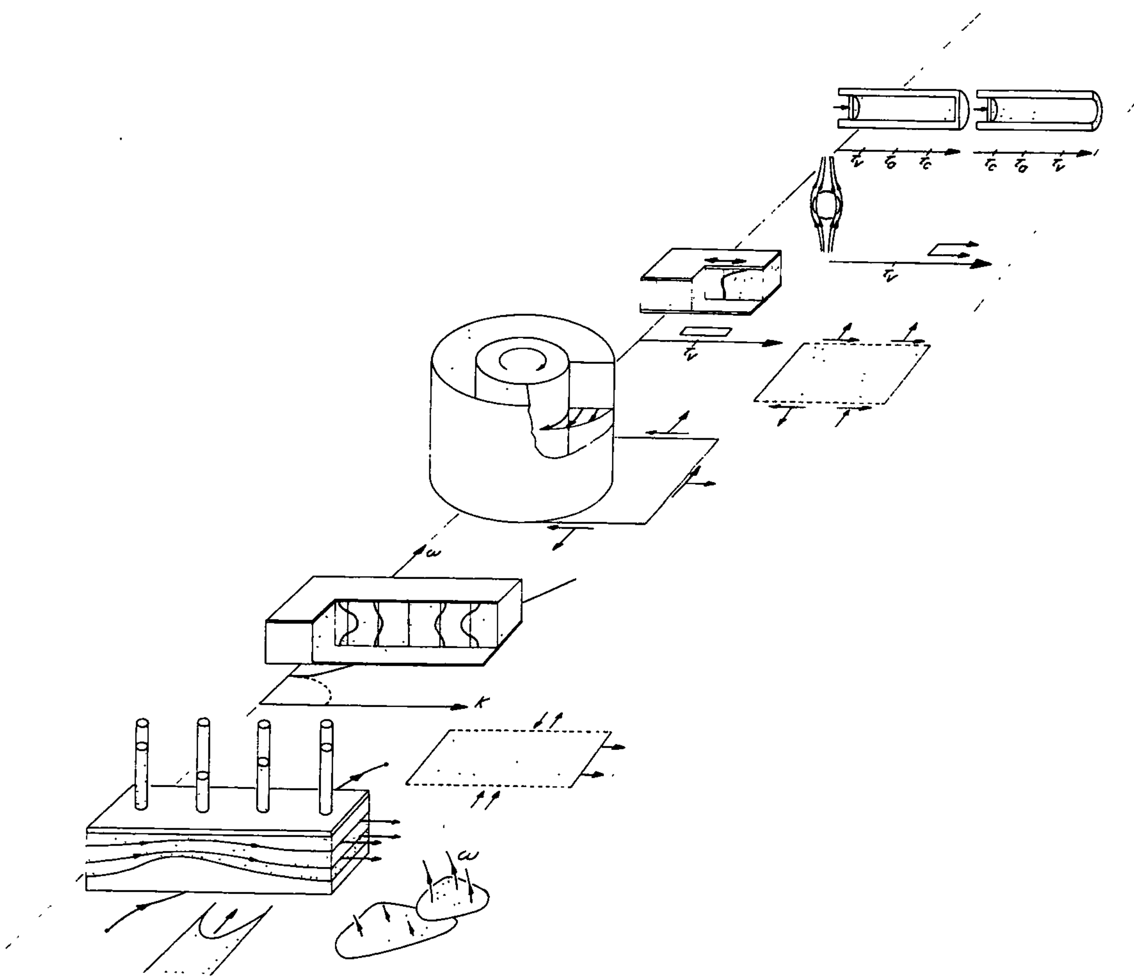
7

Laws, Approximations and Relations of Fluid Mechanics



Prob. 7.2.1 If for a volume of fixed identity (Eq. 3.7.3)

$$\int_V \alpha_i \, dV = \text{constant} \tag{1}$$

then

$$\frac{d}{dt}\int \alpha_i \, dV = 0 \tag{2}$$

From the scalar form of the Leibnitz rule (Eq. 2.6.5 with $\xi \to \alpha_i$)

$$\int_V \frac{\partial \alpha_i}{\partial t} dV + \oint_S \alpha_i \bar{v} \cdot \bar{n} \, da = 0 \tag{3}$$

where $\bar{v}$ is the velocity of the material supporting the property α_i. With the use of the Gauss theorem on the surface integral

$$\int_V \left[\frac{\partial \alpha_i}{\partial t} + \nabla\cdot(\alpha_i \bar{v})\right] dV = 0 \tag{4}$$

Because the volume of fixed identity is arbitrary

$$\frac{\partial \alpha_i}{\partial t} + \nabla\cdot \alpha_i \bar{v} = 0 \tag{5}$$

Now, if $\alpha_i = \rho\beta_i$, then Eq. (5) becomes

$$\rho\frac{\partial \beta_i}{\partial t} + \beta_i \frac{\partial \rho}{\partial t} + \beta_i \nabla\cdot\rho\bar{v} + \rho\bar{v}\cdot\nabla\beta_i = 0 \tag{6}$$

The second and third terms cancel by virtue of mass conservation, Eq. 7.2.3, leaving

$$\frac{\partial \beta_i}{\partial t} + \bar{v}\cdot\nabla\beta_i = 0 \tag{7}$$

Prob. 7.6.1 To linear terms, the normal vector is

$$\bar{n} = \bar{i}_x - \frac{\partial \xi}{\partial y}\bar{i}_y - \frac{\partial \xi}{\partial z}\bar{i}_z \tag{1}$$

and inserted into Eq. 7.6.12, this gives the surface force density to linear terms

$$(\bar{T}_s)_x = -\gamma\left(-\frac{\partial^2 \xi}{\partial y^2} - \frac{\partial^2 \xi}{\partial z^2}\right) \tag{2}$$

<u>Prob. 7.6.2</u> The initially spherical surface has a position represented by

$$F = r - (R + \xi(\theta,\phi,t)) = 0 \tag{1}$$

Thus, to linear terms in the amplitude, ξ , the normal vector is

$$\bar{n} = -\frac{\nabla F}{|\nabla F|} \approx \bar{i}_r - \frac{1}{R}\frac{\partial \xi}{\partial \theta}\bar{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial \xi}{\partial \phi}\bar{i}_\phi \tag{2}$$

It follows from the divergence operator in spherical coordinates that

$$\nabla\cdot\bar{n} = \frac{1}{r^2\sin\theta}\left[\frac{\partial r^2\sin\theta}{\partial r} - \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{R}\frac{\partial\xi}{\partial\theta}\right) - \frac{\partial}{\partial\phi}\frac{r}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\right] \tag{3}$$

Evaluation of Eq. 3 using the approximation that

$$\frac{1}{r} \approx \frac{1}{R} - \frac{\xi}{R^2} \tag{4}$$

therefore gives

$$(\tilde{T}_s)_r = \gamma\left[-\frac{2}{R} + \frac{2\xi}{R^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\xi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\xi}{\partial\phi^2}\right] \tag{5}$$

where terms that are quadradic in ξ have been dropped.

To obtain a convenient complex amplitude representation, where $\xi = Re\,\tilde{\xi}\,P_n^m(\cos\theta)\exp(-jm\phi)$, use is made of the relation, Eq. 2.16.31,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\right) - \frac{m^2}{\sin^2\theta}P_n^m(\cos\theta) = -n(n+1) \tag{6}$$

Thus, the complex amplitude of the surface force density due to surface tension is

$$(\tilde{T}_s)_r = -\frac{\gamma}{R^2}\left[(n-1)(n+2)\right]\tilde{\xi} \tag{7}$$

Actually, Eqs. 2 and 3 show that $\bar{T}_s$ also has θ and ϕ components (to linear terms in ξ). Because the surface force density is always normal to the interface, these components are balanced by pressure forces from the fluid to either side of the interface. To linear terms, the radial force balance represents the balance in the normal direction while the θ and ϕ components represent the shear balance. For an inviscid fluid it is not appropriate to include any shearing surface force density, so the stress equilibrium equations written to linear terms in the θ and ϕ directions must automatically balance.

<u>Prob. 7.6.3</u> Mass conservation requires that

$$\frac{4}{3}\pi\xi_1^3 + \frac{4}{3}\pi\xi_2^3 = 2\left(\frac{4}{3}\pi\xi_o^3\right) \Rightarrow \xi_1^3 + \xi_2^3 = 2\xi_o^3 \tag{1}$$

With the pressure outside the bubbles defined as p_o, the pressures inside the respective bubbles are

$$p_a - p_o = \frac{2\gamma}{\xi_1} \quad ; \quad p_b - p_o = \frac{2\gamma}{\xi_2} \tag{2}$$

so that the pressure difference driving fluid between the bubbles once the valve is opened is

$$p_a - p_b = 2\gamma\left[\frac{1}{\xi_1} - \frac{1}{\xi_2}\right] \tag{3}$$

The flow rate between bubbles given by differentiating Eq. 1 is then equal to Q_v and hence to the given expression for the pressure drop through the connecting tubing.

$$Q_v = -\frac{4}{3}\pi\, 3\xi_1^2 \frac{d\xi_1}{dt} = \frac{\pi R^4}{8\eta l}(p_a - p_b) = \frac{\pi R^4}{8\eta l}2\gamma\left[\frac{1}{\xi_1} - \frac{1}{\xi_2}\right] \tag{4}$$

Thus, the combination of Eqs. 1 and 4 give a first order differential equation describing the evolution of ξ_1 or ξ_2. In normalized terms, that expression is

$$\frac{d\xi_1}{dt} = \frac{1}{\xi_1^2}\left[\frac{1}{(2 - \xi_1^3)^{1/3}} - \frac{1}{\xi_1}\right] \tag{5}$$

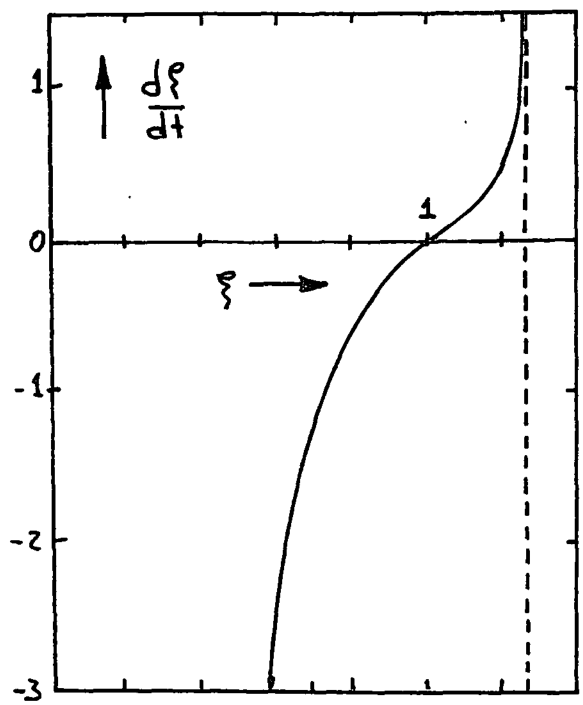
where

$$\xi_1 = \|\xi_1\|\,\xi_o \quad , \quad t = \underline{t}\left[\frac{16\eta l\xi_o^4}{R^4\gamma}\right]$$

Thus, the velocity is a function of ξ_1, and can be pictured as shown in the figure. It is therefore evident that if ξ_1 increases slightly, it will tend to further increase. The static equilibrium at $\xi_1 = \xi_o$ is unstable. Physically this results from the fact that γ is constant. As the radius of curvature of a bubble decreases, the pressure increases and forces the air into the other bubble. Note that this is not what would be found if the bubbles were replaced by most elastic membranes. The example is useful for giving a reminder of what is implied by the concept of a surface tension. Of course, if the bubble

Prob. 7.6.3 (cont.)

can not be modelled as a layer of liquid with interior and exterior interfaces comprised of the same material,   then the basic law may not apply. In the figure, note that all variables are normalized. The asymptote comes at the radius where the second bubble has completely collapsed.

