5.5.2 (cont.)

The last form is written by recognizing that in this regime $\underline{E} < 0$, and hence q_c is negative. Note that the charging rate approaches zero as the charge approaches $|q_c|$.

In regime f, the trajectories starting at the lower singularity end at the upper singularity, and hence effectively isolate the drop from trajectories beginning where there is a source of ions. To see this note that the constant for these trajectories, set by evaluating Eq. 4 where $\sin\theta = 0$ and $\cos\theta = 1$ is const. = -3q. So, these lines are

$$-\frac{1}{2}\left(r^2 - \frac{1}{r}\right)\sin^2\theta + E\left(\frac{1}{r} + \frac{1}{2}r^2\right)\sin^2\theta - 3\,q\cos\theta = -3q \tag{19}$$

Under what conditions do these lines reach the drop surface? To see, evaluate this expression at the particle surface and obtain an expression for the angle at which the trajectory meets the particle surface.

$$\frac{3E}{2}\sin^2\theta = 3q\,(\cos\theta - 1) \tag{20}$$

Graphical solution of this expression shows that there are no solutions if $\underline{E} > 0$ and $q > 0$. Thus, in regime f, the drop surface does not collect ions.

In regime i, the collection is determined by first evaluating the constant in Eq. 4 for the line passing through the critical point at $\theta = \pi$. It follows that const. = $3q$ and that

$$\underline{y}^{*2} = \frac{-12q}{1-\underline{E}} \Rightarrow y^{*2} = \frac{-12\,R^2\left(\frac{bE}{U}\right)q}{\left(1 - \frac{bE}{U}\right)\;\;q_c} \tag{21}$$

Thus, the current is

$$i_z^+ = \pi y^{*2}\,(U - bE)\rho_+ = -12\pi R^2\rho_+ b|E|\frac{q}{|q_c|} \tag{22}$$

Note that this is also the current in regimes k, l and m.

In regime g, the drop surface is shielded from trajectories coming

Prob. 5.5.2 (cont.)

from above. In regime h the critical trajectories pass through the critical points represented by Eqs. 14 and 15. Evaluation of the constant in Eq. 4 then gives

$$\text{const.} = \frac{3}{2}\left(\frac{bE}{U}\right)\frac{\mathfrak{z}^*}{\mathfrak{z}_c}\left(1 + \frac{\mathfrak{z}^2}{\mathfrak{z}^{*2}}\right) \tag{23}$$

and it follows that

$$y^{*2} = \frac{2R^2}{\left(\frac{bE}{U} - 1\right)}\left(\frac{bE}{U}\right)\left[-3\frac{\mathfrak{z}}{\mathfrak{z}_c} + \frac{3}{2}\frac{\mathfrak{z}^*}{\mathfrak{z}_c}\left(1 + \frac{\mathfrak{z}^2}{\mathfrak{z}^{*2}}\right)\right] \tag{24}$$

Thus, the current is evaluated as

$$i_3^+ = 2\pi R^2 bE\left[-3\frac{\mathfrak{z}}{\mathfrak{z}_c} + \frac{3}{2}\frac{\mathfrak{z}^*}{\mathfrak{z}_c}\left(1 + \frac{\mathfrak{z}^2}{\mathfrak{z}^{*2}}\right)\right] \tag{25}$$

Note that at the boundary between regimes g and h, where $\mathfrak{z} = \mathfrak{z}^*$, this expression goes to zero, as it should to match the null current for regime g.

As the charge approaches the boundary between regimes h and j, $\mathfrak{z} = -\mathfrak{z}^*$ and the current becomes $i_3^+ \to 12\pi R^2 bE\,\mathfrak{z}^*/\mathfrak{z}$. This suggests that the current of regime m extends into regime j. That this is the case can be seen by considering that the same critical trajectory determines the current in these latter regimes.

To determine the collection laws for the negative ions, the arguments parallel those given, with the lower signs used in going beyond Eq. 10.

Prob. 5.6.1 A statement that the initial total charge is equal to that at a later time is made by multiplying the initial volume by the initial charge density and setting it equal to the charge density at time t multiplied by the volume at that time. Here, the fact that the cloud remains uniform in its charge density is exploited.

$$\frac{4}{3}\pi(R_o^3 - R_i^3)\rho_u = \frac{4}{3}\pi R_o^3\left\{1 + \left[\frac{3\Phi_v\tau_e}{4\pi R_o^2} + 1 - \left(\frac{R_i}{R_o}\right)^3\right]\frac{t}{\tau_e} + \left(\frac{R_i}{R_o}\right)^3 - \left(\frac{3\Phi_v\tau_e}{4\pi R_o^2}\right)\frac{t}{\tau_e}\right\}\frac{\rho_u}{1+\frac{t}{\tau_e}}$$

$$= \frac{4}{3}\pi R_o^3\left[1 - \left(\frac{R_i}{R_o}\right)^2\right]\left[1 + \frac{t}{\tau_e}\right]\frac{\rho_u}{1+\frac{t}{\tau_e}}$$

Prob. 5.6.2 a) From Sec. 5.6, the rate of change of charge density for an observer moving along the characteristic line

$$\frac{d\bar{r}}{dt} = \bar{v} + b\bar{E} \tag{1}$$

is given by

$$\frac{d\rho}{dt} = -\frac{\rho^2 b}{\epsilon} \tag{2}$$

Thus, along these characteristics,

$$\rho = \frac{\rho_o}{1 + t/\tau} \quad ; \quad \tau \equiv \frac{\epsilon}{\rho_o b} \tag{3}$$

where throughout this discussion the charge density is presumed positive. The charge density at any given time depends only on the original density (where the characteristic originated) and the elapsed time. So, at any time, points from characteristic lines originating where the charge is uniform have the same charge density. Therefore, the charge-density in the cloud is uniform.

b) The integral form of Gauss' law requires that

$$\oint_S \epsilon \bar{E} \cdot \bar{n}\, da = \int_V \rho\, dV \tag{4}$$

and because the charge density is uniform in the layer, this becomes

$$E_f - E_b = \frac{\rho_o}{\epsilon} \frac{1}{(1 + t/\tau)} (z_f - z_b) \tag{5}$$

The characteristic lines for particles at the front and back of the layer are represented by

$$\frac{dz_f}{dt} = U + bE_f \quad ; \quad \frac{dz_b}{dt} = U + bE_b \tag{6}$$

These expressions combine with Eq. 5 to show that

$$\frac{d}{dt}(z_f - z_b) = \frac{1}{\tau} \frac{1}{1 + t/\tau} (z_f - z_b) \tag{7}$$

Integration gives

$$\int_{z_F - z_B}^{z_f - z_b} \frac{d(z_f - z_b)}{(z_f - z_b)} = \int_0^t \frac{d(t/\tau)}{1 + t/\tau} \tag{8}$$

and hence it follows that

$$z_f - z_b = (1 + t/\tau)(z_F - z_B) \tag{9}$$

Prob. 5.6.2(cont.)

Given the uniform charge distribution in the layer, it follows from Gauss' law that the distribution of electric field intensity is

$$E = \begin{cases} E_b & 0 < z < z_b \\ E_b + (E_f - E_b)\left[\dfrac{z - z_b}{z_f - z_b}\right] & z_b < z < z_f \\ E_f & z_f < z < \ell \end{cases} \tag{10}$$

From this it follows that the voltage, V, is related to E_f and E_b by

$$V = \int_0^{\ell} E\,dz = E_b z_b + E_b(z_f - z_b) + \frac{1}{2}(E_f - E_b)\frac{(z_f - z_b)^2}{z_f - z_b} + E_f(\ell - z_f) \tag{11}$$

From Eqs. 5 and 9,

$$E_f - E_b = \frac{\rho_o}{\epsilon}(z_F - z_B) \tag{12}$$

$$z_f - z_b = (1 + t/\tau)(z_F - z_B) \tag{13}$$

Substitution for E_b and $z_f - z_b$ as determined by these relations into Eq. 11 then gives an expression that can be solved for E_f.

$$E_f = \frac{V}{\ell} - \frac{1}{2\ell}\frac{\rho_o}{\epsilon}(z_F - z_B)^2\left(1 + \frac{t}{\tau}\right) + \frac{\rho_o}{\ell\epsilon}(z_F - z_B)z_f \tag{14}$$

d) In view of Eq. 6a, this expression makes it possible to write

$$\frac{dz_f}{dt} - \frac{(z_F - z_B)}{\ell}\frac{z_f}{\tau} = \left[U + \frac{bV}{\ell} - \frac{1}{2\ell\tau}(z_F - z_B)^2\right] - \frac{1}{2\ell\tau}(z_F - z_B)^2\frac{t}{\tau} \tag{15}$$

Solutions to this differential equation take the form

$$z_f = A\,e^{\frac{(z_F - z_B)}{\ell}\frac{t}{\tau}} + Bt + C \tag{16}$$

The coefficients of the particular solution, B and C, are found by substituting Eq. 16 into Eq. 15 to obtain

$$B = \frac{z_F - z_B}{2\tau} \tag{17}$$

$$C = \left[1 - K\frac{2\tau}{z_F - z_B}\right]\frac{\ell}{2}\ ;\ K \equiv U + \frac{bV}{\ell} - \frac{1}{2\ell\tau}(z_F - z_B)^2 \tag{18}$$

The coefficient of the homogeneous solution follows from the initial condition that when t=0, $z_f = z_F$.

$$A = z_F - \left(1 - K\frac{2\tau}{z_F - z_B}\right)\frac{\ell}{2} \tag{19}$$

Prob. 5.6.2(cont.)

