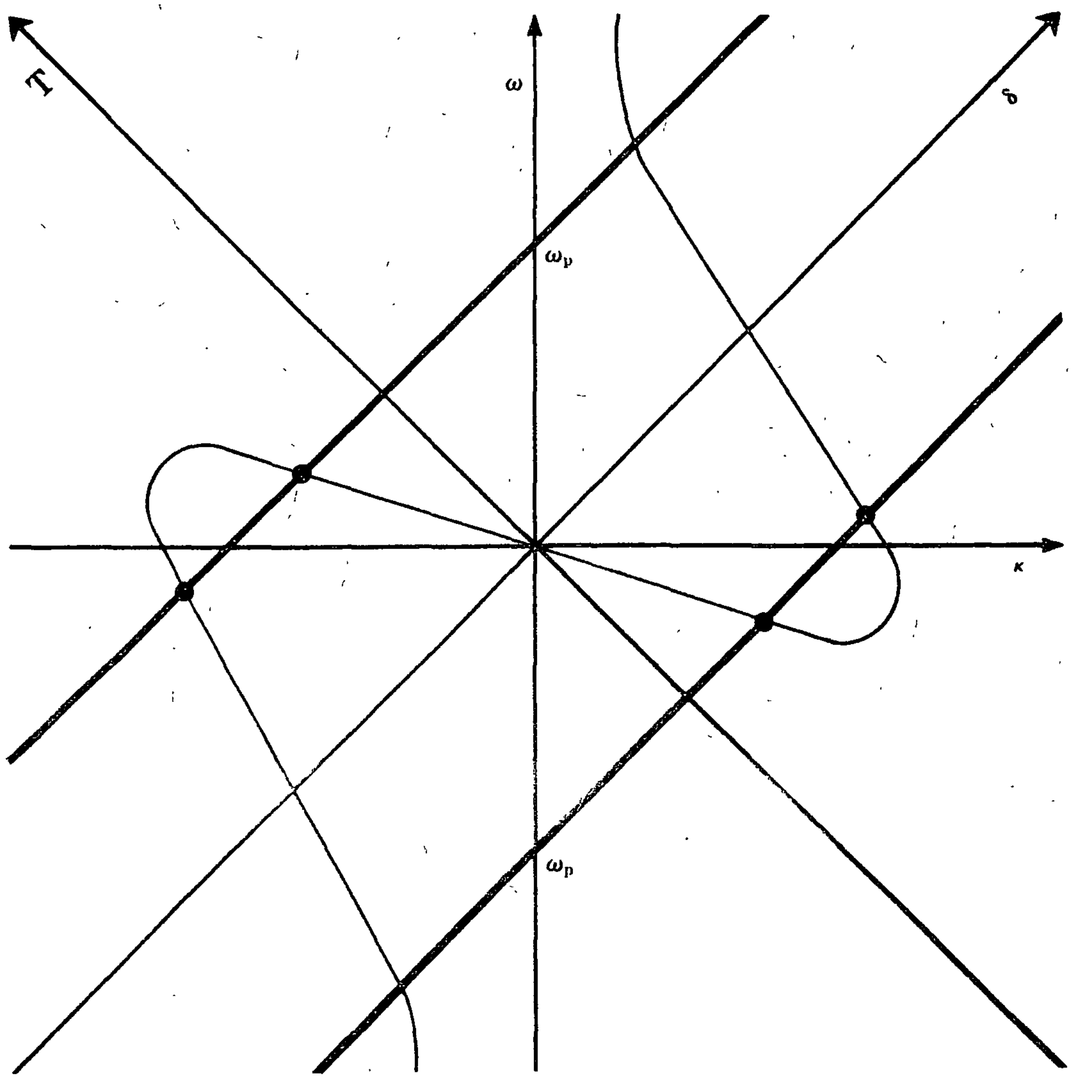
SOLUTIONS MANUAL FOR

ELECTROMECHANICAL DYNAMICS

PART III: Elastic and Fluid Media

HERBERT H. WOODSON JAMES R. MELCHER



Prepared by MARKUS ZAHN

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Mark Zahn

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PREFACE TO: SOLUTIONS MANUAL TO