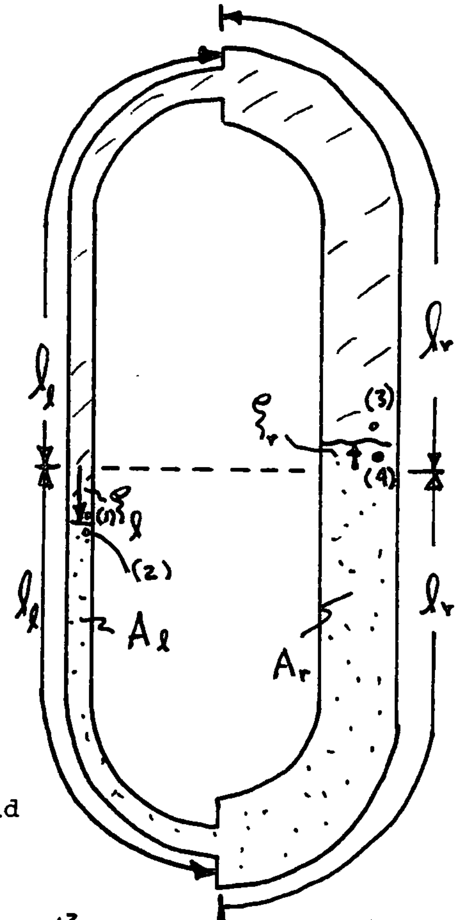


Prob. 7.8.1 Mass conservation for the lower fluid is represented by

$$[A_\ell(\ell_\ell-\xi_\ell)+A_r(\ell_r+\xi_r)]\rho_b = M_1 \qquad (1)$$

and for the upper fluid by

$$[A_\ell(\ell+\xi_\ell)+A_r(\ell_r-\xi_r)]\rho_a = M_2 \qquad (2)$$



With the assumption that the velocity has a uniform profile over a given cross-section, it follows that

$$v_\ell = \frac{A_r}{A_\ell} v_r \qquad (3)$$

while evaluation of Eqs. 1 and 2 gives

$$\xi_\ell = \frac{A_r}{A_\ell}\xi_r - \frac{M_1}{\rho_b A_\ell} + \frac{A_r}{A_\ell}\ell_r + \ell_\ell \qquad (4)$$

$$\xi_\ell = \frac{A_r}{A_\ell}\xi_r + \frac{M_2}{\rho_a A_\ell} - \frac{A_r}{A_\ell}\ell_r - \ell_\ell \qquad (5)$$

Bernoulli's equation joining points (2) and (4) through the homogeneous fluid below gives

$$p_2 - \rho_b g\xi_\ell + \frac{1}{2}\rho_b\left(\frac{d\xi_\ell}{dt}\right)^2 - \rho_b(\ell_\ell-\xi_\ell)\frac{d^2\xi_\ell}{dt^2} = p_4 + \rho_b g\xi_r + \frac{1}{2}\rho_b\left(\frac{d\xi_r}{dt}\right)^2 + \rho_b(\ell_r+\xi_r)\frac{d^2\xi_r}{dt^2} \qquad (6)$$

where the approximation made in integrating the inertial term through the transition region should be recognized. Similarly, in the upper fluid,

$$p_3 + \rho_a g\xi_r + \frac{1}{2}\rho_a\left(\frac{d\xi_r}{dt}\right)^2 - \rho_a(\ell_r-\xi_r)\frac{d^2\xi_r}{dt^2} = p_1 - \rho_a g\xi_\ell + \frac{1}{2}\rho_a\left(\frac{d\xi_\ell}{dt}\right)^2 + \rho_a(\ell_\ell+\xi_\ell)\frac{d^2\xi_\ell}{dt^2} \qquad (7)$$

These expressions are linked together at the interfaces by the stress-balance and continuity boundary conditions.

$$p_1 = p_2\,,\quad p_3 = p_4\,,\quad v_3 = v_4\,,\quad v_1 = v_2 \qquad (8)$$

Thus, subtraction of Eqs. 6 and 7 gives

$$\begin{aligned}&[\rho_b(\xi_r+\ell_r)-\rho_a(\ell_r-\xi_r)]\frac{d^2\xi_r}{dt^2} + \frac{1}{2}(\rho_b-\rho_a)\left(\frac{d\xi_r}{dt}\right)^2 + (\rho_b-\rho_a)g\xi_r \\ &= [-\rho_b(\ell_\ell-\xi_\ell)-\rho_a(\ell_\ell+\xi_\ell)]\frac{d^2\xi_\ell}{dt^2} + \frac{1}{2}(\rho_b-\rho_a)\left(\frac{d\xi_\ell}{dt}\right)^2 - (\rho_b-\rho_a)g\xi_\ell\end{aligned} \qquad (9)$$

Prob. 7.8.1(cont.)

Provided that the lengths $\ell_r \gg \xi_r$ and $\ell_\ell \gg \xi_\ell$, the equation of motion therefore takes the form

$$m\frac{d^2\xi_r}{dt^2} + \frac{1}{2}(\rho_b - \rho_a)\left[\left(\frac{d\xi_r}{dt}\right)^2 - \left(\frac{d\xi_\ell}{dt}\right)^2\right] + K\xi_r = 0 \tag{10}$$

where

$$m \equiv \frac{A_r}{A_\ell}\left[\rho_b(\ell_\ell - \xi_\ell) + \rho_a(\ell_\ell + \xi_\ell)\right] + \rho_b(\ell_r + \xi_r) + \rho_a(\ell_r - \xi_r);\ K \equiv g\left(1 + \frac{A_r}{A_\ell}\right)(\rho_b - \rho_a)$$

For still smaller amplitude motions, this expression becomes

$$\left(\frac{A_r}{A_\ell}\ell_\ell + \ell_r\right)(\rho_b + \rho_a)\frac{d^2\xi_r}{dt^2} + g\left(1 + \frac{A_r}{A_\ell}\right)(\rho_b - \rho_a)\xi_r = 0 \tag{11}$$

Thus, the system is stable if $\rho_b > \rho_a$ and given this condition, the natural frequencies are

$$\omega = \left[\frac{g(\rho_b - \rho_a)\left(1 + \frac{A_r}{A_\ell}\right)}{(\rho_b + \rho_a)\left(\frac{A_r}{A_\ell}\ell_\ell + \ell_r\right)}\right]^{1/2} \tag{12}$$

To account for the geometry, this expression obscures the simplicity of what it represents. For example, if the tube is of uniform cross-section, the lower fluid is water and the upper one air, $\rho_b \gg \rho_a$ and the natural frequency is independent of mass density (for the same reason that that of a rigid body pendulum is independent of mass, both the kinetic and potential energies are proportional to the density.) Thus, if $\ell = 1\text{m}$, the frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2g}{\ell_\ell + \ell_r}} = \frac{1}{2\pi}\sqrt{\frac{2g}{\ell}} = 0.7\ \text{Hz}$$

Prob. 7.8.2 The problem is similar to the electrical conduction problem of current flow about an insulating cavity obstructing a uniform flow. Guess that the solution is the superposition of one consistent with the uniform flow at infinity and a dipole field to account for the boundary at r=R.

$$\Theta = -Ur\cos\theta + \frac{B}{r^2}\cos\theta \tag{1}$$

Because $v_r=0$ at r=R, $B=-R^3U/2$ and it follows that

$$\Theta = -Ur\cos\theta - \frac{R^3U}{2r}\cos\theta \tag{2}$$

$$v_\theta = -\frac{3}{2}U\sin\theta \tag{3}$$

Because the air is stagnant inside the shell, the pressure there is $P_{in} = P_2 - \rho g h$. At the stagnation point where the air encounters the shell and the hole communicates the interior pressure to the outside, the application of Bernoulli's equation gives

$$\frac{1}{2}\rho v_\theta^2 + \rho g h + p = P_2 \tag{4}$$

where h measures the height from the "ground" plane. In view of Eq. 3 and evaluated in spherical coordinates, this expression becomes

$$p - P_{in} = -\frac{1}{2}\rho v_\theta^2 = -\frac{9}{8}\rho U^2 \sin^2\theta \tag{5}$$

(5)

To find the force tending to lift the shell off the "ground", compute

$$f_x = -\int_S p n_x da = -\int_{-\pi/2}^{\pi/2}\int_0^{\pi}(p - P_{in})n_x R^2 \sin\theta \, d\theta \, d\phi \tag{6}$$

Because $n_x = \sin\theta$, this expression gives

$$f_x = -R^2 2\int_0^{\pi} -\frac{9}{8}\rho U^2 \sin^4\theta \, d\theta \tag{7}$$

so that the force is

$$f_x = \rho\pi R^2\left(\frac{27}{64}\right)U^2 \tag{8}$$

Prob. 7.8.3 First, use Eq. 7.8.5 to relate the pressure in the essentially static interior region to the velocity in the cross-section A.