The position of the back edge of the charge layer follows from this expression and Eq. 9.

$$z_b = z_f - (z_F - z_B)(1 + t/\tau) \tag{20}$$

Normalization of these last two expressions in accordance with

$$\underline{t} \equiv t/\tau \,, \quad \underline{V} \equiv \tau b V/\ell^2 \,, \quad \underline{U} = U/(bV/\ell) \tag{21}$$

$$(\underline{z}_f, \underline{z}_F, \underline{z}_b, \underline{z}_B) = (z_f, z_F, z_b, z_B)/\ell$$

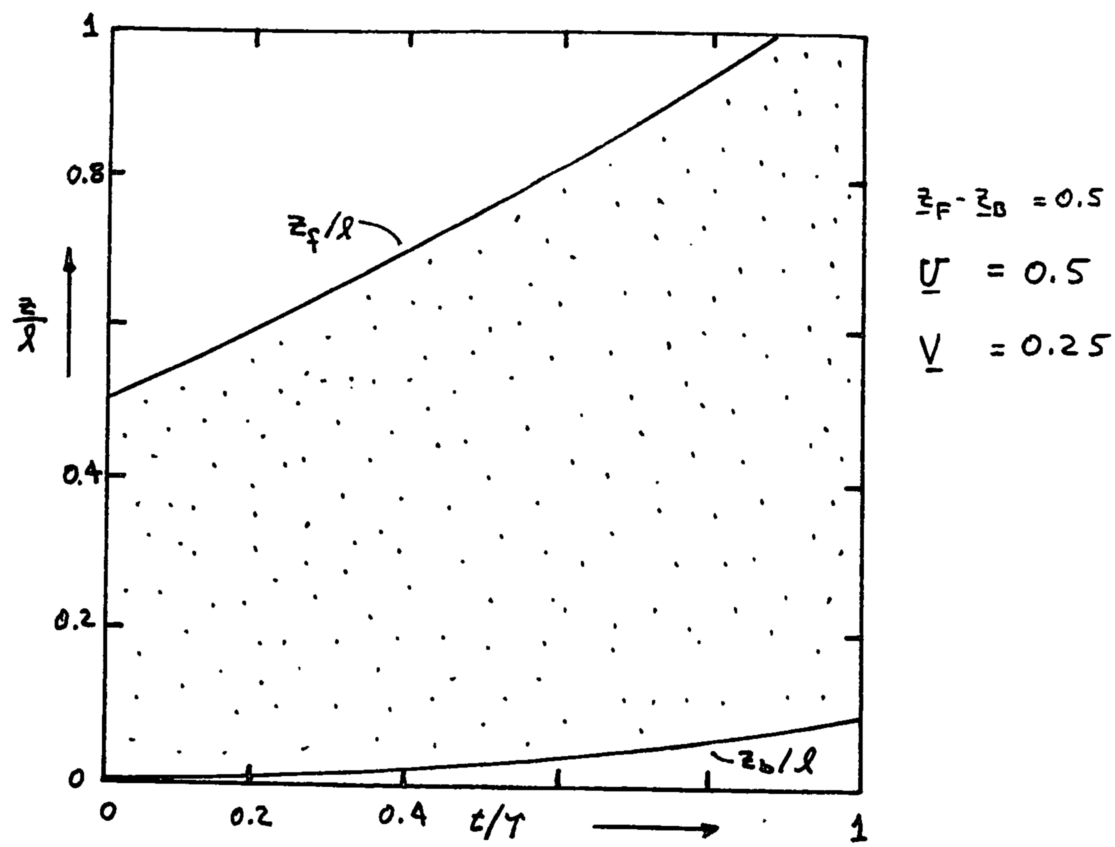
results in

$$\underline{z}_f = \left\{ \underline{z}_F - \frac{1}{2} + \frac{\underline{V}}{\underline{z}_F - \underline{z}_B}\left[\underline{U} + 1 - \frac{(\underline{z}_F - \underline{z}_B)^2}{2\underline{V}}\right]\right\} e^{(\underline{z}_F - \underline{z}_B)\underline{t}} + \frac{1}{2}(\underline{z}_F - \underline{z}_B)\underline{t} + \left\{\frac{1}{2} - \frac{\underline{V}}{\underline{z}_F - \underline{z}_B}\left[\underline{U} + 1 - \frac{(\underline{z}_F - \underline{z}_B)^2}{2\underline{V}}\right]\right\} \tag{22}$$

and

$$\underline{z}_b = \underline{z}_f - (\underline{z}_F - \underline{z}_B)(1 + \underline{t}) \tag{23}$$

The evolution of the charge layer is illustrated in the figure.



Prob. 5.7.1 The characteristic equations are Eqs. 5.6.2 and 5.6.3, written as

$$\frac{d\rho}{dt} = -\frac{\rho^2 b}{\epsilon} \tag{1}$$

$$\frac{dz}{dt} = U + bE \tag{2}$$

It follows from Eq. 1 that

$$\int_1^{\rho/\rho_o} \frac{d(\rho/\rho_o)}{(\rho/\rho_o)^2} = \int_0^t \frac{\rho_o b}{\epsilon} dt \Rightarrow \frac{\rho}{\rho_o} = \frac{1}{1 + t/\tau} \; ; \tau \equiv \frac{\epsilon}{\rho_o b} \tag{3}$$

Charge conservation requires that

$$J = \rho(bE + U) = \frac{i}{A} \tag{4}$$

where i/A is a constant. This is used to evaluate the right hand side of Eq. 2, which then becomes

$$\frac{dz}{dt} = \frac{J}{\rho} = \frac{i}{A\rho_o}\left(1 + \frac{t}{\tau}\right) \tag{5}$$

where Eq. 3 has been used. Integration then gives

$$\int_0^z dz = \int_0^{t/\tau} \frac{i\tau}{A\rho_o}\left(1 + \frac{t}{\tau}\right) d\left(\frac{t}{\tau}\right) = \frac{i\tau}{2A\rho_o}\left[\left(1 + \frac{t}{\tau}\right)^2 - 1\right] \tag{6}$$

Thus,

$$\left(1 + \frac{t}{\tau}\right)^2 = 2\frac{z}{l}\,\frac{i_o}{iR_e} + 1 \tag{7}$$

Finally, substitution into Eq. 3 gives the desired dependence on z.

$$\frac{\rho}{\rho_o} = \left[1 + \left(\frac{2z}{l}\right)\left(\frac{i_o}{iR_e}\right)\right]^{-1/2} \tag{8}$$

Prob. 5.9.1 For uniform distributions, Eqs. 9 and 10 become

$$\frac{d\rho_+}{dt} = \beta n - \frac{\alpha\rho_+\rho_-}{q} \tag{1}$$

$$\frac{d\rho_-}{dt} = \beta n - \frac{\alpha\rho_+\rho_-}{q} \tag{2}$$

$$\frac{dn}{dt} = -\frac{\beta n}{q} + \frac{\alpha\rho_+\rho_-}{q^2} \tag{3}$$

Subtraction of Eqs. 1 and 2 shows that

$$\frac{d}{dt}(\rho_+ - \rho_-) = 0 \tag{4}$$

and given the initial conditions it follows that

$$\rho_+ = \rho_- \tag{5}$$

Note that there being no net charge is consistent with $\bar{E}=0$ in Gauss' law.

(b) Multiplication of Eq. 3 by q and addition to Eq. 1, incorporating Eq. 5, then gives

$$\frac{d}{dt}(\rho_+ + qn) = 0 \tag{6}$$

The constant of integration follows from the initial conditions.

$$\rho_+ + qn = qn_o \tag{7}$$

Introduced into Eq. 3, this expression results in the desired equation for n(t).

$$\frac{dn}{dt} = -\frac{\beta}{q}n + \alpha(n_o - n)^2 \tag{8}$$

Introduced into Eq. 1 it gives an expression for $\rho_+(t)$.

$$\frac{d\rho_+}{dt} = -\frac{\beta}{q}\rho_+ - \frac{\alpha}{q}\rho_+^2 + \beta n_o \tag{9}$$

(c) The stationary state follows from Eq. 8 .

$$n = \left(n_o + \frac{\beta}{2\alpha q}\right) - \sqrt{\left(n_o + \frac{\beta}{2\alpha q}\right)^2 - n_o^2} \tag{10}$$

(d) The first terms on the right in Eqs. 8 and 9 dominate at early times making it clear that the characteristic time for the transients is $\tau_{th} = q/\beta$.

<u>Prob. 5.10.1</u> With $\rho_F(x_0, z_0, 0)$ defined as the charge distribution when t=0, the general solution is

$$\rho_f = \rho_F(x_0, z_0, 0)\, e^{-t/\tau} \quad ; \quad \tau \equiv \epsilon/\sigma \tag{1}$$