ELECTROMECHANICAL DYNAMICS, PART III:

ELASTIC AND FLUID MEDIA

This manual presents, in an informal format, solutions to the problems found at the ends of chapters in Part III of the book, Electromechanical Dynamics. It is intended as an aid for instructors, and in special circumstances, for use by students. A sufficient amount of explanatory material is included such that the solutions, together with problem statements, are in themselves a teaching aid. They are substantially as found in the records for the undergraduate and graduate courses 6.06, 6.526, and 6.527, as taught at Massachusetts Institute of Technology over a period of several years.

It is difficult to give proper credit to all of those who contributed to these solutions, because the individuals involved range over teaching assistants, instructors, and faculty who have taught the material over a period of more than four years. However, special thanks are due the authors, Professor J. R. Melcher and Professor H. H. Woodson, who gave me the opportunity and incentive to write this manual. This work has greatly increased the value of my graduate education, in addition to giving me the pleasure of working with these two men.

The manuscript was typed by Mrs. Evelyn M. Holmes, whom I especially thank for her sense of humor, advice, patience and expertise which has made this work possible.

Of most value during the course of this work was the understanding of my girl friend, then fiancée, and now my wife, Linda, in spite of the competition for time.

Markus Zahn

Cambridge, Massachusetts
October, 1969

PROBLEM 11.1

Part a

We add up all the volume force densities on the elastic material, and with the help of equation 11.1.4, we write Newton's law as

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} - \rho g \tag{a}$$

where we have taken $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0$. Since this is a static problem, we let $\frac{\partial}{\partial t} = 0$. Thus,

$$\frac{\partial T_{11}}{\partial x_1} = \rho g. \tag{b}$$

From 11.2.32, we obtain

$$T_{11} = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} \tag{c}$$

Therefore

$$(2G + \lambda) \frac{\partial^2 \delta_1}{\partial x_1^2} = \rho g \tag{d}$$

Solving for δ_1, we obtain

$$\delta_1 = \frac{\rho g}{2(2G+\lambda)} x_1^2 + C_1 x + C_2 \tag{e}$$

where C_1 and C_2 are arbitrary constants of integration, which can be evaluated by the boundary conditions

$$\delta_1(0) = 0 \tag{f}$$

and

$$T_{11}(L) = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} (L) = 0 \tag{g}$$

since $x_1 = L$ is a free surface. Therefore, the solution is

$$\delta_1 = \frac{\rho g \, x_1}{2(2G+\lambda)} [x_1 - 2L]. \tag{f}$$

Part b

Again applying 11.2.32

PROBLEM 11.1 (Continued)

$$T_{11} = (2G+\lambda)\,\frac{\partial \delta_1}{\partial x_1} = \rho g[x_1 - L)$$

$$T_{12} = T_{21} = 0$$

$$T_{13} = T_{31} = 0$$

$$T_{22} = \lambda\,\frac{\partial \delta_1}{\partial x_1} = \frac{\lambda \rho g}{(2G+\lambda)}\,[x_1 - L] \qquad \text{(g)}$$

$$T_{33} = \lambda\,\frac{\partial \delta_1}{\partial x_1} = \frac{\lambda \rho g}{(2G+\lambda)}\,[x_1 - L]$$

$$T_{32} = T_{23} = 0$$

$$\bar{\bar{T}} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \qquad \text{(h)}$$

PROBLEM 11.2

Since the electric force only acts on the surface at $x_1 = -L$, the equation of motion for the elastic material $(-L \le x_1 \le 0)$ is from Eqs. (11.1.4) and (11.2.32),

$$\rho\,\frac{\partial^2 \delta_1}{\partial t^2} = (2G+\lambda)\,\frac{\partial^2 \delta_1}{\partial x_1^2} \qquad \text{(a)}$$