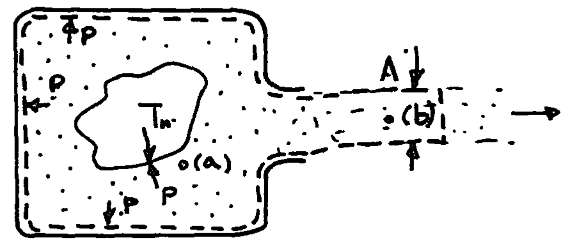
$$p_a + \frac{1}{2}\rho v_a^2 = p_b + \frac{1}{2}\rho v_b^2 \Rightarrow T_n + 0 = 0 + \frac{1}{2}\rho U^2 \tag{1}$$

Second, use the pressure from Sec. 7.4 to write the integral momentum conservation statement of Eq. 7.3.2 as

$$f_x = \oint_S p\, n_x da \quad = -\oint_S \rho v_x \bar{v}\cdot\bar{n}\, da = -\rho A U^2 \tag{2}$$

Applied to the surface shown in the figure, this equation becomes

$$f_x = -AU^2\rho \tag{3}$$



The combination of Eqs. 1 and 3 eliminates U as an unknown and gives the required result.

Prob. 7.9.1 See 8.17 for treatment of more general situation which becomes this one in the limit of no volume charge density.

Prob. 7.9.2 (a) By definition, given that the equilibrium velocity is $\bar{v} = \Omega r \bar{i}_\theta$, the vorticity follows as

$$\bar{\omega} = \nabla \times \bar{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\bar{i}_z = 2\Omega \bar{i}_z \tag{1}$$

(b) The equilibrium pressure follows from the radial component of the force equation

$$\rho(\bar{v}\cdot\nabla\bar{v})_r + \nabla p = 0 \Rightarrow -\rho\Omega^2 r + \frac{\partial p}{\partial r} = 0 \tag{2}$$

Integration gives

$$p = p_o + \frac{1}{2}\rho\Omega^2 r^2 \tag{3}$$

(c) With the laboratory frame of reference given the primed variables, the appropriate equations are

$$\nabla' \cdot \bar{v}' = 0 \tag{4}$$

$$\rho\left(\frac{\partial \bar{v}'}{\partial t'} + \bar{v}'\cdot\nabla\bar{v}'\right) + \nabla' p' = 0 \tag{5}$$

With the recognition that p' and v_θ' have equilibrium parts, these are first linearized to obtain

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r' v_r') + \frac{1}{r'}\frac{\partial v_\theta'}{\partial \theta'} + \frac{\partial v_z'}{\partial z'} = 0 \tag{6}$$

$$\rho\left(\frac{\partial v_r'}{\partial t'} + \Omega\frac{\partial v_r'}{\partial \theta'} - 2\Omega v_\theta'\right) + \frac{\partial p'}{\partial r'} = 0 \tag{7}$$

$$\rho\left(\frac{\partial v_\theta'}{\partial t'} + \Omega\frac{\partial v_\theta'}{\partial \theta'} + 2\Omega v_r'\right) + \frac{1}{r}\frac{\partial p'}{\partial \theta'} = 0 \tag{8}$$

$$\rho\frac{\partial v_z'}{\partial t'} + \frac{\partial p'}{\partial z'} = 0 \tag{9}$$

The transformation of the derivatives is facilitated by the diagram of the dependences of the independent variables given to the right. Thus

$$\begin{tikzcd} r \arrow[r] & r' \\ \theta \arrow[r] \arrow[rdd] & \theta' \\ z \arrow[r] & z' \\ t \arrow[r] & t' \end{tikzcd}$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial t'} = \frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial r'} = \frac{\partial}{\partial r}, \text{ etc.} \tag{10}$$

Because the variables in Eqs. 6-9 are already linearized, the perturbation

Prob. 7.9.2 (cont.)

part of the azimuthal velocity in the laboratory frame is the same as that in the rotating frame. Thus

$$v_r' = v_r\,,\; v_\theta' \equiv \Omega r + v_\theta'\Big|_{part} = \Omega r + v_\theta\,,\; v_z' = v_z\,,\; p' = p \tag{10}$$

Expressed in the rotating frame of reference, Eqs. 6-9 become

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{11}$$

$$\rho\left(\frac{\partial v_r}{\partial t} - 2\Omega v_\theta\right) + \frac{\partial p}{\partial r} = 0 \tag{12}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + 2\Omega v_r\right) + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \tag{13}$$

$$\rho\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z} = 0 \tag{14}$$

(d) In the *rotating frame* of reference, it is now assumed that variables take the complex amplitude form

$$\begin{bmatrix} \bar{v} \\ p \end{bmatrix} = \mathrm{Re}\begin{bmatrix} \hat{v} \\ \hat{p} \end{bmatrix} e^{j(\omega t - m\theta - kz)} \tag{15}$$

Then, it follows from Eqs. 22-24 that

$$\hat{v}_r = \frac{1}{\rho}\,\frac{\frac{2jm}{r}\Omega\hat{p} - j\omega\frac{d\hat{p}}{dr}}{(2\Omega)^2 - \omega^2} \tag{16}$$

$$\hat{v}_\theta = -\frac{1}{\rho}\,\frac{\frac{m\omega}{r}\hat{p} - 2\Omega\frac{d\hat{p}}{dr}}{(2\Omega)^2 - \omega^2} \tag{17}$$

$$\hat{v}_z = \frac{k}{\omega\rho}\hat{p} \tag{18}$$

Substitution of these expressions into the continuity equation, Eq. 11, then gives the desired expression for the complex pressure.

$$r^2\frac{d^2\hat{p}}{dr^2} + r\frac{d\hat{p}}{dr} - \hat{p}\left(m^2 + \gamma^2 r^2\right) = 0 \tag{19}$$

where

Prob. 7.9.2 (cont.)

$$\gamma^2 \equiv k^2\left[1 - \frac{(2\Omega)^2}{\omega^2}\right]$$

(e) With the replacement $k^2 \to \gamma^2$, Eq. 19 is the same expression for $\hat{p}$ in cylindrical coordinates as in Sec. 2.16. Either by inspection or by using Eq. 2.16.25, it follows that

$$\hat{p} = \hat{p}^\alpha \frac{H_m(j\gamma\beta)J_m(j\gamma r) - J_m(j\gamma\beta)H_m(j\gamma r)}{H_m(j\gamma\beta)J_m(j\gamma\alpha) - J_m(j\gamma\beta)H_m(j\gamma\alpha)}$$

$$+\hat{p}^\beta \frac{J_m(j\gamma\alpha)H_m(j\gamma r) - H_m(j\gamma\alpha)J_m(j\gamma r)}{J_m(j\gamma\alpha)H_m(j\gamma\beta) - H_m(j\gamma\alpha)J_m(j\gamma\beta)} \tag{20}$$

From Eq. 16, first evaluated using this expression and then evaluated at $r = \alpha$ and $r = \beta$ respectively, it follows that

$$\begin{bmatrix} \hat{v}_r^\alpha \\ \hat{v}_r^\beta \end{bmatrix} = \frac{j\omega}{\rho(4\Omega^2 - \omega^2)} \begin{bmatrix} f_m(\beta,\alpha,\gamma) + \frac{2\Omega m}{\omega\alpha} & g_m(\alpha,\beta,\gamma) \\ g_m(\beta,\alpha,\gamma) & f_m(\alpha,\beta,\gamma) + \frac{2m\Omega}{\beta\omega} \end{bmatrix} \begin{bmatrix} \hat{p}^\alpha \\ \hat{p}^\beta \end{bmatrix} \tag{21}$$

The inverse of this is the desired transfer relation.

$$\begin{bmatrix} \hat{p}^\alpha \\ \hat{p}^\beta \end{bmatrix} = \frac{\rho(4\Omega^2 - \omega^2)}{j\omega D} \begin{bmatrix} f_m(\alpha,\beta,\gamma) + \frac{2m\Omega}{\beta\omega} & -g_m(\alpha,\beta,\gamma) \\ -g_m(\beta,\alpha,\gamma) & f_m(\beta,\alpha,\gamma) + \frac{2\Omega m}{\omega\alpha} \end{bmatrix} \begin{bmatrix} \hat{v}^\alpha \\ \hat{v}^\beta \end{bmatrix} \tag{22}$$

where

$$D \equiv \left[f_m(\beta,\alpha,\gamma) + \frac{2\Omega m}{\omega\alpha}\right]\left[f_m(\alpha,\beta,\gamma) + \frac{2m\Omega}{\beta\omega}\right] - g_m(\beta,\alpha,\gamma)g_m(\alpha,\beta,\gamma)$$

<u>Prob. 7.11.1</u> For a weakly compressible gas without external force densities, the equations of motion are Eqs. 7.1.3, 7.4.4 (with $\vec{F}_{ex}=0$) and Eq. 7.10.3.

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{v} = 0 \tag{1}$$

$$\rho\left[\frac{\partial\bar{v}}{\partial t} + \bar{v}\cdot\nabla\bar{v}\right] + \nabla p = 0 \tag{2}$$

$$\rho = \rho_o + (p - p_r)/a^2 \tag{3}$$

where ρ_o, a^2 and p_r are constants determined by the static equilibrium. With primes used to indicate perturbations from this equilibrium, the linearized forms of these expressions are

$$\frac{1}{a^2}\frac{\partial p'}{\partial t} + \rho_o\nabla\cdot\bar{v}' = 0 \tag{4}$$

$$\rho_o\frac{\partial\bar{v}'}{\partial t} + \nabla p' = 0 \tag{5}$$

where Eqs. 1 and 3 have been combined.

The divergence of Eq. 5 combines with the time derivative of Eq. 4 to eliminate $\nabla\cdot\bar{v}'$ and give an expression for p' alone.

$$\frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} = \nabla^2 p' \tag{6}$$

For solutions of the form $p = \mathrm{Re}\,\hat{p}(r)P_n^m(\cos\theta)e^{j(\omega t - m\phi)}$, Eq. 6 reduces to (See Eqs. 2.16.30-2.16.34)

$$P_n^m\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\hat{p}}{dr}\right) + \frac{\hat{p}}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n^m}{d\theta}\right) - \frac{m^2P_n^m\hat{p}}{r^2\sin^2\theta} + \frac{\omega^2}{a^2}P_n^m\hat{p} = 0 \tag{7}$$

In view of Eq. 2.16.31, the second and third terms are $-n(n+1)P_n^m\hat{p}$ so that this expression reduces to

$$r^2\frac{d^2\hat{p}}{dr^2} + 2r\frac{d\hat{p}}{dr} + \left[\frac{\omega^2r^2}{a^2} - n(n+1)\right]\hat{p} = 0 \tag{8}$$

Given the solutions to this expression, it follows from Eq. 5 that

$$\hat{v}_r = \frac{j}{\omega\rho_o}\frac{d\hat{p}}{dr} \tag{9}$$

provides the velocity components.