on the lines

$$\begin{aligned} x &= x_0 \\ z &= \frac{Ux}{d}t + z_0 \end{aligned} \tag{2}$$

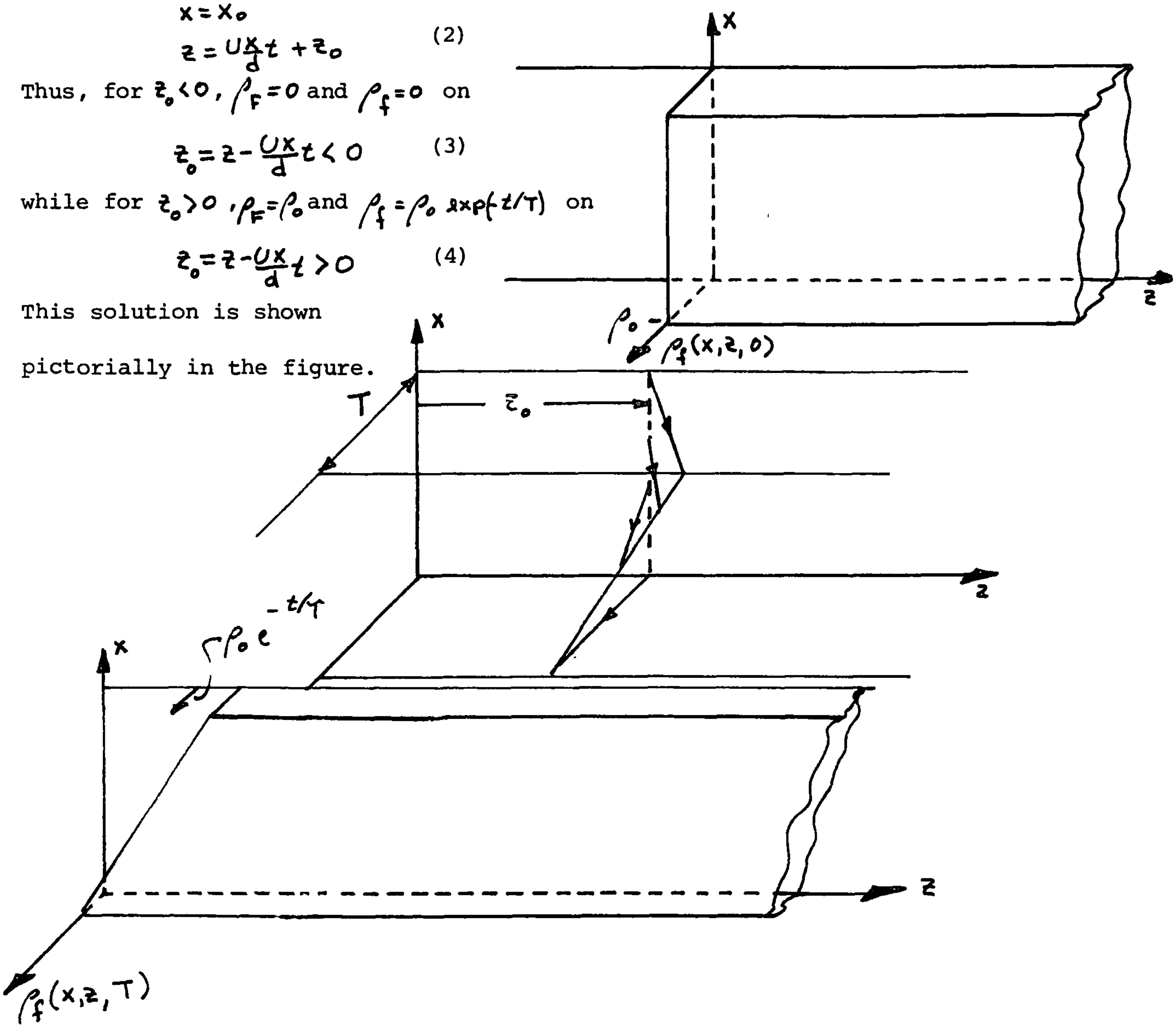
Thus, for $z_0 < 0$, $\rho_F = 0$ and $\rho_f = 0$ on

$$z_0 = z - \frac{Ux}{d}t < 0 \tag{3}$$

while for $z_0 > 0$, $\rho_F = \rho_0$ and $\rho_f = \rho_0\, \exp(-t/\tau)$ on

$$z_0 = z - \frac{Ux}{d}t > 0 \tag{4}$$

This solution is shown pictorially in the figure.



Prob. 5.10.2 With the understanding that time is measured along a characteristic line, the charge density is

$$\rho = \rho(t = t_a, z = 0)\, e^{-(t-t_a)/\tau} \quad ; \quad \tau \equiv \epsilon/\sigma \tag{1}$$

where t_a is the time when the characteristic passed through the plane z=0, as shown in the figure. The solution to the characteristic equations is

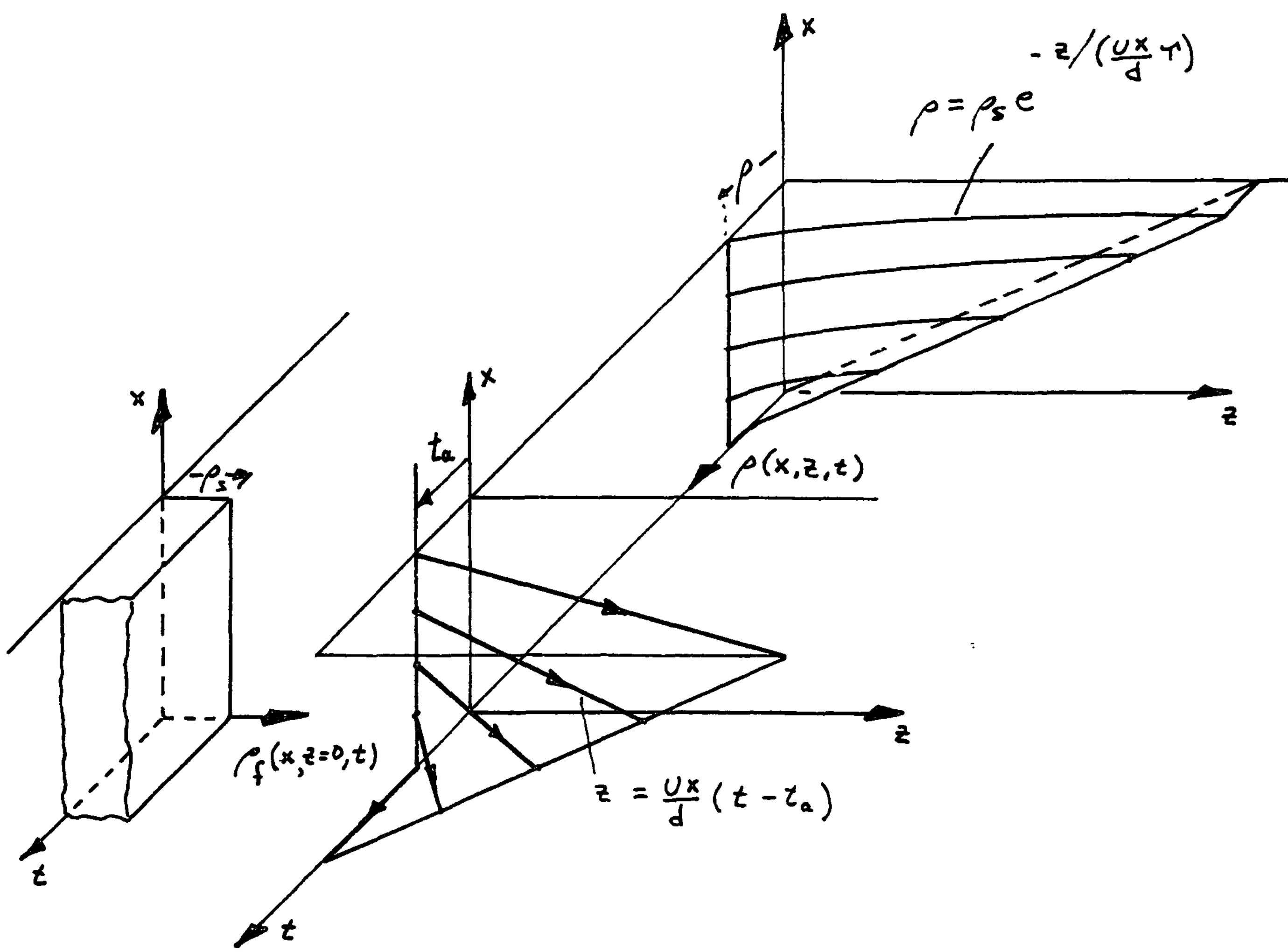
$$x = \text{constant} \tag{2}$$

$$z = \frac{Ux}{d}(t - t_a) \tag{3}$$

Thus, substitution for $t - t_a$ in Eq. 1 gives the charge density as

$$\rho = \begin{cases} \rho_s\, e^{-z/(Ux\tau/d)} & ; 0 < z < Uxt/d \\ 0 & ; Uxt/d < z \end{cases} \tag{4}$$

The time varying boundary condition at z=0, the characteristic lines and the charge distribution are illustrated in the figure. Note that once the wave-front has passed, the charge density remains constant in time.



Prob. 5.10.3 With it understood that

$$q = \int_V \rho_f \, dV \tag{1}$$

the integral form of Gauss' law is

$$\oint_S \epsilon \bar{E} \cdot \bar{n} \, da = q \tag{2}$$

and conservation of charge in integral form is

$$\oint_S \sigma \bar{E} \cdot \bar{n} \, da + \frac{dq}{dt} = 0 \tag{3}$$

Because ϵ and σ are uniform over the enclosing surface, S, these combine to eliminate $\bar{E}$ and require

$$\frac{dq}{dt} + \frac{q}{\tau} = 0 \quad ; \; \tau \equiv \epsilon/\sigma \tag{4}$$

Thus, the charge decays with the relaxation time.