The boundary conditions are

$$\delta_1(0,t) = 0$$

and

$$M\,\frac{\partial^2 \delta_1(-L,t)}{\partial t^2} = aD(2G+\lambda)\,\frac{\partial \delta_1}{\partial x_1}(-L,t) + f^e \qquad \text{(b)}$$

f^e is the electric force in the x_1 direction at $x_1 = -L$, and may be found by using the Maxwell Stress Tensor $T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij}\,\varepsilon E_k E_k$ to be (see Appendix G for discussion of stress tensor),

$$f^e = -\frac{\varepsilon}{2}\,E^2\,aD$$

with

$$E = \frac{V_o + V_1 \cos \omega t}{d + \delta_1(-L,t)} \qquad \text{(c)}$$

PROBLEM 11.2 (continued)

Expanding f^e to linear terms only, we obtain

$$f^e = -\frac{\varepsilon a D}{2}\left[\frac{V_o^2}{d^2} + \frac{2V_oV_1 \cos \omega t}{d^2} - \frac{2V_o^2}{d^3}\delta_1(-L,t)\right] \tag{d}$$

We have neglected all second order products of small quantities.

Because of the constant bias V_o, and the sinusoidal nature of the perturbations, we assume solutions of the form

$$\delta_1(x_1,t) = \delta_1(x_1) + \text{Re}\ \hat{\delta}e^{j(\omega t-kx_1)} \tag{e}$$

where

$$\hat{\delta} << \delta_1(x_1) << L$$

The relationship between ω and k is readily found by substituting (e) into (a), from which we obtain

$$k = \pm\frac{\omega}{v_p} \text{ with } v_p = \sqrt{\frac{2G+\lambda}{\rho}} \tag{f}$$

We first solve for the equilibrium configuration which is time independent. Thus

$$\frac{\partial^2\delta_1(x_1)}{\partial x_1^2} = 0 \tag{g}$$

This implies

$$\delta_1(x_1) = C_1x_1 + C_2$$

Because $\delta_1(0) = 0$, $C_2 = 0$.

From the boundary condition at $x_1 = -L$ ((b) & (d))

$$aD(2G+\lambda)C_1 - \frac{\varepsilon}{2}\frac{V_o^2}{d^2}aD = 0 \tag{h}$$

Therefore

$$\delta_1(x_1) = +\frac{\varepsilon}{2}\frac{V_o^2}{d^2(2G+\lambda)}x_1 \tag{i}$$

Note that $\delta_1(x_1 = -L)$ is negative, as it should be.

For the time varying part of the solution, using (f) and the boundary condition

$$\delta(0,t) = 0$$

PROBLEM 11.2 (continued)

we can let the perturbation δ_1 be of the form

$$\delta_1(x_1,t) = \text{Re}\ \hat{\delta}\ \sin kx_1\ e^{j\omega t} \tag{j}$$

Substituting this assumed solution into (b) and using (d), we obtain

$$+ Mw^2\ \hat{\delta}\ \sin kL = aD(2G+\lambda)k\ \hat{\delta}\ \cos kL - \frac{\varepsilon aDV_oV_1}{d^2} - \frac{\varepsilon aDV_o^2}{d^3}\hat{\delta}\ \sin kL \tag{k}$$

Solving for $\hat{\delta}$, we have

$$\hat{\delta} = -\frac{\varepsilon aDV_oV_1}{d^2\left[Mw^2\ \sin kL - aD(2G+\lambda)k\ \cos kL + \frac{\varepsilon aDV_o^2}{d^3}\sin kL\right]}$$

Thus, because $\hat{\delta}$ has been shown to be real,

$$\delta_1(-L,t) = -\frac{\varepsilon}{2}\ \frac{V_o^2\ L}{d^2(2G+\lambda)} - \hat{\delta}\ \sin kL\ \cos \omega t \tag{m}$$

Part b

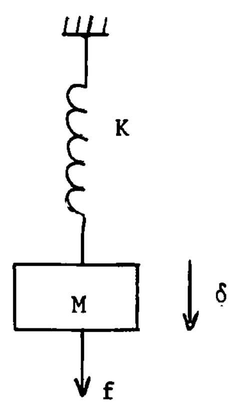
If $k\ell \ll 1$, we can approximate the sinusoidal part of (m) as

$$\delta_1(-L,t) = \frac{\varepsilon aDV_oV_1\ \cos \omega t}{d^2\left[Mw^2 - \frac{aD(2G+\lambda)}{L} + \frac{\varepsilon aDV_o^2}{d^3}\right]} \tag{n}$$

We recognize this as a force-displacement relation for a mass on the end of a spring.