Substitution into Eq. 8 shows that with $u \equiv \frac{\omega r}{a}$, solutions to Eq. 8 are

$$j_n(u) \equiv \sqrt{\frac{\pi}{2u}}\,J_{n+\frac{1}{2}}(u) \quad ; \quad h_n(u) \equiv \sqrt{\frac{\pi}{2u}}\,H_{n+\frac{1}{2}}(u)$$

(j_n and h_n are spherical Bessel functions of first and third kind. See Abramowitz, M. and Stegun, I.A., Handbook of Mathematical Functions, National

Prob. 7.11.1 (cont.)

Bureau of Standards, 1964, p437.) As is clear from its definition, $h_n(u)$ is singular as $u \to 0$.

The appropriate linear combination of these solutions can be written by inspection as

$$\hat{p} = \hat{p}^{\alpha} \frac{\left[\dfrac{j_n\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega \beta}{a}\right)} - \dfrac{h_n\left(\frac{\omega r}{a}\right)}{h_n\left(\frac{\omega \beta}{a}\right)}\right]}{\left[\dfrac{j_n\left(\frac{\omega \alpha}{a}\right)}{j_n\left(\frac{\omega \beta}{a}\right)} - \dfrac{h_n\left(\frac{\omega \alpha}{a}\right)}{h_n\left(\frac{\omega \beta}{a}\right)}\right]} + \hat{p}^{\beta} \frac{\left[\dfrac{j_n\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega \alpha}{a}\right)} - \dfrac{h_n\left(\frac{\omega r}{a}\right)}{h_n\left(\frac{\omega \alpha}{a}\right)}\right]}{\left[\dfrac{j_n\left(\frac{\omega \beta}{a}\right)}{j_n\left(\frac{\omega \alpha}{a}\right)} - \dfrac{h_n\left(\frac{\omega \beta}{a}\right)}{h_n\left(\frac{\omega \alpha}{a}\right)}\right]} \qquad (10)$$

Thus, from Eq. 9 it follows that

$$\hat{v}_r = \frac{j}{\omega \rho_o} \left\{ \frac{\omega}{a} \frac{j_n'\left(\frac{\omega r}{a}\right) h_n\left(\frac{\omega \beta}{a}\right) - h_n'\left(\frac{\omega r}{a}\right) j_n\left(\frac{\omega \beta}{a}\right)}{j_n\left(\frac{\omega \alpha}{a}\right) h_n\left(\frac{\omega \beta}{a}\right) - h_n\left(\frac{\omega \alpha}{a}\right) j_n\left(\frac{\omega \beta}{a}\right)} \hat{p}^{\alpha} - \frac{\omega}{a} \frac{j_n'\left(\frac{\omega r}{a}\right) h_n\left(\frac{\omega \alpha}{a}\right) - h_n'\left(\frac{\omega r}{a}\right) j_n\left(\frac{\omega \alpha}{a}\right)}{h_n\left(\frac{\omega \beta}{a}\right) j_n\left(\frac{\omega \alpha}{a}\right) - j_n\left(\frac{\omega \beta}{a}\right) h_n\left(\frac{\omega \alpha}{a}\right)} \hat{p}^{\beta} \right\} \qquad (11)$$

where j_n' and h_n' signify derivatives with respect to the arguments. Evaluation of Eq. 11 at the respective boundaries gives transfer relations

$$\begin{bmatrix} \hat{v}_r^{\alpha} \\ \hat{v}_r^{\beta} \end{bmatrix} = \frac{j}{\omega \rho_o} \begin{bmatrix} f_n(\beta, \alpha) & g_n(\alpha, \beta) \\ g_n(\beta, \alpha) & f_n(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} \qquad (12)$$

where

$$f_n(x,y) \equiv -\frac{\omega}{a} \frac{h_n\left(\frac{\omega}{a}x\right) j_n'\left(\frac{\omega}{a}y\right) - j_n\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega y}{a}\right)}{j_n\left(\frac{\omega}{a}x\right) h_n\left(\frac{\omega}{a}y\right) - j_n\left(\frac{\omega y}{a}\right) h_n\left(\frac{\omega}{a}x\right)}$$

$$g_n(x,y) \equiv -\frac{\omega}{a} \frac{h_n\left(\frac{\omega}{a}x\right) j_n'\left(\frac{\omega}{a}x\right) - j_n\left(\frac{\omega}{a}x\right) h_n'\left(\frac{\omega}{a}x\right)}{j_n\left(\frac{\omega}{a}x\right) h_n\left(\frac{\omega}{a}y\right) - j_n\left(\frac{\omega}{a}y\right) h_n\left(\frac{\omega}{a}x\right)}$$

Prob. 7.11.1 (cont.)

Inversion of these relations gives

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = -j\omega\rho_o \frac{\begin{bmatrix} f_n(\alpha,\beta) & -g_n(\alpha,\beta) \\ -g_n(\beta,\alpha) & f_n(\beta,\alpha) \end{bmatrix}}{f_n(\beta,\alpha)f_n(\alpha,\beta)-g_n(\beta,\alpha)g_n(\alpha,\beta)} \begin{bmatrix} \hat{v}_r^{\alpha} \\ \hat{v}_r^{\beta} \end{bmatrix} \tag{13}$$

and this expression becomes

$$\begin{bmatrix} \hat{p}^{\alpha} \\ \hat{p}^{\beta} \end{bmatrix} = -j\omega\rho_o \begin{bmatrix} F_n(\beta,\alpha) & G_n(\alpha,\beta) \\ G_n(\beta,\alpha) & F_n(\alpha,\beta) \end{bmatrix} \begin{bmatrix} \hat{v}_r^{\alpha} \\ \hat{v}_r^{\beta} \end{bmatrix} \tag{14}$$

where

$$F_n(x,y) \equiv \frac{a}{\omega} \frac{h_n\left(\frac{\omega y}{a}\right) j_n'\left(\frac{\omega x}{a}\right) - j_n\left(\frac{\omega y}{a}\right) h_n'\left(\frac{\omega x}{a}\right)}{j_n'\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega y}{a}\right) - j_n'\left(\frac{\omega y}{a}\right) h_n'\left(\frac{\omega x}{a}\right)}$$

$$G_n(x,y) \equiv -\frac{a}{\omega} \frac{h_n\left(\frac{\omega x}{a}\right) j_n'\left(\frac{\omega x}{a}\right) - j_n\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega x}{a}\right)}{h_n'\left(\frac{\omega x}{a}\right) j_n'\left(\frac{\omega y}{a}\right) - j_n'\left(\frac{\omega x}{a}\right) h_n'\left(\frac{\omega y}{a}\right)}$$

With a rigid wall at r=R it follows from Eq. 14 that there can then only be a response if

$$F_n(0,R) = \frac{a}{\omega} \frac{j_n\left(\frac{\omega R}{a}\right)}{j_n'\left(\frac{\omega R}{a}\right)} \longrightarrow \infty \tag{15}$$

so that the desired eigenvalue equation is

$$j_n'\left(\frac{\omega R}{a}\right) = 0 \tag{16}$$

This is easy to see without the transfer relations because in this case Eq. 10 is replaced by simply

$$\hat{p} = \hat{p}^{\alpha} \frac{j_n\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega R}{a}\right)} \tag{17}$$

so that it follows from Eq. 9 that

$$\hat{v}_r = \frac{j}{\omega\rho_o} \hat{p}^{\alpha} \left(\frac{\omega}{a}\right) \frac{j_n'\left(\frac{\omega r}{a}\right)}{j_n\left(\frac{\omega R}{a}\right)}$$

For $\hat{p}^{\alpha}$ to be finite at r=R but $\hat{v}_r^{\alpha}=0$ there, Eq. 16 must hold. Roots to this expression are tabulated (Abromowitz and Stegun, p468).

Prob. 7.12.1 It follows from Eq. (f) of Table 7.9.1 in the limit $\beta \to 0$ that

$$\hat{p}^{\alpha} = j(\omega - kU)\rho F_m(0,R)\hat{v}_r^{\alpha} \tag{1}$$

where

$$F_m(0,R) \to \frac{J_m(j\gamma R)}{j\gamma R\, J_m'(j\gamma R)} \tag{2}$$

It follows that there can be a finite pressure response at the wall even if there is no velocity there if

$$\begin{aligned} \gamma R = 0 \quad &\Rightarrow \quad \omega - kU = \pm a k \quad (n=0) \\ J_m'(j\gamma R) = 0 \quad &\Rightarrow \quad j\gamma R = \alpha_n \,, \quad n \neq 0, \pm 1, \pm 2 \cdots \end{aligned} \tag{3}$$

The zero mode is the principal mode (propagation down to zero frequency)

$$k = \frac{\omega}{U \pm a} \tag{4}$$

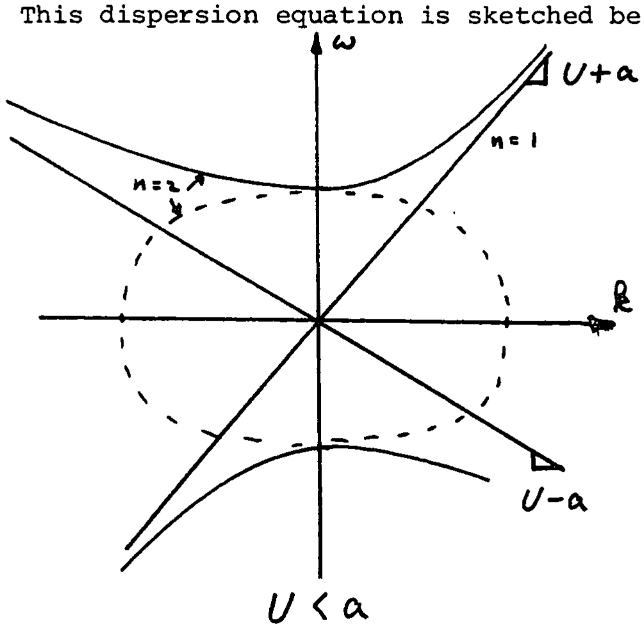
while the higher order modes have a dispersion equation that follows from the roots of Eq. 3b and the definition of γ .