Prob. 5.12.1 (a) Basic laws are

$$\nabla \times \bar{E} = 0 \Rightarrow \bar{E} = -\nabla \Phi \tag{1}$$

$$\nabla \cdot \epsilon \bar{E} = \rho_f \tag{2}$$

$$\nabla \cdot \bar{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \tag{3}$$

The first and second are substituted into the last with the conduction current as given to obtain an expression for the potential

$$\sigma_x \frac{\partial^2 \Phi}{\partial x^2} + \sigma_y \frac{\partial^2 \Phi}{\partial y^2} + \sigma_z \frac{\partial^2 \Phi}{\partial z^2} + \epsilon \frac{\partial}{\partial t}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right) = 0 \tag{4}$$

With the substitution of the complex amplitude form, this requires of the potential that

$$\frac{d^2 \hat{\Phi}}{dx^2} - \gamma^2 \hat{\Phi} = 0 \tag{5}$$

where

$$\gamma^2 \equiv \left[k_y^2(\sigma_y + j\omega\epsilon) + k_z^2(\sigma_z + j\omega\epsilon)\right] / (\sigma_x + j\omega\epsilon)$$

Although γ is now complex, solution of Eq. 5 is the same as in Sec. 2.16, except that the time dependance has been assumed.

$$\hat{\Phi} = \hat{\Phi}^\alpha \frac{\sinh \gamma x}{\sinh \gamma \Delta} - \hat{\Phi}^\beta \frac{\sinh \gamma (x - \Delta)}{\sinh \gamma \Delta} \tag{6}$$

from which it follows that

$$\hat{J}_x = -(j\omega\epsilon + \sigma_x)\gamma \left[\hat{\Phi}^\alpha \frac{\cosh \gamma x}{\sinh \gamma \Delta} - \hat{\Phi}^\beta \frac{\cosh \gamma (x - \Delta)}{\sinh \gamma \Delta}\right] \tag{7}$$

Evaluation at the (α, β) surfaces, where $x = \Delta$ and $x = 0$, respectively, then gives the required transfer relations

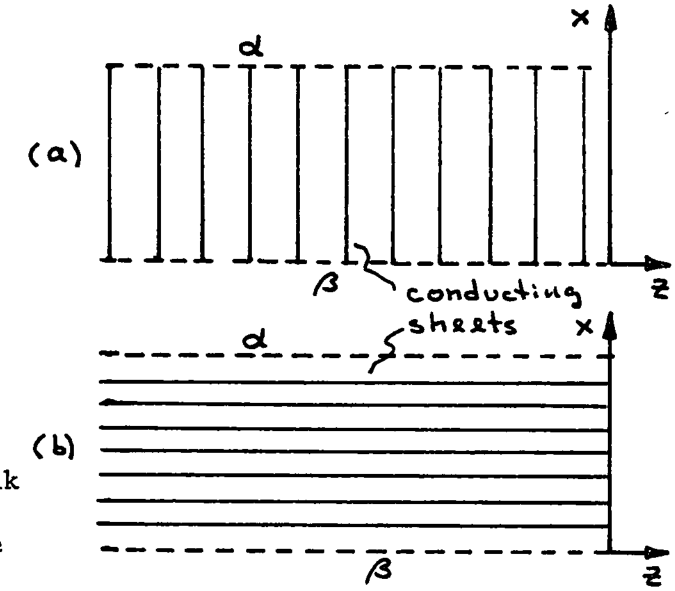
$$\begin{bmatrix} \hat{J}_x^\alpha \\ \hat{J}_x^\beta \end{bmatrix} = (j\omega\epsilon + \sigma_x)\gamma \begin{bmatrix} -\coth \gamma\Delta & \frac{1}{\sinh \gamma\Delta} \\ \frac{-1}{\sinh \gamma\Delta} & \coth \gamma\Delta \end{bmatrix} \begin{bmatrix} \hat{\Phi}^\alpha \\ \hat{\Phi}^\beta \end{bmatrix} \tag{8}$$

Prob. 5.12.1(cont.)

(b) In this limit, the medium might be composed of finely dispersed wires extending in the x direction and insulated from each other, as shown in the figure. With σ_y and $\sigma_z \to 0$,

$$\gamma^2 = j\omega\epsilon k^2/(\sigma_x + j\omega\epsilon) \to j\omega\epsilon k^2/\sigma_x$$

as $\omega \to 0$.



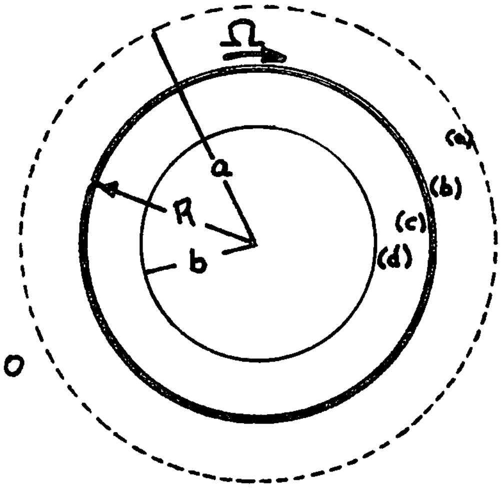
That this factor is complex means that the entries in Eq. 8 are complex. Thus, there is a phase shift (in space and/or in time depending on the nature of the excitations) of the potential in the bulk relative to that on the boundaries. The amplitude of γ gives an indication of the extent to which the potential penetrates into the volume. As $\omega \to 0, \gamma \to 0$, which points to an "infinite" penetration at zero frequency. That is, regardless of the spatial distribution of the potential at one surface, at zero frequency it will be reproduced at the other surface regardless of wavelength in the directions y and z. Regardless of k, the transfer relations reduce to

$$\begin{bmatrix} \hat{J}_x^{\alpha} \\ \hat{J}_x^{\beta} \end{bmatrix} \to \frac{\sigma_x}{\Delta}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \hat{\Phi}^{\alpha} \\ \hat{\Phi}^{\beta} \end{bmatrix} \qquad (9)$$

The "wires" carry the potential in the x direction without loss of spatial resolution.

(c) With no conduction in the x direction but finely dispersed conducting sheets in y-z planes, $\gamma^2 \to k^2(1 + \sigma_o/j\omega\epsilon)$. Thus, the fields do not penetrate in the x direction at all in the limit $\omega \to 0$. In the absence of time varying excitations, the y-z planes relax to become equipotentials and effectively shield the surface potentials from the material volume.

Prob. 5.13.1 a) Boundary conditions are



$$\hat{\Phi}^a = \hat{V}_o \tag{1}$$

$$\hat{\Phi}^b = \hat{\Phi}^c \tag{2}$$

Charge conservation for the sheet requires that

$$\frac{1}{R}\frac{\partial}{\partial\theta}(\sigma_s E_\theta) + \left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)(D_r^b - D_r^c) = 0$$

where

$$\hat{E}_\theta = jm\hat{\Phi}$$

In terms of complex amplitudes,

$$\frac{\sigma_s m^2}{R^2}\hat{\Phi}^b + j(\omega - m\Omega)(\hat{D}_r^b - \hat{D}_r^c) = 0 \tag{3}$$

Finally, there is the boundary condition

$$\hat{\Phi}^d = 0 \tag{4}$$

Transfer relations for the two regions follow from Table 2.16.2. They are written with Eqs. 1,2,and 4 taken into account.

$$\begin{bmatrix}\hat{D}_r^a\\ \hat{D}_r^b\end{bmatrix} = \epsilon_o\begin{bmatrix} f_m(R,a) & g_m(a,R)\\ g_m(R,a) & f_m(a,R)\end{bmatrix}\begin{bmatrix}\hat{V}_o\\ \hat{\Phi}^b\end{bmatrix} \tag{5}$$

$$\begin{bmatrix}\hat{D}_r^c\\ \hat{D}_r^d\end{bmatrix} = \epsilon_o\begin{bmatrix} f_m(b,R) & g_m(R,b)\\ g_m(b,R) & f_m(R,b)\end{bmatrix}\begin{bmatrix}\hat{\Phi}^b\\ 0\end{bmatrix} \tag{6}$$

Substitution of Eqs. 5b and 6a into Eq. 3 gives

$$\frac{\sigma_s m^2}{R^2}\hat{\Phi}^b + j(\omega - m\Omega)\epsilon_o\left\{g_m(R,a)\hat{V}_o + \hat{\Phi}^b\left[f_m(a,R) - f_m(b,R)\right]\right\} = 0 \tag{7}$$

or

$$\hat{\Phi}^b = \frac{-j\, S_e \hat{V}_o g_m(R,a) R}{m^2 + j\, S_e[f_m(a,R) - f_m(b,R)]R} \tag{8}$$

where $S_e \equiv \epsilon_o(\omega - m\Omega)R/\sigma_s$.

Prob. 5.13.1 (cont.)

b) The torque is

$$\Upsilon_z = (2\pi R^2 \ell)\,\tfrac{1}{2}\,\mathrm{Re}\,\hat{D}_r^b\,\hat{E}_\theta^{b*} \tag{9}$$

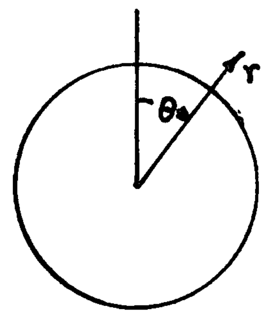
Because $\hat{E}_\theta = jm\hat{\Phi}/R$ and because of Eq. 5b, this expression becomes

$$\Upsilon_z = \pi R^2 \ell\,\mathrm{Re}\left[\epsilon_0 g_m(R,a)\hat{V}_o \frac{(-jm)}{R}\hat{\Phi}^{b*}\right] \tag{10}$$

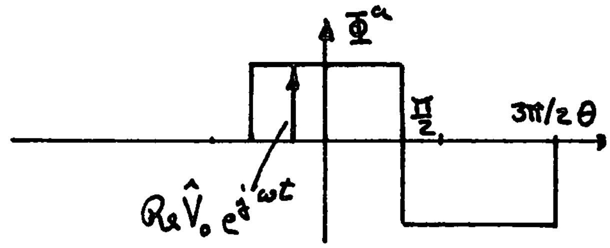
Substitution from Eq. 8 then gives the desired expression

$$\Upsilon_z = \frac{\pi R^2 \ell\,\epsilon_0 |\hat{V}_o|^2 g_m^2(R,a)\,S_e\,m^3}{m^4 + S_e^2\left[f_m(a,R) - f_m(b,R)\right]^2 R^2} \tag{11}$$

Prob. 5.13.2 With the (θ, r) coordinates defined as shown, the potential is the function of θ shown to the right. This function is represented by



$$\Phi^a = \mathrm{Re}\left[\sum_{m=-\infty}^{+\infty} \hat{V}_m e^{-jm\theta}\right] e^{j\omega t} \tag{1}$$



The multiplication of both sides by $e^{jn\theta}$ and integration over one period then gives

$$2\pi\hat{V}_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{V}_o e^{jn\theta} d\theta - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \hat{V}_o e^{jn\theta} d\theta \tag{2}$$

which gives (n → m)

$$\hat{V}_m = \frac{2\hat{V}_o}{\pi}\,\frac{\sin\left(\frac{m\pi}{2}\right)}{m} \tag{3}$$

Looking ahead, the current to the upper center electrode is

$$\hat{i} = j\omega\hat{q} = j\omega W \int_0^{\pi} \sum_{-\infty}^{+\infty} (\hat{D}_x^d)_m e^{-jm\theta} d\theta = 2j\omega W \sum_{-\infty}^{+\infty} \frac{\hat{D}_{xm}^b}{m} \sin\left(\frac{m\pi}{2}\right) \tag{4}$$

It then follows from Eqs. 6b and 8 that

$$\hat{i} = \frac{4\omega W \epsilon_0}{\pi} \sum_{m=-\infty}^{+\infty} j^m \frac{\sin^2\left(\frac{m\pi}{2}\right)}{m^2}\, \frac{g_m(b,R)\,S_{em}\hat{V}_o\, g_m(R,a)}{m^2 + j S_{em}\left[f_m(a,R) - f_m(b,R)\right]R} \tag{5}$$

where $S_{em} \equiv (\omega - m\Omega)R\epsilon_0/\sigma_s$.