Part c

We thus can model (n) as



PROBLEM 11.2 (Continued)

where

$$f = -\frac{\varepsilon a D V_o V_1 \cos \omega t}{d^2}$$

and

$$K = \frac{aD(2G+\lambda)}{L} - \frac{\varepsilon a D V_o^2}{d^3}$$

We see that the electrical force acts like a negative spring constant.

PROBLEM 11.3

Part a

From (11.1.4), we have the equation of motion in the x_2 direction as

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = \frac{\partial T_{21}}{\partial x_1} \tag{a}$$

From (11.2.32),

$$T_{21} = G \left[\frac{\partial \delta_2}{\partial x_1}\right] \tag{b}$$

Therefore, substituting (b) into (a), we obtain an equation for δ_2

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = G \frac{\partial^2 \delta_2}{\partial x_1^2} \tag{c}$$

We assume solutions of the form

$$\delta_2 = \text{Re}\, \hat{\delta}_2\, e^{j(\omega t - kx_1)} \tag{d}$$

where from (c) we obtain

$$k = \pm \frac{w}{v_p} \qquad v_p^2 = \frac{G}{\rho}$$

Thus we let

$$\delta_2 = \text{Re}\left[\delta_a\, e^{j(\omega t - kx_1)} + \delta_b\, e^{j(\omega t + kx_1)}\right] \tag{e}$$

$$\text{with } k = \frac{\omega}{v_p}$$

The boundary conditions are

$$\delta_2(\ell, t) = \delta_o e^{j\omega t} \tag{f}$$

PROBLEM 11.3 (continued)

and

$$\left.\frac{\partial\delta_2}{\partial x_1}\right|_{x_1=0} = 0 \qquad (g)$$

since the surface at $x_1 = 0$ is free.

Therefore

$$\delta_a e^{-jk\ell} + \delta_b e^{jk\ell} = \delta_o \qquad (h)$$

and

$$-jk\,\delta_a + jk\,\delta_b = 0 \qquad (i)$$

Solving, we obtain

$$\delta_a = \delta_b = \frac{\delta_o}{2\cos k\ell} \qquad (j)$$

Therefore

$$\delta_2(x_1,t) = \mathrm{Re}\left[\frac{\delta_o}{\cos k\ell}\cos kx_1\, e^{j\omega t}\right] = \frac{\delta_o}{\cos k\ell}\cos kx_1 \cos\omega t \qquad (k)$$

and

$$T_{21}(x_1,t) = -\mathrm{Re}\left[\frac{G\delta_o k}{\cos k\ell}\sin kx_1\, e^{j\omega t}\right] \qquad (\ell)$$

$$= -\frac{G\delta_o k}{\cos k\ell}\sin kx_1 \cos\omega t$$

Part b

In the limit as ω gets small

$$\delta_2(x_1,t) \to \mathrm{Re}[\delta_o e^{j\omega t}] \qquad (m)$$

In this limit, δ_2 varies everywhere in phase with the source. The slab of elastic material moves as a rigid body. Note from (ℓ) that the force per unit area at $x_1 = \ell$ required to set the slab into motion is $T_{21}(\ell,t) = \rho\ell \frac{d^2}{dt^2}(\delta_o \cos\omega t)$ or the mass $/(x_2-x_3)$ area times the rigid body acceleration.