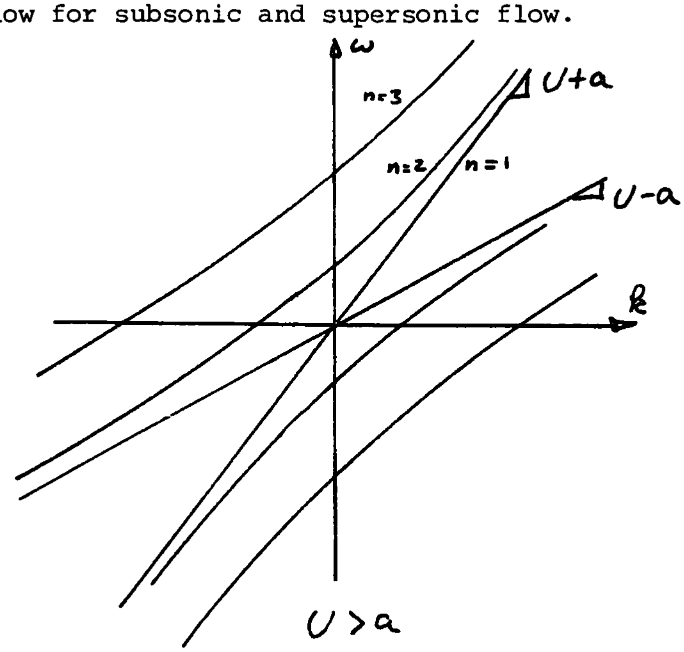
$$-\frac{a^2 \alpha_n^2}{R^2} = a^2 k^2 - (\omega - kU)^2 \tag{5}$$

Solution of k gives the wavenumbers of the spatial modes

$$k = \frac{-\omega U \pm \sqrt{a^2\omega^2 - (a^2 - U^2)a^2\frac{\alpha_n^2}{R^2}}}{(a^2 - U^2)} \tag{6}$$

This dispersion equation is sketched below for subsonic and supersonic flow.





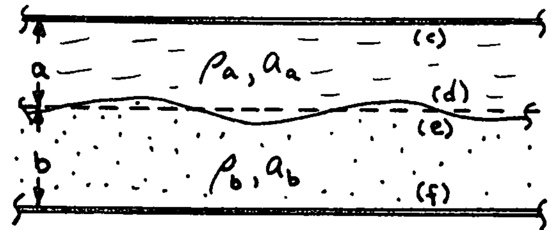
Prob. 7.12.2 Boundary conditions are

$$\hat{v}_x^c = 0 \tag{1}$$

$$\hat{v}_x^d = \hat{v}_x^e \tag{2}$$

$$\hat{p}^d = \hat{p}^e \tag{3}$$

$$\hat{v}_x^f = 0 \tag{4}$$



With these conditions incorporated from the outset, the transfer relations (Eqs. (c) of Table 7.9.1) for the respective regions are

$$\begin{bmatrix} \hat{p}^c \\ \hat{p}^d \end{bmatrix} = \frac{j\omega\rho_a}{\gamma_a}\begin{bmatrix} -\coth\gamma_a a & \dfrac{1}{\sinh\gamma_a a} \\ \dfrac{-1}{\sinh\gamma_a a} & \coth\gamma_a a \end{bmatrix}\begin{bmatrix} 0 \\ \hat{v}_x^d \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \hat{p}^d \\ \hat{p}^f \end{bmatrix} = \frac{j\omega\rho_b}{\gamma_b}\begin{bmatrix} -\coth\gamma_b b & \dfrac{1}{\sinh\gamma_b b} \\ \dfrac{-1}{\sinh\gamma_b b} & \coth\gamma_b b \end{bmatrix}\begin{bmatrix} \hat{v}_x^d \\ 0 \end{bmatrix} \tag{6}$$

where $\gamma_a^2 = k^2 - \omega^2/a_a^2$ and $\gamma_b^2 = k^2 - \omega^2/a_b^2$. By equating Eqs. 5b and 6a it follows that

$$j\frac{\omega\rho_a}{\gamma_a}\coth\gamma_a a = -j\frac{\omega\rho_b}{\gamma_b}\coth\gamma_b b \tag{7}$$

With the definitions of γ_a and γ_b, this expression is the desired dispersion equation relating ω and k. Given a real ω , the wavenumbers of the spatial modes are in general complex numbers satisfying the complex equation, Eq. 7. For long waves, a principal mode propagates through the system with a phase velocity that combines those of the two regions. That is, for $|\gamma_a a| \ll 1$ and $|\gamma_b b| \ll 1$ Eq. 7 becomes

$$\frac{\rho_a}{\gamma_a^2 a} = -\frac{\rho_b}{\gamma_b^2 b} \Rightarrow \frac{a}{\rho_a}\left[k^2 - \left(\frac{\omega}{a_a}\right)^2\right] = -\frac{b}{\rho_b}\left[k^2 - \left(\frac{\omega}{a_b}\right)^2\right] \tag{8}$$

and it follows that

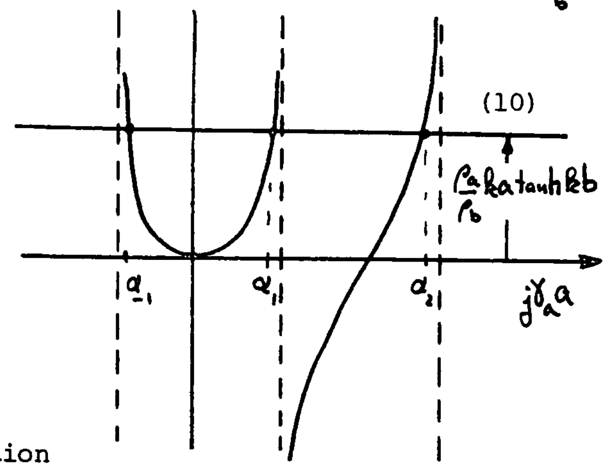
$$k = \pm\frac{\omega}{a_c} \ ; \ a_c \equiv \sqrt{\frac{\left[\dfrac{a}{\rho_a} + \dfrac{b}{\rho_b}\right]}{\left[\dfrac{a}{\rho_a a_a^2} + \dfrac{b}{\rho_b a_b^2}\right]}} \tag{9}$$

Prob. 7.12.2(cont.)

A second limit is of interest for propagation of acoustic waves in a gas over a liquid. The liquid behaves in a quasi-static fashion for the lowest order modes because on time scales of interest waves propagate through the liquid essentially instantaneously. Thus, the liquid acts as a massive load comprising one wall of a guide for the waves in the air. In this limit, $a_a \ll a_b$ and $k^2 \gg \frac{\omega^2}{a_b^2}$

$\Rightarrow \gamma_b \approx k$ and Eq. 7 becomes

$$\frac{\rho_b}{\rho_a(ka)} \coth kb = -\frac{\coth \gamma_a a}{\gamma_a a} \qquad (10)$$



This expression can be solved graphically, as illustrated in the figure, because Eq. 10 can be written so as to make evident real roots.

$$\frac{\rho_a}{\rho_b} ka \tanh kb = (j\gamma_a a)\tan(j\gamma_a a) \qquad (11)$$

Given these roots, it follows from the definition of γ_a that the wavenumbers of the associated spatial modes are

$$k = \pm\left(\frac{\omega^2}{a_a^2} - \frac{\alpha_n^2}{a^2}\right)^{1/2} \qquad (11)$$

<u>Prob. 7.13.1</u> The objective here is to establish some rapport for the elastic solid. Whether subjected to shear or normal stresses, it can deform in such a way as to balance these stresses with no further displacements. Thus, it is natural to expect stresses to be related to displacements rather than velocities. (Actually strains rather than strain-rates.) That a linearized description does not differentiate between $\bar{\xi}(\bar{r}_0,t)$ interpreted as the displacement of the particle that is at $\bar{r}_0$ or was at $\bar{r}_0$ (and is now at $\bar{r}_0+\bar{\xi}(\bar{r}_0,t)$) can be seen by simply making a Taylor's expansion.

$$\xi_i(\bar{r}_0+\bar{\xi},t) = \xi_i(\bar{r}_0) + \left.\frac{\partial \xi_i}{\partial x_j}\right|_{\bar{r}_0} \xi_j + \cdots \tag{1}$$

Terms that are quadradic or more in the components of $\bar{\xi}$ are negligible

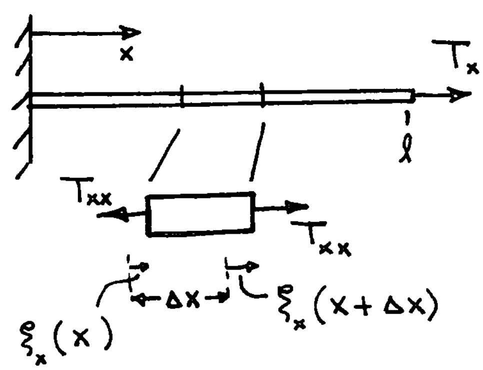
Because the measured result is observed for various spacings, d, the suggestion is that an incremental slice of the material, shown analogously in Fig. 7.13.1, can be described by

$$T_{zx} = G_s\left[\frac{\xi_z(x+\Delta x) - \xi_z(x)}{\Delta x}\right] \tag{2}$$

In the limit $\Delta x \to 0$, the one-dimensional shear-stress displacement relation follows

$$T_{zx} = G_s \frac{\partial \xi_z}{\partial x} \tag{3}$$

For dilatational motions, it is helpful to discern what can be expected by considering the one-dimensional extension of the thin rod shown in the sketch. That the measured result holds independent of the initial length, ℓ, suggests that the relation should hold for a section of length Δx as well. Thus,

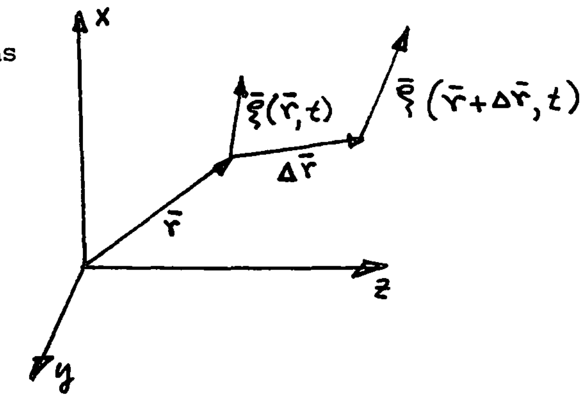


$$T_{xx} = E_s\left[\frac{\xi_x(x+\Delta x) - \xi_x(x)}{\Delta x}\right] \tag{4}$$

In the limit $\Delta x \to 0$, the stress- displacement relation for a <u>thin</u> rod follows.

$$T_{xx} = E_s \frac{\partial \xi_x}{\partial x} \tag{5}$$

Prob. 7.14.1 Consider the relative deformations of material having the initial relative displacement $\Delta\bar{r}$, as shown in the sketch.