Prob. 5.13.2 (cont.)

If the series is truncated at m=±1, this expression becomes one analogous to the one in the text.

$$\hat{i} = j\frac{4}{\pi}\omega w\epsilon_o\, g_1(b,R)\,g_1(R,a)\left\{\frac{S_{e1}}{1+jS_{e1}[f_1(a,R)-f_1(b,R)]R} - \frac{S_{e-1}}{1+jS_{e-1}[f_1(a,R)-f_1(b,R)]R}\right\} \qquad (6)$$

or

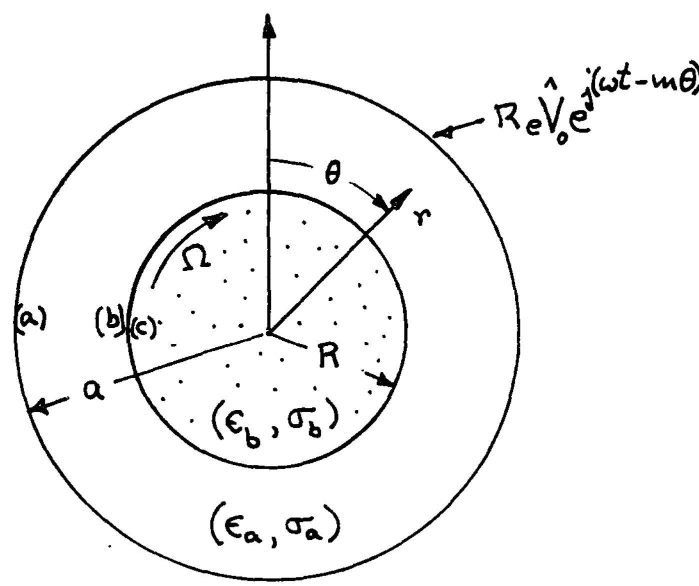
$$|\hat{i}| = \frac{-\frac{4}{\pi}\omega w\epsilon_o|\hat{V}_o|\, g_1(b,R)\,g_1(R,a)\left(\frac{2R\epsilon_o}{\sigma_s}\Omega\right)}{\sqrt{\{1+S_{e1}^2[f_1(a,R)-f_1(b,R)]^2R^2\}\{1+S_{e-1}^2[f_1(a,R)-f_1(b,R)]^2R^2\}}} \qquad (7)$$

<u>Prob. 5.14.1</u> Bulk relations for the two regions, with surfaces designated as in the figure, are

$$\begin{bmatrix}\hat{D}_r^a\\ \hat{D}_r^b\end{bmatrix} = \epsilon_a\begin{bmatrix} f_m(R,a) & g_m(a,R)\\ g_m(R,a) & f_m(a,R)\end{bmatrix}\begin{bmatrix}\hat{\Phi}^a\\ \hat{\Phi}^b\end{bmatrix} \qquad (1)$$

and

$$\hat{D}_r^c = \epsilon_b f_m(0,R)\,\hat{\Phi}^b \qquad (2)$$



Integration of the Maxwell stress over a surface enclosing the rotor amounts to a multiplication of the average traction in the θ direction by the surface area, and then to obtain a torque, by the lever arm, R.

$$T_z = \frac{1}{2}\mathrm{Re}\left[2\pi l R^2\, \hat{E}_\theta^{b*}\hat{D}_r^b\right] \qquad (3)$$

Because $\hat{E}_\theta^b = +j\frac{m}{R}\hat{\Phi}^b$, introduction of Eq. 1b into Eq. 3 makes it possible to write this torque in terms of the driving potential $\hat{\Phi}^a = \hat{V}_o$ and the potential on the surface of the rotor.

Prob. 5.14.1(cont.)

$$\tau_z = \pi R^2 \ell \epsilon_a g_m(R,a) \mathcal{R}e\left(j\frac{m}{R}\hat{V}_o^* \hat{\Phi}^b\right) \tag{4}$$

There are two boundary conditions at the surface of the rotor. The potential must be continuous, so

$$\hat{\Phi}^b = \hat{\Phi}^c \tag{5}$$

and charge must be conserved.

$$j(\omega - \Omega m)(\hat{D}_r^b - \hat{D}_r^c) + \left(\frac{\sigma_a}{\epsilon_a}\hat{D}_r^b - \frac{\sigma_b}{\epsilon_b}\hat{D}_r^c\right) = 0 \tag{6}$$

Substitution of Eqs. 1b and 2, again using the boundary condition $\hat{\Phi}^a = \hat{V}_o$ and Eq. 5, then gives an expression that can be solved for the rotor surface potential.

$$\hat{\Phi}^b = \frac{-\hat{V}_o g_m(R,a)[\epsilon_a j(\omega - \Omega m) + \sigma_a]}{\sigma_a f_m(a,R) - \sigma_b f_m(0,R) + j(\omega - \Omega m)(\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R))} \tag{7}$$

Substitution of Eq. 7 into Eq. 4 shows that the torque is

$$\tau_z = \frac{\pi R^2 \ell m \epsilon_a g_m(R,a) \mathcal{R}e[\epsilon_a(\omega - \Omega m) - j\sigma_a]|\hat{V}_o|^2}{R[\sigma_a f_m(a,R) - \sigma_b f_m(0,R)][1 + j S_e]} \tag{8}$$

where $$S_e \equiv \frac{(\omega - \Omega m)(\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R))}{\sigma_a f_m(a,R) - \sigma_b f_m(0,R)}$$

Rationalization of Eq. 8 show that the real part is

$$\tau_z = \frac{-\pi R \ell \epsilon_a |\hat{V}_o|^2 (\epsilon_a \sigma_b - \sigma_a \epsilon_b) g_m^2(R,a) f_m(0,R) m}{[\sigma_a f_m(a,R) - \sigma_b f_m(0,R)][\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R)]} \frac{S_e}{1 + S_e^2} \tag{9}$$

Note that $f_m(0,R)$ is negative, so this expression takes the same form as Eq. 5.14.11.

Prob. 5.14.2 (a) Boundary conditions at the rotor surface require continuity of potential and conservation of charge.

$$[\![\Phi]\!] = 0 \tag{1}$$

$$\frac{\partial \sigma_f}{\partial t} + \Omega \frac{\partial \sigma_f}{\partial \theta} = \sigma \frac{\partial \Phi^a}{\partial r} \tag{2}$$

where Gauss' law gives $\sigma_f = \epsilon_a E_r^a - \epsilon_b E_r^b$

Potentials in the fluid and within the rotor are respectively

$$\Phi = E(t) r \cos\theta + P_x(t) \frac{\cos\theta}{r} + P_y(t) \frac{\sin\theta}{r} \; ; \quad r > b \tag{3}$$

$$\Phi = Q_x(t) r \cos\theta + Q_y(t) r \sin\theta \tag{4}$$

These are substituted into Eqs. 1 and 2, which are factored according to whether terms have a sin θ or cos θ dependence. Thus, each expression gives rise to two equations in P_x, P_y, Q_x and Q_y. Elimination of Q_x and Q_y reduces the four expressions to two.

$$(\epsilon_a + \epsilon_b)\frac{dP_x}{dt} + (\epsilon_a + \epsilon_b)\Omega P_y + \sigma P_x = -b^2(\epsilon_b - \epsilon_a)\frac{dE}{dt} + \sigma b^2 E \tag{5}$$

$$(\epsilon_a + \epsilon_b)\frac{dP_y}{dt} - (\epsilon_a + \epsilon_b)\Omega P_x - (\epsilon_b - \epsilon_a) E \Omega b^2 + \sigma P_y = 0 \tag{6}$$

To write the mechanical equation of motion, the electric torque per unit length is computed.

$$T = b \int_0^{2\pi} \frac{\epsilon_a}{b} \frac{\partial \Phi^a}{\partial r} \frac{\partial \Phi^a}{\partial \theta} b \, d\theta \tag{7}$$

Substitution from Eq. 3 and integration gives

$$T = 2 \epsilon_a \pi E P_y \tag{8}$$

Thus, the torque equation is

$$I \frac{d\Omega}{dt} + B\Omega = -2 \epsilon_a \pi E P_y \tag{9}$$

The first of the given equations of motion is obtained from this one by using the normalization that is also given. The second and third relations follow by similarly normalizing Eqs. 5 and 6.

Prob. 5.14.2(cont.)

(b) Steady rotation with $\underline{E}$=1 reduces the equations of motion to

$$\Omega = P_y \tag{10}$$

$$\Omega P_y + P_x = H_e^2 \tag{11}$$

$$-\Omega P_x + P_y = f H_e^2 \Omega \tag{12}$$

Elimination among these for Ω results in the expression

$$H_e^2(1+f)\Omega = (1+\Omega^2)\Omega \tag{13}$$

One solution to this expression is the static equilibrium $\Omega = 0$.