Part c

The slab can resonate if we can have a finite displacement, even as $\delta_o \to 0$. This can happen if the denominator of (k) vanishes

$$\cos k\ell = 0 \qquad (n)$$

or

$$\omega = \frac{(2n+1)\pi v_p}{2\ell} \qquad n = 0,1,2,\ldots \qquad (o)$$

PROBLEM 11.3 (continued)

The lowest frequency is for n = 0

$$\text{or } \omega_{low} = \frac{\pi v_p}{2\ell} \tag{p}$$

PROBLEM 11.4

Part a

We have that

$$\tau_i = T_{ij}n_j = \alpha\delta_{ij}n_j$$

It is given that the T_{ij} are known, thus the above equation may be written as three scalar equations $(T_{ij} - \alpha\delta_{ij})n_j = 0$, or:

$$\begin{aligned} (T_{11} - \alpha)n_1 + T_{12}n_2 + T_{13}n_3 &= 0 \\ T_{21}n_1 + (T_{22} - \alpha)n_2 + T_{23}n_3 &= 0 \\ T_{31}n_1 + T_{32}n_2 + (T_{33} - \alpha)n_3 &= 0 \end{aligned} \tag{a}$$

Part b

The solution for these homogeneous equations requires that the determinant of the coefficients of the n_i's equal zero.

Thus

$$\begin{aligned} &(T_{11} - \alpha)[(T_{22} - \alpha)(T_{33} - \alpha) - (T_{23})^2] \\ &\quad - T_{12}[T_{12}(T_{33} - \alpha) - T_{13}T_{23}] \\ &\quad + T_{13}[T_{12}T_{23} - T_{13}(T_{22} - \alpha)] = 0 \end{aligned} \tag{b}$$

where we have used the fact that

$$T_{ij} = T_{ji}. \tag{c}$$

Since the T_{ij} are known, this equation can be solved for α.

Part c

Consider $T_{12} = T_{21} = T_o$, with all other components equal to zero. The determinant of coefficients then reduces to

$$-\alpha^3 + T_o^2\alpha = 0 \tag{d}$$

for which

$$\alpha = 0 \tag{e}$$

or

$$\alpha = \pm T_o \tag{f}$$

The $\alpha = 0$ solution indicates that with the normal in the x_3 direction, there is no normal stress. The $\alpha = \pm T_o$ solution implies that there are two surfaces where the net traction is purely normal with stresses $\pm T_o$, respectively, as

PROBLEM 11.4 (continued)

found in example 11.2.1. Note that the normal to the surface for which the shear stress is zero can be found from (a), since α is known, and it is known that $|\overline{n}| = 1$.

PROBLEM 11.5

From Eqs. 11.2.25 - 11.2.28, we have

$$e_{11} = \frac{1}{E}[T_{11} - \nu(T_{22} + T_{33})] \quad \text{(a)}$$

$$e_{22} = \frac{1}{E}[T_{22} - \nu(T_{33} + T_{11})] \quad \text{(b)}$$

$$e_{33} = \frac{1}{E}[T_{33} - \nu(T_{11} + T_{22})] \quad \text{(c)}$$

and

$$e_{ij} = \frac{T_{ij}}{2G} \qquad i \neq j \quad \text{(d)}$$

These relations must still hold in a primed coordinate system, where we can use the transformations

$$T'_{ij} = a_{ik}a_{j\ell}T_{k\ell} \quad \text{(e)}$$

and

$$e'_{ij} = a_{ik}a_{j\ell}e_{k\ell} \quad \text{(f)}$$

For an example, we look at e'_{11}

$$e'_{11} = a_{1k}a_{1\ell}e_{k\ell} = \frac{1}{E}[T'_{11} - \nu(T'_{22} + T'_{33})] \quad \text{(g)}$$

This may be rewritten as

$$a_{1k}a_{1\ell}e_{k\ell} = \frac{1}{E}[(1 + \nu)a_{1k}a_{1\ell}T_{k\ell} - \nu\, \delta_{k\ell}T_{k\ell}] \quad \text{(h)}$$

where we have used the relation from Eq.(8.2.23), page G10 or 439.

$$a_{pr}a_{ps} = \delta_{ps} \quad \text{(i)}$$

Consider some values of k and ℓ where $k \neq \ell$.

Then, from the stress-strain relation in the unprimed frame,

$$a_{1k}a_{1\ell}e_{k\ell} = a_{1k}a_{1\ell}\frac{T_{k\ell}}{2G} = \frac{a_{1k}a_{1\ell}}{E}(1+\nu)T_{k\ell} \quad \text{(j)}$$

Thus

$$\frac{1}{2G} = \frac{1+\nu}{E} \quad \text{(k)}$$

or $E = 2G(1+\nu)$ which agrees with Eq. (g) of example 11.2.1.