Taylor's expansion gives

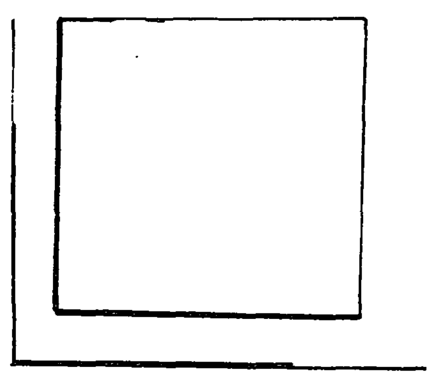
$$\xi_i(\bar{r}+\Delta\bar{r})-\xi_i(\bar{r})=\xi_i(\bar{r})+\frac{\partial\xi_i}{\partial x_j}\Delta x_j-\xi_i(\bar{r}) \quad (1)$$

Terms are grouped so as to identify the rotational part of the deformation and exclude it from the definition of the strain.

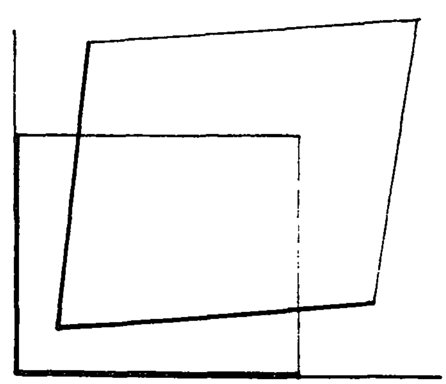
$$\xi_i(\bar{r}+\Delta\bar{r})-\xi_i(\bar{r})=\frac{1}{2}\left[\frac{\partial\xi_i}{\partial x_j}-\frac{\partial\xi_j}{\partial x_i}\right]\Delta x_j + e_{ij}\Delta x_j\,;\ e_{ij}\equiv\frac{1}{2}\left[\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_j}{\partial x_i}\right] \quad (2)$$

Thus, the strain is defined as describing that part of the deformation that can be expected to be directly related to the local stress.

• The sketches below respectively show the change in shape of a rectangle attached to the material as it suffers pure dilatational and shear deformations.



normal strain



shear strain

Prob. 7.15.1 Arguments follow those given, with $\overset{o}{e}_{ij} \rightarrow e_{ij}$. To make Eq. 6.5.17 become Eq. (b) of the table, it is clear that

$$k_1 - k_2 = 2G_r \,;\; k_2 = \lambda_s \tag{1}$$

The new coefficient is related to G and E by considering the thin rod experiment. Because the transverse stress components were zero, the normal component of stress and strain are related by

$$\begin{bmatrix} T_{xx} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_2 \\ k_2 & k_1 & k_2 \\ k_2 & k_2 & k_1 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix} \tag{2}$$

Given T_{xx}, the longitudinal and transverse strain components are determined from these three equations. Solution for e_{xx} gives

$$e_{xx} = \frac{(k_1 + k_2)}{k_1(k_1 + k_2) - 2k_2^2} T_{xx} \tag{3}$$

and comparison of this expression to that for the thin rod shows that

$$E_r = \frac{(2G_r + \lambda_s)(G_r + \lambda_s) - \lambda_s^2}{G_r + \lambda_s} \tag{4}$$

Solution of this expression for λ_s gives Eq. (f) of the table.

It also follows from Eqs. 2 that

$$-\frac{e_{yy}}{e_{xx}} = -\frac{(k_2^2 - k_1 k_2)}{(k_1 + k_2)(k_1 - k_2)} = \frac{k_2}{k_1 + k_2} \tag{5}$$

With k_1 and k_2 expressed using Eqs. 1 and then the expression for λ_s in terms of G and E, Eq. g of the table follows.

Prob. 7.15.2 In general

$$e_{ij} = a_{ik} a_{jl} e_{kl} \tag{1}$$

In particular, the sum of the diagonal elements in the primed frame is

$$e'_{nn} = a_{nk} a_{nl} e_{kl} \tag{2}$$

It follows from Eq. 3.9.14 and the definition of a_{ij} that $a_{ki}a_{kj} = \delta_{ij}$. Thus, Eq. 2 becomes the statement to be proven

$$e'_{nn} = \delta_{kl} e_{kl} = e_{nn} \tag{3}$$

Prob. 7.15.3 From Eq. 7.15.20 it follows that

$$S_{ij} = \begin{bmatrix} p & 0 & \frac{\eta U}{d} \\ 0 & p & 0 \\ \frac{\eta U}{d} & 0 & p \end{bmatrix} \tag{1}$$

Thus, Eq. 7.15.5 becomes

$$\begin{bmatrix} p-T & 0 & \frac{\eta U}{d} \\ 0 & p-T & 0 \\ \frac{\eta U}{d} & 0 & p-T \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \tag{2}$$

which reduces to

$$-(p-T)^3 + \left(\frac{\eta U}{d}\right)^2 (p-T) = 0 \tag{3}$$

Thus, the principal stresses are

$$T = p \;, \; T = p \pm \sqrt{\eta U/d} \tag{4}$$

From Eq. 7.15.5c it follows that

$$n_1 = \pm n_3 \tag{5}$$

so that the normal vectors to the two nontrivial principal planes are

$$\bar{n} = \frac{1}{\sqrt{2}}(\bar{i}_x \pm \bar{i}_z) \tag{6}$$

Prob. 7.16.1 Equation d of the table states Newton's law for incremental motions. Substitution of Eq. b for T_{ij} and of Eq. a for e_{ij} gives

$$\frac{\partial T_{ij}}{\partial x_j} = G_s \frac{\partial}{\partial x_j}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) + \lambda_s \frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) \tag{1}$$

Manipulations are now made with the vector identity

$$\nabla \times \nabla \times \bar{\xi} = \nabla(\nabla\cdot\bar{\xi}) - \nabla^2 \bar{\xi} \tag{2}$$

in mind. In view of the desired form of the equation of motion, Eq. 1 is written as

$$\frac{\partial T_{ij}}{\partial x_j} = (2G_s+\lambda)\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) - 2G_s\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) + G_s\frac{\partial}{\partial x_j}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \tag{3}$$

Half of the second term cancels with the last, so that the expression becomes

$$\frac{\partial T_{ij}}{\partial x_j} = (2G_s+\lambda)\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) - G_s\left[\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_k}{\partial x_k}\right) - \frac{\partial^2 \xi_i}{\partial x_j \partial x_j}\right] \tag{4}$$

In vector form, this is equivalent to

$$\nabla\cdot\bar{\bar{T}} = (2G_s+\lambda_s)\nabla(\nabla\cdot\bar{\xi}) - G_s\left[\nabla(\nabla\cdot\bar{\xi}) - \nabla^2\bar{\xi}\right] \tag{5}$$

Finally, the identity of Eq. 2 is used to obtain

$$\nabla\cdot\bar{\bar{T}} = (2G_s+\lambda_s)\nabla(\nabla\cdot\bar{\xi}) - G_s\nabla\times\nabla\times\bar{\xi} \tag{6}$$

and the desired equation of incremental motion is obtained.

Prob. 7.18.1 Because $\bar{A}_s$ is solenoidal, $\nabla\times\nabla\times\bar{A}_s = -\nabla^2\bar{A}_s$ and so substitution of $\bar{\xi}$ into the equation of motion gives

$$\nabla\times\left[\rho\frac{\partial^2\bar{A}_s}{\partial t^2} - G_s\nabla^2\bar{A}_s - \bar{G}\right] - \nabla\left[\rho\frac{\partial^2\psi_s}{\partial t^2} - (2G+\lambda)\nabla^2\psi_s + \mathcal{E}\right] = 0 \tag{1}$$

The equation is therefore satisfied if

$$\frac{\partial^2\bar{A}_s}{\partial t^2} = v_s^2\nabla^2\bar{A}_s + \frac{\bar{G}}{\rho} \quad ; \quad v_s \equiv \sqrt{G_s/\rho} \tag{2}$$

$$\frac{\partial^2\psi_s}{\partial t^2} = v_c^2\nabla^2\psi_s - \frac{\mathcal{E}}{\rho} \quad ; \quad v_c \equiv \sqrt{(2G_s+\lambda_s)/\rho} \tag{3}$$

That $\bar{A}_s$ represent rotational (shearing) motions is evident from taking the curl of the deformation

$$\nabla\times\bar{\xi} = \nabla\times\left[\nabla\times\bar{A}_s\right] - \nabla\times\nabla\psi_s = -\nabla^2\bar{A}_s \tag{4}$$

Similarly, the divergence is represented by ψ_s alone. These classes of deformation propagate with distinct velocities and are uncoupled in the material volume. However, at a boundary there is in general coupling between the two modes.

Prob. 7.18.2 Subject to no external forces, the equation of motion for the particle is simply

$$\frac{4}{3}\rho_p \pi R^3 \frac{dU}{dt} + 6\pi\eta R U = 0 \qquad (1)$$

Thus, with U_o the initial velocity,

$$U = U_o \exp(-t/\tau) \qquad (2)$$

where $\tau \equiv (2/9)(\rho_p R^2/\eta)$

Prob.7.19.1 There are two ways to obtain the stress tensor. First, observe that the divergence of the given S_{ij} is the mechanical force density on the right in the incompressible force equation.

$$\frac{\partial S_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(-p\,\delta_{ij}) + G_s \frac{\partial}{\partial x_j}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \qquad (1)$$

Because $\partial\xi_j/\partial x_j \equiv \nabla\cdot\bar{\xi}=0$, this expression becomes

$$\frac{\partial S_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + G_s \frac{\partial^2 \xi_i}{\partial x_j \partial x_j} \qquad (2)$$

which is recognized as the right hand side of the force equation.

As a second approach, simply observe from Eq. (b) of Table P7.16.1 that the required S_{ij} is obtained if $\lambda_s \nabla\cdot\bar{\xi} \rightarrow -p$ and e_{ij} is as given by Eq. (a) of that Table.