Another is possible if H_e^2 exceeds the critical value

$$H_e^2 = 1/(1+f) \Rightarrow \frac{\epsilon_a \epsilon_b \mathcal{E}^2}{\sigma \eta} = 1 \tag{14}$$

in which case Ω is given by

$$\Omega = \sqrt{(1+f)H_e^2 - 1} \tag{15}$$

Prob. 5.15.1 From Eq. 8 of the solution to Prob. 5.13.8, the temporal modes are found by setting the denominator equal to zero. Thus,

$$m^2 + j(\omega - m\Omega)\frac{\epsilon_o R^2}{\sigma_s}\left[f_m(a,R) - f_m(b,R)\right] = 0 \tag{1}$$

Solution for ω then gives

$$\omega = m\Omega + \frac{j\sigma_s}{\epsilon_o R^2 m^2\left[f_m(a,R) - f_m(b,R)\right]} \tag{2}$$

where $f_m(a,R) > 0$ and $f_m(b,R) < 0$ so that the imaginary part of ω represents decay.

Prob. 5.15.2 The temporal modes follow from the equation obtained by setting the denominator of Eq. 7 from the solution to Prob. 5.14.1 equal to zero.

$$\sigma_a f_m(a,R) - \sigma_b f_m(0,R) + j(\omega - \Omega m)\left[\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R)\right] = 0 \tag{1}$$

Solved for ω, this gives the desired eigenfrequencies.

$$\omega = \Omega m + j\frac{\sigma_a f_m(a,R) - \sigma_b f_m(0,R)}{\epsilon_a f_m(a,R) - \epsilon_b f_m(0,R)} \tag{2}$$

Note that $f_m(a,R) > 0$ while $f_m(0,R) < 0$, so the frequencies represent decay.

Prob. 5.15.3 The conservation of charge boundary condition takes the form

$$\nabla_\Sigma \cdot \bar{K} + \frac{\partial \sigma_f}{\partial t} = 0 \tag{1}$$

where the surface current density is

$$\bar{K} = \bar{i}_\theta(\sigma_s E_\theta^a) + \bar{i}_\phi(\sigma_s E_\phi + \sigma_f \Omega R \sin\theta) \tag{2}$$

Using Eq. (2) to evaluate Eq. (1) and writing $\bar{E}$ in terms of the potential, Φ, the conservation of charge boundary condition becomes

$$\frac{1}{R}\frac{\partial}{\partial\theta}(\sigma_s E_\theta^a \sin\theta) + \frac{\sigma_s}{R}\frac{\partial E_\phi}{\partial\phi} + \left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)(\sigma_f \sin\theta) = 0 \tag{3}$$

With the substitution of the solutions to Laplace's equation in spherical coordinates

$$\Phi = Re\,\hat{\Phi}(r) P_n^m(\cos\theta)\, e^{-jm\phi} e^{j\omega t} \tag{4}$$

the boundary condition stipulates that

$$\frac{-\sigma_s \sin\theta}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\hat{\Phi}^a P_n^m\right) - \frac{m^2\hat{\Phi}^a P_n^m}{\sin^2\theta}\right] + j(\omega - m\Omega)\sin\theta\,\hat{\sigma}_f^a P_n^m = 0 \tag{5}$$

By definition, the operator in square brackets is

$$-n(n+1)\hat{\Phi}^a P_n^m \tag{6}$$

and so the boundary condition becomes simply

$$\frac{\sigma_s}{R^2}\hat{\Phi}^a n(n+1) + j(\omega - m\Omega)\hat{\sigma}_f^a = 0 \tag{7}$$

In addition, the potential is continuous at the boundary r = R.

$$\hat{\Phi}^a = \hat{\Phi}^b \tag{8}$$

Transfer relations representing the fields in the volume regions are Eqs. 4.8.18 and 4.8.19. For the outside region $\beta \to$ (a) while for the inside region, $\alpha \to$ (b). Thus, Eq. (7), which can also be written as

Prob. 5.15.3 (cont.)

$$\frac{\sigma_s}{R^2} n(n+1)\hat{\Phi}^a + j(\omega - m\Omega)(\hat{D}_r^a - \hat{D}_r^b) = 0 \tag{9}$$

becomes, with substitution for $\hat{D}_r^a$ and $\hat{D}_r^b$, and use of Eq. (8),

$$\frac{\sigma_s}{R^2} n(n+1)\hat{\Phi}^a + j(\omega - m\Omega)\left[\frac{\epsilon_a(n+1)}{R}\hat{\Phi}^a + \frac{\epsilon_b n}{R}\hat{\Phi}^a\right] = 0 \tag{10}$$

This expression is homogeneous in the amplitude $\hat{\Phi}^a$, (there is no drive) and it follows that the natural modes satisfy the dispersion equation

$$\omega = m\Omega + j\frac{\sigma_s n(n+1)}{R[\epsilon_b n + \epsilon_a(n+1)]} \tag{11}$$

where (n,m) are the integer mode numbers in spherical coordinates.

In a uniform electric field, surface charge on the spherical surface would assume the same distribution as on a perfectly conducting sphere.... a cos θ distribution. Hence, the associated mode which describes the build up or decay of this distribution is n = 1, m = 0. The time constant for charging or discharging a particle where the conduction is primarily on the surface is therefore

$$\tau = R(2\epsilon_a + \epsilon_b)/2\sigma_s \tag{12}$$

Prob. 5.15.4 The desired modes of charge relaxation are the homogeneous response. This can be found by considering the system without excitations. Thus, for the exterior region,

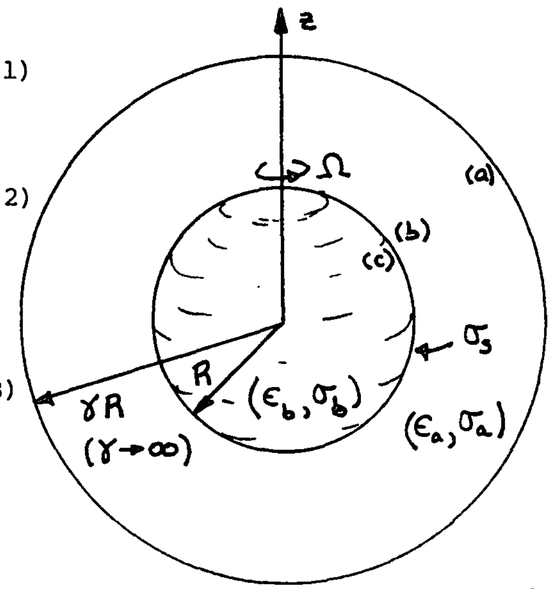
$$\hat{D}_r^b = \epsilon_a f_n(\infty, R)\hat{\Phi}^b = \epsilon_a \frac{(n+1)}{R}\hat{\Phi}^b \tag{1}$$

while for the interior region,

$$\hat{D}_r^c = \epsilon_b f_n(0, R)\hat{\Phi}^c = -\frac{\epsilon_b n}{R}\hat{\Phi}^c \tag{2}$$

At the interface, the potential must be continuous, so

$$\hat{\Phi}^c = \hat{\Phi}^b \tag{3}$$



The second boundary condition combines conservation of charge and Gauss' law. To express this in terms of complex amplitudes, first observe that charge conservation requires that the accumulation of surface charge either is the result of a net divergence of surface current in the region of surface conduction, or results from a difference of conduction current from the volume regions.

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}\right)\sigma_f = -\nabla_\Sigma \cdot \bar{K}_f' - \bar{n}\cdot[\![\bar{J}_f']\!] = \sigma_s \nabla_\Sigma^2 \Phi^b - \left(\frac{\sigma_a}{\epsilon_a}D_r^b - \frac{\sigma_b}{\epsilon_b}D_r^c\right) \tag{4}$$

where

$$\nabla_\Sigma^2 = \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

For solutions having the complex amplitude form in spherical coordinates,

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) = -n(n+1) \tag{5}$$

so, with the use of Gauss' law, Eq. 4 becomes

$$j(\omega - m\Omega)(\hat{D}_r^b - \hat{D}_r^c) = \frac{-n(n+1)\sigma_s}{R^2}\hat{\Phi}^b + \left(\frac{\sigma_b}{\epsilon_b}\hat{D}_r^c - \frac{\sigma_a}{\epsilon_a}\hat{D}_r^b\right) \tag{6}$$

Substitution of Eqs. 1-3 into this expression gives an equation that is homogeneous in $\hat{\Phi}^b$. The coefficient of $\hat{\Phi}^b$ must therefore vanish. Solved for $j\omega$, the resulting expression is

$$j\omega = j\Omega m - \left[\frac{n(n+1)\sigma_s}{R} + \sigma_a(n+1) + \sigma_b n\right] \Big/ \left[\epsilon_a(n+1) + \epsilon_b n\right] \tag{7}$$

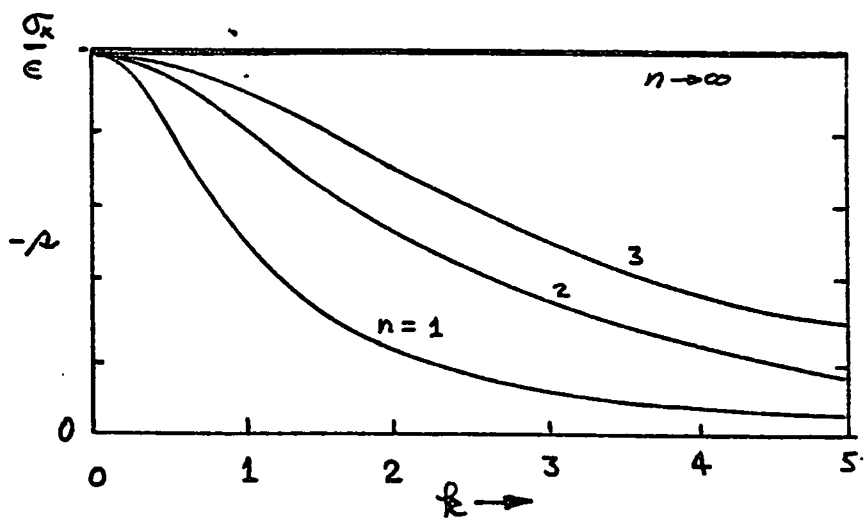
Prob. 5.15.5 (a) With the potentials in the transfer relations of Prob. 5.12.1 constrained to zero, the response cannot be finite unless the determinant of the coefficients is infinite. This condition is met if $\sinh \gamma\Delta = 0$. Roots to this expression are $\gamma\Delta = jn\pi$, n = 1, 2, ... and it follows that the required eigenfrequency equation is the expression for γ^2 with $\gamma^2 = -(n\pi/\Delta)^2$.