PROBLEM 11.6

Part a

Following the analysis in Eqs. 11.4.16 - 11.4.26, the equation of motion for the bar is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4 \xi}{\partial x_1^4} = 0 \tag{a}$$

where ξ measures the bar displacement in the x_2 direction, T_2 in Eq. 11.4.26 = 0 as the surfaces at $x_2 = \pm b$ are free. The boundary conditions for this problem are that at $x_1 = 0$ and at $x_1 = L$

$$T_{21} = 0 \quad \text{and} \quad T_{11} = 0 \tag{b}$$

as the ends are free.

We assume solutions of the form

$$\xi = \text{Re}\ \hat{\xi}(x)e^{j\omega t} \tag{c}$$

As in example 11.4.4, the solutions for $\hat{\xi}(x)$ are

$$\hat{\xi}(x) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1 \tag{d}$$

with

$$\alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4}$$

Now, from Eqs. 11.4.18 and 11.4.21,

$$T_{21} = \frac{(x_2^2 - b^2)E}{2}\frac{\partial^3 \xi}{\partial x_1^3} \tag{e}$$

which implies

$$\frac{\partial^3 \xi}{\partial x_1^3} = 0 \tag{f}$$

at $x_1 = 0$, $x_1 = L$

and

$$T_{11} = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2} \tag{g}$$

which implies

$$\frac{\partial^2 \xi}{\partial x_1^2} = 0 \tag{h}$$

at $x_1 = 0$ and $x_1 = L$

PROBLEM 11.6 (continued)

With these relations, the boundary conditions require that

$$\begin{aligned}
- A \qquad\qquad\qquad\qquad &+ C &= 0 \\
- A \cos \alpha L + B \sin \alpha L &+ C \cosh \alpha L + D \sinh \alpha L &= 0 \\
- B \qquad\qquad &+ D &= 0 \\
- A \sin \alpha L - B \cos \alpha L &+ C \sinh \alpha L + D \cosh \alpha L &= 0
\end{aligned} \qquad (i)$$

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, and D equal zero. Performing this operation, we obtain

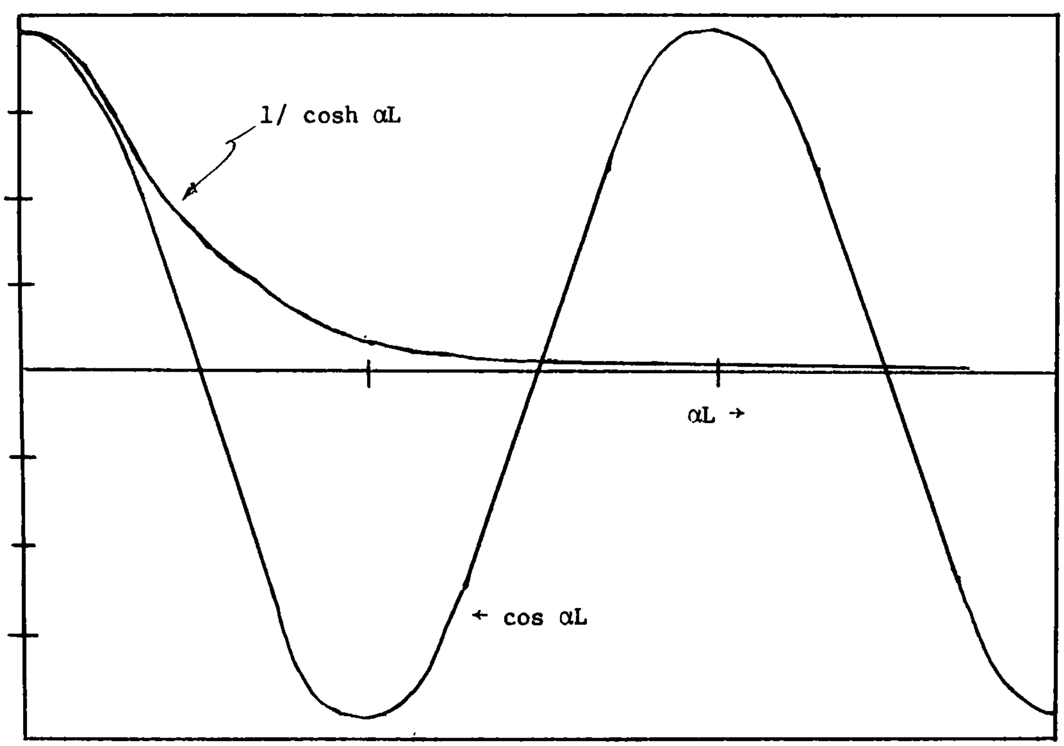
$$\cos \alpha L \cosh \alpha L = 1 \qquad (j)$$