One way to make the analogy is to write out the equations of motion in terms of complex amplitudes.

$$j\omega\rho\hat{v}_x = -\frac{d\hat{p}}{dx} + \eta\left(\frac{d^2\hat{v}_x}{dx^2} - k^2\hat{v}_x\right) \quad\Big|\quad (j\omega)^2\rho\hat{\xi}_x = -\frac{d\hat{p}}{dx} + G_s\left(\frac{d^2\hat{\xi}_x}{dx^2} - k^2\hat{\xi}_x\right) \qquad (3)$$

$$j\omega\rho\hat{v}_y = jk\hat{p} + \eta\left(\frac{d^2\hat{v}_y}{dx^2} - k^2\hat{v}_y\right) \quad\Big|\quad (j\omega)^2\rho\hat{\xi}_y = jk\hat{p} + G_s\left(\frac{d^2\hat{\xi}_y}{dx^2} - k^2\hat{\xi}_y\right) \qquad (4)$$

$$\frac{d\hat{v}_x}{dx} - jk\hat{v}_y = 0 \quad\Big|\quad \frac{d\hat{\xi}_x}{dx} - jk\hat{\xi}_y = 0 \qquad (5)$$

$$\hat{S}_{xx} = -\hat{p} + \eta\frac{d\hat{v}_x}{dx} \quad\Big|\quad \hat{S}_{xx} = -\hat{p} + G_s\frac{d\hat{\xi}_x}{dx} \qquad (6)$$

$$\hat{S}_{yx} = \eta\left(\frac{d\hat{v}_y}{dx} - jk\hat{v}_x\right) \quad\Big|\quad \hat{S}_{yx} = G_s\left(\frac{d\hat{\xi}_y}{dx} - jk\hat{\xi}_x\right) \qquad (7)$$

The given substitution then turns the left side equations (for the incompressible fluid mechanics) into those on the right (for the incompressible solid mechanics).

Prob. 7.19.2 The laws required to represent the elastic displacements and stresses are given in Table P7.16.1. In terms of $\bar{A}_s$ and Ψ_s as defined in Prob. 7.18.1, Eq. (e) becomes

$$\nabla\times\left[\rho\frac{\partial^2\bar{A}_s}{\partial t^2}+G_s\nabla\times\nabla\times\bar{A}_s\right]-\nabla\left[\rho\frac{\partial^2\Psi_s}{\partial t^2}-(2G_s+\lambda_s)\nabla^2\Psi_s\right]=0 \tag{1}$$

Given that because $\nabla\cdot\bar{A}_s=0$, $\nabla\times\nabla\times\bar{A}_s=-\nabla^2\bar{A}_s$, this expression is satisfied if

$$\frac{\partial^2\bar{A}_s}{\partial t^2}=\frac{G_s}{\rho}\nabla^2\bar{A}_s\Rightarrow\frac{\partial^2 A}{\partial t^2}=\frac{G_s}{\rho}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial^2\Psi_s}{\partial t^2}=\frac{2G_s+\lambda_s}{\rho}\nabla^2\Psi_s\Rightarrow\frac{\partial^2\Psi}{\partial t^2}=\frac{2G_s+\lambda_s}{\rho}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right) \tag{3}$$

In the second equations, $\bar{A}_s=A(x,y,t)\bar{i}_z$, $\Psi_s=\Psi(x,y,t)$, to represent the two-dimensional motions of interest.

Given solutions to Eqs. 2 and 3, $\bar{\xi}$ is evaluated.

$$\xi_x=\frac{\partial A}{\partial y}-\frac{\partial\Psi}{\partial x} \tag{4}$$

$$\xi_y=-\frac{\partial A}{\partial x}-\frac{\partial\Psi}{\partial y} \tag{5}$$

The desired stress components then follow from Eqs. (a) and (b) from Table P7.16.1.

$$S_{xx}=(2G_s+\lambda_s)\frac{\partial\xi_x}{\partial x}+\lambda_s\frac{\partial\xi_y}{\partial y} \tag{6}$$

$$S_{yx}=G_s\left(\frac{\partial\xi_y}{\partial x}+\frac{\partial\xi_x}{\partial y}\right) \tag{7}$$

In particular, solutions of the form $A=\mathrm{Re}\,\hat{A}(x)\,e^{j(\omega t-ky)}$ and

Prob. 7.19.2 (cont.)

$\psi = R_e\,\hat{\psi}(x)\,e^{j(\omega t - ky)}$ are substituted into Eqs. 2 and 3 to obtain

$$\frac{d^2\hat{A}}{dx^2} - \gamma_s^2\hat{A} = 0 \tag{8}$$

$$\frac{d^2\hat{\psi}}{dx^2} - \gamma_c^2\hat{\psi} = 0 \tag{9}$$

where $\gamma_s^2 = k^2 - \omega^2\rho/G_s$ and $\gamma_c^2 = k^2 - \omega^2\rho/(2G_s + \lambda_s)$.

With the proviso that γ_s and γ_c have positive real parts,

$$\hat{A} = \hat{A}_1\,e^{\pm\gamma_s x}\ ;\ \hat{\psi} = \hat{\psi}_1\,e^{\pm\gamma_c x} \tag{10}$$

are solutions appropriate to infinite half spaces. The upper signs refer to a lower half space while the lower ones refer to an upper half space.

It follows from Eq. 10 that the displacements of Eqs. 4 and 5 are

$$\hat{\xi}_x = -jk\hat{A}_1\,e^{\pm\gamma_s x} \mp \gamma_c\,\hat{\psi}_1\,e^{\pm\gamma_c x} \tag{11}$$

$$\hat{\xi}_y = \mp\gamma_s\,\hat{A}_1\,e^{\pm\gamma_s x} + jk\,\hat{\psi}_1\,e^{\pm\gamma_c x} \tag{12}$$

These expressions are now used to trade-in the $(\hat{A}_1, \hat{\psi}_1)$ on the displacements evaluated at the interface.

$$\begin{bmatrix} \hat{\xi}_x^{\alpha,\beta} \\ \hat{\xi}_y^{\alpha,\beta} \end{bmatrix} \begin{bmatrix} -jk & \mp\gamma_c \\ \mp\gamma_s & jk \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{\psi}_1 \end{bmatrix} \tag{13}$$

Inversion of this expression gives

Prob. 7.19.2 (cont.)

$$\begin{bmatrix} \hat{A}_1 \\ \hat{\psi}_1 \end{bmatrix} = \frac{1}{k^2 - \gamma_c\gamma_s}\begin{bmatrix} jk & \pm\gamma_c \\ \pm\gamma_s & -jk \end{bmatrix}\begin{bmatrix} \hat{\xi}_x^{\alpha,\beta} \\ \hat{\xi}_y^{\alpha,\beta} \end{bmatrix} \tag{14}$$

In terms of complex amplitudes, Eqs. 6 and 7 are

$$\hat{S}_{xx} = (2G_s + \lambda_s)\frac{d\hat{\xi}_x}{dx} - jk\lambda_s\hat{\xi}_y \tag{15}$$

$$\hat{S}_{yx} = G_s\left[\frac{d\hat{\xi}_y}{dx} - jk\hat{\xi}_x\right] \tag{16}$$

and these in turn are evaluated using Eqs. 11 and 12. The resulting expressions are evaluated at $x=0$ to give

$$\begin{bmatrix} \hat{S}_{xx}^{\alpha,\beta} \\ \hat{S}_{yx}^{\alpha,\beta} \end{bmatrix} = \begin{bmatrix} \mp 2jG_s k\gamma_s & \left[-(2G_s+\lambda_s)\gamma_c^2 + k^2\lambda_s\right] \\ -G_s(\gamma_s^2 + k^2) & \pm jG_s k\, 2\gamma_c \end{bmatrix}\begin{bmatrix} \hat{A}_1 \\ \hat{\psi}_1 \end{bmatrix} \tag{17}$$

Finally, the transfer relations follow by replacing the column matrix on the right by the right-hand side of Eq. 14, and multiplying out the two 2x2 matrices. Note that the definitions, $v_c^2 \equiv (2G_s+\lambda_s)/\rho$, $v_s^2 \equiv G_s/\rho$ (and hence $v_c^2 - 2v_s^2 = \lambda_s/\rho$) from Prob. 7.18.1 have been used.

<u>Prob. 7.19.3</u> (a) The boundary conditions are on the stress. Because only perturbations are involved, $\hat{S}^{\alpha\beta}_{xx}$ and $\hat{S}^{\alpha\beta}_{yx}$ are therefore zero. It follows that the determinant of the coefficients of $(\hat{\xi}^{\alpha\beta}_x, \hat{\xi}^{\alpha\beta}_y)$ is therefore zero. Thus, the desired dispersion equation is

$$\gamma_s v_c^2(\gamma_c^2 - k^2) v_s^2 \gamma_c (\gamma_s^2 - k^2) + k^2 v_s^2(\gamma_s^2 + k^2 - 2\gamma_c\gamma_s)[k^2(v_c^2 - 2v_s^2) - v_c^2\gamma_c^2 + 2\gamma_c\gamma_s v_s^2] = 0 \tag{1}$$

This simplifies to the given expression provided that the definitions of γ_s^2 and γ_c^2 are used to eliminate v_c^2 through the condition $v_s^2(\gamma_s^2 - k^2) = v_c^2(\gamma_c^2 - k^2)$.

(b) Substitution of $\gamma_s^2 = k^2 - \omega^2/v_s^2$, $\gamma_c^2 = k^2 - \omega^2/v_c^2$ into the square of the dispersion equation gives

$$\left(2k^2 - \frac{\omega^2}{v_s^2}\right)^4 - 16\left(k^2 - \frac{\omega^2}{v_s^2}\right)\left(k^2 - \frac{\omega^2}{v_c^2}\right)k^4 = 0 \tag{2}$$

Division by k^8 gives

$$(2 - \underline{\omega}^2)^4 - 16(1 - \underline{\omega}^2)\left(1 - \frac{v_s^2}{v_c^2}\underline{\omega}^2\right) = 0 \tag{3}$$

where $\underline{\omega} \equiv \omega / k v_s$ and it is clear that the only parameter is v_s / v_c. Multiplied out, this expression becomes the given polynomial.

(c) Given a valid root to Eq. 3 (one that makes $\mathrm{Re}\,\gamma_s > 0$ and $\mathrm{Re}\,\gamma_c > 0$, $\underline{\omega} = \alpha$, it follows that

$$\omega = \alpha v_s k \tag{4}$$

Thus, the phase velocity, αv_s, is independent of k.

(d) From Prob. 7.18.1

$$\frac{v_s^2}{v_c^2} = G_s/(2G_s + \lambda_s) \tag{5}$$

Prob. 7.19.3 (cont.)

while from Eq. g of Table P7.16.1

$$E_s = (\nu_s + 1) 2 G_s \tag{6}$$

Thus, E_s is eliminated from Eq. f of that table to give

$$\lambda_s = 2\nu_s G_s / (1 - 2\nu_s) \tag{7}$$

The desired expression follows from substitution of this expression for λ_s in Eq. 5.