$$s \equiv j\omega = -\frac{\left[\sigma_x\left(\frac{n\pi}{\Delta}\right)^2 + \sigma_y k_y^2 + \sigma_z k_z^2\right]}{\epsilon\left[k^2 + \left(\frac{n\pi}{\Delta}\right)^2\right]}; \quad k^2 = k_y^2 + k_z^2 \tag{1}$$

(b) Note that if $\sigma_x = \sigma_y = \sigma_z \equiv \sigma$, this expression reduces to $-\sigma/\epsilon$ regardless of n. The discrete modes degenerate into a continuum of modes representing the charge relaxation process in a uniform conductor. (c) For $\sigma_y \to 0$ and $\sigma_z \to 0$, Eq. 1 reduces to

$$s = -\frac{\sigma_x}{\epsilon}\left(\frac{n\pi}{\Delta}\right)^2 \Big/ \left[k^2 + \left(\frac{n\pi}{\Delta}\right)^2\right] \tag{2}$$

Thus, the eigenfrequencies as shown in Fig. P5.15.5a depend on k with the mode number as a parameter.

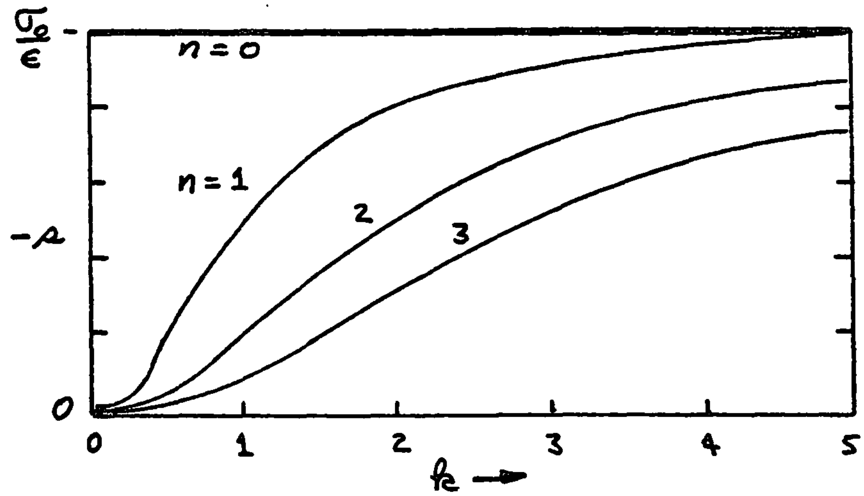


Prob. 5.15.5(cont.)

(d) With $\sigma_x = 0, \sigma_y = \sigma_z = \sigma_o$, Eq. 1 reduces to

$$s = -\frac{\sigma_o}{\epsilon} k^2 \Big/ \left[k^2 + \left(\frac{n\pi}{\Delta}\right)^2 \right] \tag{3}$$

and the eigenfrequencies depend on k as shown in Fig. P5.15.5b.



Prob. 5.17.1 In the upper region, solutions to Laplace's equation take the form

$$\hat{\Phi}_n = \hat{\Phi}^a \frac{\sinh k_n x}{\sinh k_n d} - \hat{\Phi}^b \frac{\sinh k_n(x-d)}{\sinh k_n d} \tag{1}$$

It follows from this fact alone and Eqs. 5.17.17-5.17.19 that in region I, where $\hat{\Phi}^a = 0$

$$\Phi = -Re\, j\frac{\hat{V}_o d\epsilon}{\sigma_s}\sum_{n=-1}^{-\infty} \frac{(\omega-k_nU)[e^{j(k_n-\beta)\ell}-1]e^{j(\omega t-k_n z)}}{(k_n-\beta)D'(\omega,k_n)} \frac{\sinh k_n(x-d)}{\sinh k_n d} \tag{2}$$

Similarly, in region II, where $\hat{\Phi}^a = \hat{V}_o$

$$\begin{aligned}\Phi = -Re\, j\frac{\hat{V}_o d\epsilon}{\sigma_s}\Bigg\{&\sum_{n=-1}^{-\infty}\frac{(\omega-k_nU)e^{j(k_n-\beta)\ell-jk_n z}\sinh k_n(x-d)}{(k_n-\beta)D'(\omega,k_n)\sinh k_n d}\\ &+\frac{(\omega-\beta U)e^{-j\beta z}\sinh\beta(x-d)}{D(\omega,\beta)\sinh\beta d}\\ &+\sum_{n=1}^{\infty}\frac{(\omega-k_nU)e^{-jk_n z}\sinh k_n(x-d)}{(k_n-\beta)D'(\omega,k_n)}\Bigg\}e^{j\omega t} + Re\,\hat{V}_o e^{j(\omega t-\beta z)}\frac{\sinh\beta x}{\sinh\beta d}\end{aligned} \tag{3}$$

and in region III, where $\hat{\Phi}^a = 0$

$$\Phi = Re\, j\frac{\hat{V}_o d\epsilon}{\sigma_s}\Bigg\{\sum_{n=1}^{\infty}\frac{(\omega-k_nU)[e^{j(k_n-\beta)\ell}-1]e^{j(\omega t-k_n z)}}{(k_n-\beta)D'(\omega,k_n)}\frac{\sinh k_n(x-d)}{\sinh k_n d} \tag{4}$$

In the lower region, $\hat{\Phi}^d = 0$ throughout, so

$$\hat{\Phi}_n = \hat{\Phi}^b \frac{\sinh k_n(x+d)}{\sinh k_n d} \tag{5}$$

and in region I

$$\Phi = Re\, j\frac{\hat{V}_o d\epsilon}{\sigma_s}\sum_{n=-1}^{-\infty}\frac{(\omega-k_nU)[e^{j(k_n-\beta)\ell}-1]e^{j(\omega t-k_n z)}}{(k_n-\beta)D'(\omega,k_n)}\frac{\sinh k_n(x+d)}{\sinh k_n d} \tag{6}$$

in region II

Prob. 5.17.1 (cont.)

$$\Phi = \mathcal{R}e \, \frac{j\hat{V}_o d\epsilon}{\sigma_s}\left\{\sum_{n=-1}^{-\infty} \frac{(\omega - k_n U)[e^{j(k_n-\beta)\ell}]e^{-jk_n z}}{(k_n-\beta) D'(\omega, k_n)} \frac{\sinh k_n(x+d)}{\sinh k_n d}\right.$$

$$+ \frac{(\omega-\beta U) e^{-j\beta z}}{D(\omega,\beta)} \frac{\sinh\beta(x+d)}{\sinh\beta d} \tag{7}$$

$$\left. + \sum_{n=1}^{\infty} \frac{(\omega - k_n U) e^{-jk_n z}}{(k_n-\beta) D'(\omega,k_n)} \frac{\sinh k_n(x+d)}{\sinh k_n d}\right\} e^{j\omega t}$$

and in region III

$$\Phi = -\mathcal{R}e \, \frac{j\hat{V}_o d\epsilon}{\sigma_s}\left\{\sum_{n=1}^{\infty} \frac{(\omega - k_n U)[e^{j(k_n-\beta)\ell} - 1] e^{-jk_n z}}{(k_n-\beta) D'(\omega,k_n)}\right\} e^{j\omega t} \tag{8}$$

<u>Prob. 5.17.2</u> The relation between Fourier transforms has already been determined in Sec. 5.14, where the response to a single complex amplitude was found. Here, the single traveling wave on the (a) surface is replaced by

$$\Phi^a(z,t) = \mathcal{R}e\left\{\hat{V}_o[u_{-1}(z) - u_{-1}(z-\ell)]\, e^{j(\omega t - \beta z)}\right\} = \mathcal{R}e\, \hat{\Phi}(z)\, e^{j\omega t} \tag{1}$$

where

$$\hat{\Phi}^a = \hat{V}_o[u_{-1}(z) - u_{-1}(z-\ell)]\, e^{-j\beta z} \tag{2}$$

Thus, the Fourier transform of the driving potential is

$$\hat{\tilde{\Phi}}^a = \int_{-\infty}^{+\infty} \hat{\Phi}\, e^{jkz} dz = \int_0^{\ell} \hat{V}_o\, e^{-j(\beta-k)z} dz = \frac{\hat{V}_o[e^{j(k-\beta)\ell} - 1]}{j(k-\beta)} \tag{3}$$

It follows that the transform of the potential in the (b) surface is given by Eq. 5.14.8 with $\hat{V}_o \rightarrow \hat{\tilde{\Phi}}$, and a=b=d.

$$\hat{\tilde{\Phi}}^b = \frac{1}{\cosh kd}\, \frac{\sigma_a}{\sigma_a+\sigma_b}\left[\frac{1 + j(\omega - kU)\frac{\epsilon_a}{\sigma_a}}{1 + j(\omega - kU)\frac{(\epsilon_a+\epsilon_b)}{\sigma_a+\sigma_b}}\right]\hat{\tilde{\Phi}}^a \tag{4}$$

where $\hat{\tilde{\Phi}}^a$ is given by Eqs. 1 and 2. The spatial distribution follows by taking the inverse Fourier transform.