Thus,

$$\beta = \alpha L = \left[\omega^2 \left(\frac{3\rho}{Eb^2}\right)\right]^{1/4} L \qquad (k)$$

Part b

The roots of $\cos \beta = \dfrac{1}{\cosh \beta}$ follow from the figure.



Note from the figure that the roots αL are essentially the roots $3\pi/2$, $5\pi/2$, ... of $\cos \alpha L = 0$.

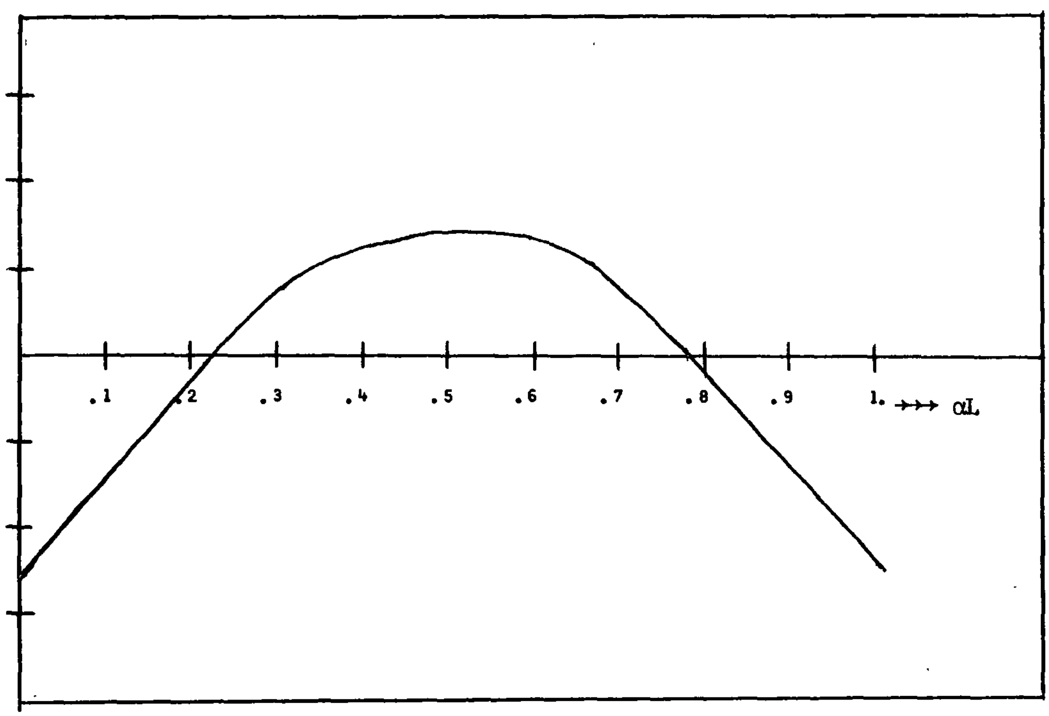
PROBLEM 11.6 (continued)

Part c

It follows from (i) that the eigenfunction is

$$\hat{\xi} = A'[(\sin \alpha x_1 + \sinh \alpha x_1)(\sin \alpha L + \sinh \alpha L) + (\cos \alpha L - \cosh \alpha L)(\cos \alpha x_1 + \cosh \alpha x_1) \qquad (\ell)$$

where A' is an arbitrary amplitude. This expression is found by taking one of the constants A ... D as known, and solving for the others. Then, (d) gives the required dependence on x_1 to within an arbitrary constant. A sketch of this function is shown in the figure.