Prob. 7.19.4 (a) With the force density included, Eq. 1 becomes

$$\nabla^2 \left(\rho \frac{\partial A_v}{\partial t} - \eta \nabla^2 A_v - G \right) = 0 \tag{1}$$

In terms of complex amplitudes, this expression in turn is

$$\left(\frac{d^2}{dx^2} - k^2 \right) \left[\frac{d^2 \hat{A}}{dx^2} - \gamma^2 \hat{A} + \frac{\hat{G}(x)}{\eta} \right] = 0 \tag{2}$$

The solution that makes the quantity in brackets [ ] vanish is now called $\hat{A}_p(x)$ and the total solution is $\hat{A} = \hat{A}_H + \hat{A}_p$ with associated velocity and stress functions of the form $\hat{v}_x = (\hat{v}_x)_H + (\hat{v}_x)_p$ and $\hat{S}_{xx} = (\hat{S}_{xx})_H + (\hat{S}_{xx})_p$. The transfer relations, Eq. 7.19.13, still relate the homogeneous solutions, so

$$\begin{bmatrix} \hat{S}^\alpha_{xx} - (\hat{S}^\alpha_{xx})_p \\ \hat{S}^\beta_{xx} - (\hat{S}^\beta_{xx})_p \\ \hat{S}^\alpha_{yx} - (\hat{S}_{yx})_p \\ \hat{S}^\beta_{yx} - (\hat{S}^\beta_{yx})_p \end{bmatrix} = \eta [P_{ij}] \begin{bmatrix} \hat{v}^\alpha_x - (\hat{v}^\alpha_x)_p \\ \hat{v}^\beta_x - (\hat{v}^\beta_x)_p \\ \hat{v}^\alpha_y - (\hat{v}^\alpha_y)_p \\ \hat{v}^\beta_y - (\hat{v}^\beta_y)_p \end{bmatrix} \tag{3}$$

With the particular stress solutions shifted to the right and the velocity components separated, this expression is equivalent to that given.

(b) For the example where $\hat{G} = F_o x$,

Prob. 7.19.4 (cont.)

$$(\hat{v}_x)_p = \frac{-jkF_o}{\eta\gamma^2}x\ ;\ (\hat{v}_y)_p = \frac{-F_o}{\eta\gamma^2}\ ;\ (\hat{p})_p = 0\ ;\ (\hat{S}_{xx})_p = \frac{-2jkF_o}{\gamma^2}\ ;\ (\hat{S}_{yx})_p = -\frac{k^2F_o}{\gamma^2}x \tag{4}$$

Thus, evaluation of Eq. 3 gives

$$\begin{bmatrix} \hat{S}_{xx}^{\alpha} \\ \hat{S}_{xx}^{\beta} \\ \hat{S}_{yx}^{\alpha} \\ \hat{S}_{yx}^{\beta} \end{bmatrix} = \eta[P_{ij}]\begin{bmatrix} \hat{v}_x^{\alpha} \\ \hat{v}_x^{\beta} \\ \hat{v}_y^{\alpha} \\ \hat{v}_y^{\beta} \end{bmatrix} - \frac{kF_o}{\gamma^2}\begin{bmatrix} 2j \\ 2j \\ k\Delta \\ 0 \end{bmatrix} - \frac{F_o}{\gamma^2}[P_{ij}]\begin{bmatrix} -jk\Delta \\ 0 \\ -1 \\ -1 \end{bmatrix} \tag{5}$$

<u>Prob. 7.19.5</u> The temporal modes follow directly from Eq. 13, because the velocities are zero at the respective boundaries. Thus, unless the root happens to be trivial, for the response to be finite, F=0. Thus, with $\Delta = d$, the required eigenfrequency equation is

$$\frac{2\gamma}{k}(1 - \cosh\gamma d \cosh kd) + \sinh\gamma d \sinh kd\left[\left(\frac{\gamma}{k}\right)^2 + 1\right] = 0 \tag{1}$$

where once γ is found from this expression, the frequency follows from the definition

$$\gamma \equiv \sqrt{k^2 + \frac{j\omega\rho}{\eta}} \tag{2}$$

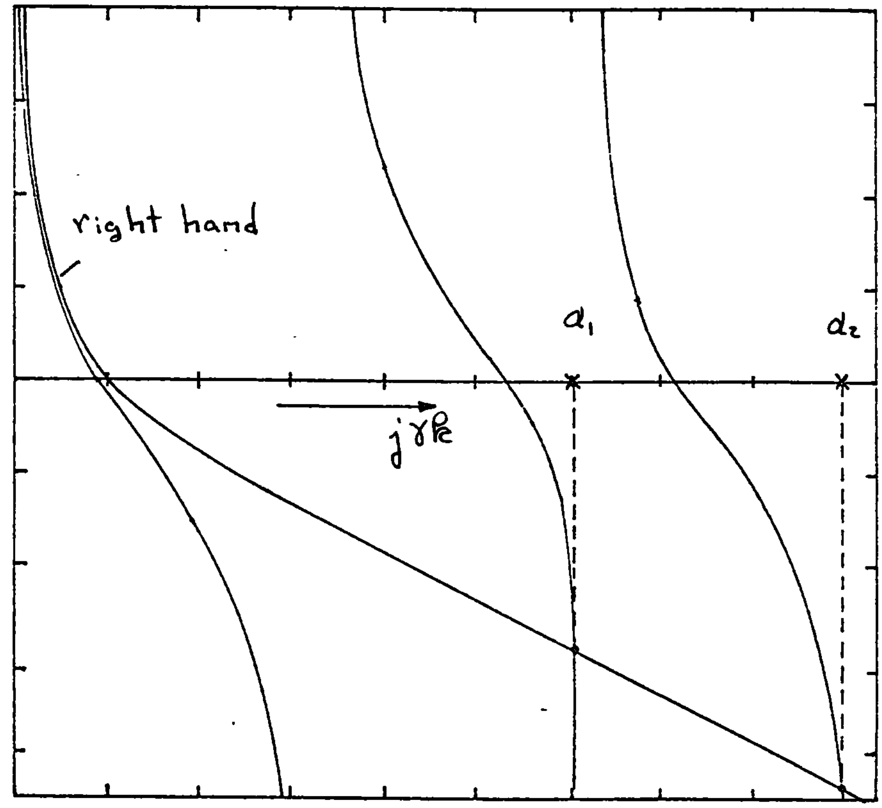
Note that Eq. 1 can be written as

$$\frac{\cos\left(j\frac{\gamma}{k}kd\right)\cosh kd - 1}{\sin\left(\frac{j\gamma}{k}kd\right)\sinh kd} = \frac{1 - \left(\frac{j\gamma}{k}\right)^2}{2\left(\frac{j\gamma}{k}\right)} \tag{3}$$

The right-hand side of this expression can be plotted once and for all, as shown in the figure. To plot the left-hand side as a function of $j\gamma/k$, it is necessary to specify kd. For the case where kd=1, the plot is as shown in the figure. From the graphical solution, roots $j\gamma/k = \alpha_n$ follow. The corresponding eigenfrequency follows from Eq. 2 as

Prob. 7.19.5 (cont.)

$$j\omega_n = \frac{\eta k^2}{\rho}(\alpha_n^2 - 1)$$



Prob. 7.20.1 The analogy is clear if the force and stress equations are compared. The appropriate fluid equation in the creep flow limit is Eq. 7.18.12.

$$\nabla p = \eta \nabla^2 \bar{v} \quad \Big| \quad \nabla p = G_s \nabla^2 \bar{\xi} \tag{1}$$

$$S_{ij} = -p + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad \Big| \quad S_{ij} = -p + G_s\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \tag{2}$$

To see that this limit is one in which times of interest are long compared to the time for propagation of either a compressional or a shear wave through a length of interest, write Eq. (e) of Table P7.16.1 in normalized form

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{2G_s + \lambda_s}{\rho \ell^2}\tau^2 \nabla(\nabla \cdot \bar{\xi}) - \frac{G_s \tau^2}{\ell^2}\nabla \times \nabla \times \bar{\xi} \tag{3}$$

where (see Prob. 7.18.1 exploration of wave dynamics)

$$t = \underline{t}\,\tau \quad ; \quad v_c \equiv \sqrt{(2G_s + \lambda_s)/\rho}$$

$$(x, y, z) = (\underline{x}, \underline{y}, \underline{z})\ell \quad ; \quad v_s \equiv \sqrt{G_s/\rho}$$

and observe that the inertial term is ignorable if

$$\frac{(\ell/\tau)^2}{v_c^2} << 1 \quad ; \quad \frac{(\ell/\tau)^2}{v_s^2} << 1 \tag{4}$$

Prob. 7.20.1 (cont.)

With the identification $p \equiv (2G_s+\lambda_s)\nabla\cdot\bar{\xi}$, the fully quasistatic elastic equations result. Note that in this limit, it is understood that $\nabla\cdot\bar{v} = 0$ and $\nabla\cdot\bar{\xi} = 0$

Prob. 7.21.1 In Eq. 7.20.17, $\tilde{v}_r^\alpha = 0$, $\tilde{v}_\theta^\beta = 0$ and $n = 1$. Thus,

$$\tilde{\Lambda}_1 = \frac{UR^2}{2}, \; \tilde{\Lambda}_2 = \frac{R^2}{4}U, \; \tilde{\Lambda}_3 = -\frac{3R^2U}{4}, \; \tilde{\Lambda}_4 = 0 \tag{1}$$

and so Eq. 7.20.13 becomes

$$\tilde{\Lambda} = \frac{R^2U}{2}\left[\left(\frac{r}{R}\right)^2 + \frac{1}{2}\left(\frac{R}{r}\right) - \frac{3}{2}\left(\frac{r}{R}\right)\right] \tag{2}$$

The θ dependence is given by Eq. 10 as $\sin\theta P_1^1(\cos\theta)$ so finally the desired stream function is Eq. 5.5.5.

Prob. 7.21.2 The analogy discussed in Prob. 7.20.1 applies so that the transfer relations are directly applicable (with the appropriate substitutions) to the evaluation of Eq. 7.21.1. Thus, $U \rightarrow \Xi$ and $\eta \rightarrow G_s$. Just as the rate of fall of a sphere in a highly viscous fluid can be used to deduce the viscosity through the use of Eq. 4, the shear modulus can be deduced by observing the displacement of a sphere subject to the force f_z.

$$f_z = 6\pi G_s R \Xi \tag{1}$$