Prob. 5.17.2(cont.)

$$\hat{\Phi}^b = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{\hat{\Phi}}^b e^{-jkz}\,dk \tag{5}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{V_o\left[\epsilon_a(\omega - kU) - j\sigma_a\right]\left[e^{j(l-z)k}e^{-j\beta l} - e^{-jkz}\right]dk}{(k-\beta)\,D(\omega,k)}$$

where

$$D(\omega,k) \equiv \cosh kd\left[(\sigma_a+\sigma_b) + j(\omega - kU)(\epsilon_a+\epsilon_b)\right]$$

Singularities of the integrand given by $D(\omega,k)=0$ are either

$$\cosh(kd) = 0 \Rightarrow jkd = \pm(|2n|-1)\pi/2 \Rightarrow k_n = \mp\frac{(|2n|-1)}{d}\frac{\pi}{2}j,\ n=\pm 1 \cdots \pm\infty \tag{6}$$

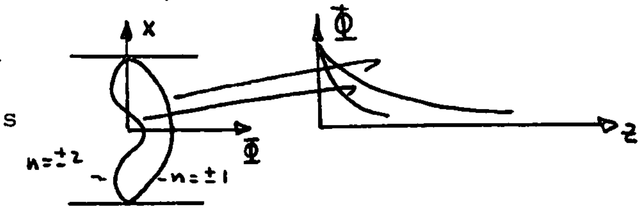
or

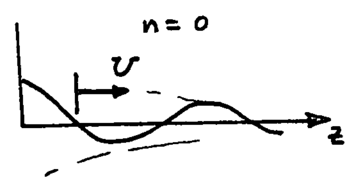
$$(\sigma_a+\sigma_b) + j(\omega - kU)(\epsilon_a+\epsilon_b) = 0 \Rightarrow k = \frac{\omega}{U} - j\frac{(\sigma_a+\sigma_b)}{(\epsilon_a+\epsilon_b)}\frac{1}{U} \equiv k_0 \tag{7}$$

With the transverse coordinate, x, taken as having its origin on the moving sheet, the distribution of potential is in general given by $(\hat{\hat{\Phi}}^a = 0)$

$$\hat{\Phi} = \begin{cases} -\hat{\hat{\Phi}}^b \dfrac{\sinh k(x-d)}{\sinh kd} & ;\ x>0 \\ \hat{\hat{\Phi}}^b \dfrac{\sinh k(x+d)}{\sinh kd} & ;\ x<0 \end{cases} \tag{8}$$

Thus, the n $\neq$ 0 modes, which are either purelly growing or decaying with an exponential dependence in the longitudinal direction, have the sinusoidal transverse dependence sketched. Note that these are the modes expected from Laplace's equation in the absence of a sheet. They have no derivative in the x direction at the sheet surface, and therefore represent modes with no net surface charge on the sheet.





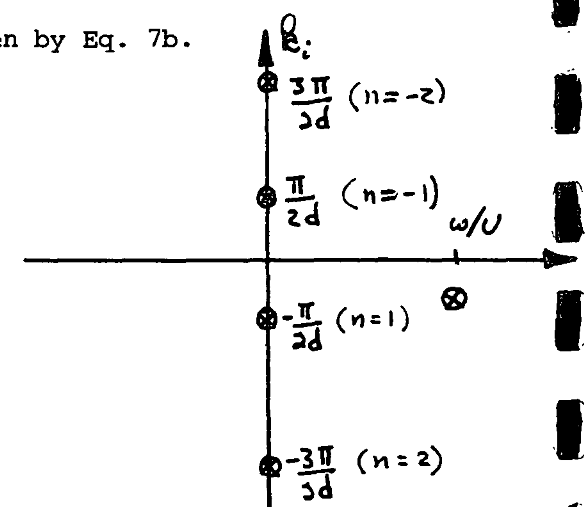
These modes, which are uncoupled from the sheet, are possible because of the symmetry of the configuration obtained by making a=b. The n=0 mode is the only one involving the charge relaxation on the sheet. Because the wavenumber is complex, the transverse dependence is neither purelly exponential or sinusoidal. In fact, the transverse dependence can no longer be represented by a single amplitude, since all positions in a given z plane do not have the

Prob. 5.17.2(cont.)

same phase. By using the identity $\sinh(u+jv) = \sinh u \cos v + j\cosh u \sin v$, the magnitude of the transverse dependence in the upper region given by Eq. 8 can be shown to be

$$\left|\frac{\sinh k(x-d)}{\sinh kd}\right| = \sqrt{\frac{\sinh^2 k_r(x-d)\cos^2 k_i(x-d) + \cosh^2 k_r(x-d)\sin^2 k_i(x-d)}{\sinh^2 k_r d\cos^2 k_i d + \cosh^2 k_r d \sin^2 k_i d}} \qquad (9)$$

where the real and imaginary parts of k are given by Eq. 7b.



In the complex k plane, the poles of Eq. 5 are as shown in the sketch. Note that $k=\beta$ is not a singular point because the numerator contains a zero also at $k=\beta$. In using the Residue theorem, the contour is closed in the upper half plane for $z < 0$ and in the lower half for $z > \ell$. For the intermediate region, II, the term multiplying $\exp jk(\ell - z)$ must be closed from above while that multiplying $\exp -jkz$ is closed from below. Thus, in region I, $z<0$,

$$\Phi^b = \mathrm{Re}\, e^{j\omega t}\hat{V}_o \sum_{n=-1}^{-\infty} \frac{[j(\omega - k_n U)\epsilon_a + \sigma_a][e^{j(\ell-z)k_n} e^{-j\beta\ell} - e^{-jk_n z}]}{-j(k_n-\beta)(-1)^n d[(\sigma_a+\sigma_b)+j(\omega-k_n U)(\epsilon_a+\epsilon_b)]} \qquad (10)$$

in region II, the integral is split as described and the"pole"at $k=\beta$ is now actually a singularity, and hence makes a contribution. $0<z<\ell$

$$\Phi^b = \mathrm{Re}\,\hat{V}_o e^{j\omega t}\Bigg\{\sum_{n=-1}^{-\infty} \frac{[j(\omega-k_n U)\epsilon_a + \sigma_a]\, e^{j(\ell-z)k_n}\, e^{-j\beta\ell}}{j(-1)^{n+1} d(k_n-\beta)[(\sigma_a+\sigma_b)+j(\omega-k_n U)(\epsilon_a+\epsilon_b)]}$$

$$+ \frac{[\sigma_b\varepsilon_a - \sigma_a\varepsilon_b]\, e^{-jk_o z}}{\cosh k_o d\,[U(\epsilon_a+\epsilon_b)^2](k_o-\beta)} \qquad (11)$$

$$+\sum_{n=1}^{\infty}\frac{[j(\omega-k_n U)\epsilon_a+\sigma_a]\, e^{-jk_n z}}{j(k_n-\beta)(-1)^n d[(\sigma_a+\sigma_b)+j(\omega-k_n U)(\epsilon_a+\epsilon_b)]} + \frac{[j\epsilon_a(\omega-\beta U)+\sigma_a]\, e^{-j\beta z}}{D(\omega,\beta)}\Bigg\}$$

Finally in region III, $z > \ell$,

Prob. 5.17.2(cont.)

$$\hat{\Phi}^b = -\mathcal{R}_e \hat{V}_o e^{j\omega t}\left\{\sum_{n=1}^{\infty} \frac{[j(\omega - k_n U)\epsilon_a + \sigma_a][e^{j(\ell - z)k_n} e^{-j\beta\ell} - e^{-jk_n z}]}{j(k_n - \beta)(-1)^n d[(\sigma_a + \sigma_b) + j(\omega - k_n U)(\epsilon_a + \epsilon_b)]}\right. \tag{12}$$

$$\left. - j\left[\frac{\epsilon_a\sigma_b - \epsilon_b\sigma_a}{\epsilon_a + \epsilon_b}\right] \frac{e^{j(\ell - z)k_o} e^{-j\beta\ell} - e^{-jk_o z}}{\cosh k_o d \; U\,(\epsilon_a + \epsilon_b)(k_o - \beta)}\right\}$$

The total force follows from an evaluation of

$$f = \frac{1}{4\pi}\mathcal{R}_e \int_{-\infty}^{+\infty} \hat{E}_z^b [\hat{D}_x^b - \hat{D}_x^c]^* dk = \frac{\mathcal{R}_e}{4\pi}\int_{-\infty}^{+\infty} jk\hat{\Phi}^b [\hat{D}_x^b - \hat{D}_x^c]^* dk \tag{13}$$

Use of Eqs. 5.14.8 and 5.14.9 for $\hat{\Phi}^b$ and $[\![\hat{D}_x]\!]$ results in

$$f = -\frac{\mathcal{R}_e}{4\pi}\int_{-\infty}^{+\infty} \frac{jk^2[j(\omega - kU)\epsilon_a + \sigma_a]\hat{\Phi}^a \hat{\Phi}^{a*}(\epsilon_a\sigma_b - \epsilon_b\sigma_a)\,dk}{\sinh kd \cosh kd[(\sigma_a + \sigma_b)^2 + (\omega - kU)^2(\epsilon_a + \epsilon_b)^2]} \tag{14}$$

The real part is therefore simply

$$f = \frac{\epsilon_a\sigma_b - \epsilon_b\sigma_a}{4\pi}\int_{-\infty}^{+\infty} \frac{k^2(\omega - kU)\epsilon_a \hat{\Phi}^a \hat{\Phi}^{a*}\,dk}{\sinh kd \cosh kd[(\sigma_a + \sigma_b)^2 + (\omega - kU)^2(\epsilon_a + \epsilon_b)^2]} \tag{15}$$

where the square of the driving amplitudes follows from Eq. 3.

$$\hat{\Phi}^a \hat{\Phi}^{a*} = \frac{4|\hat{V}_o|^2 \sin^2\left[\frac{(k - \beta)\ell}{2}\right]}{(k - \beta)^2} \tag{16}$$