PROBLEM 11.7

As in problem 11.6, the equation of motion for the elastic beam is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4 \xi}{\partial x_1^4} = 0 \qquad \text{(a)}$$

The four boundary conditions for this problem are:

$$\xi(x_1 = 0) = 0 \qquad \xi(x_1 = L) = 0$$

$$\delta_1(0) = -x_2 \frac{\partial \xi}{\partial x_1}\bigg|_{x_1 = 0} = 0 \quad \delta_1(L) = -x_2 \frac{\partial \xi}{\partial x_1}\bigg|_{x_1 = L} = 0 \qquad \text{(b)}$$

We assume solutions of the form

$\xi(x_1,t) = \text{Re}\ \hat{\xi}(x_1)e^{j\omega t}$, and as in problem 11.6, the solutions for (c) $\hat{\xi}(x_1)$ are

$$\hat{\xi}(x_1) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1$$

$$\text{with } \alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4} \qquad \text{(d)}$$

Applying the boundary conditions, we obtain

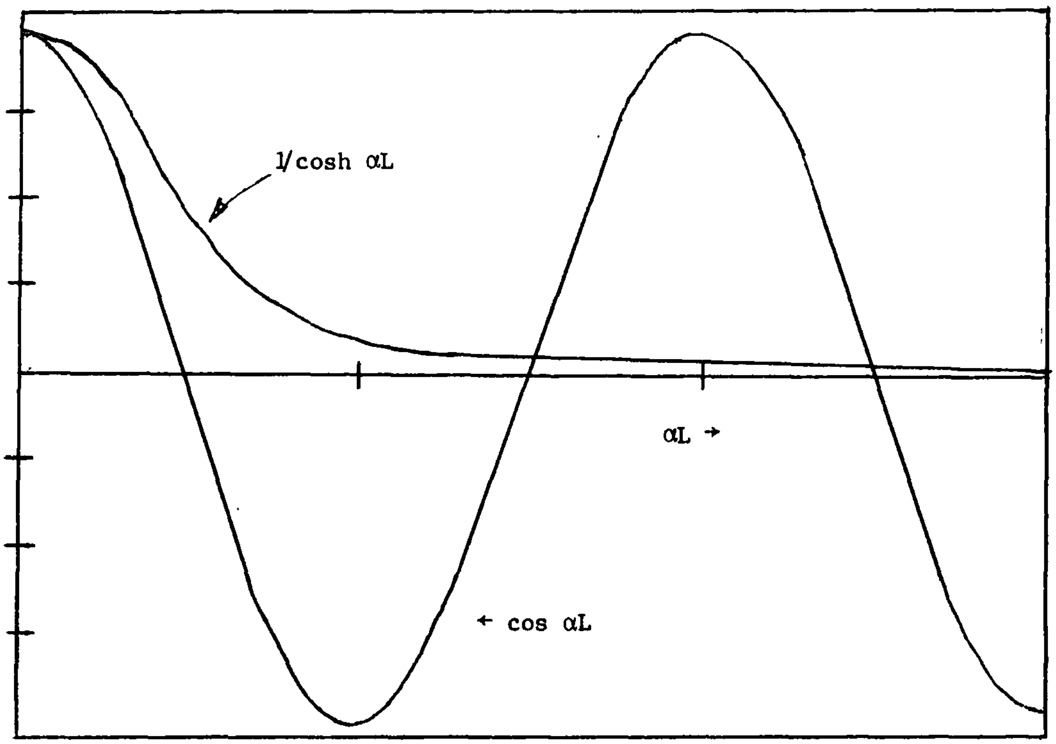
$$\begin{aligned}
B \qquad\qquad\qquad\qquad + D \qquad &= 0\\
A \sin \alpha L + B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L &= 0 \qquad \text{(e)}\\
A \qquad\qquad + C \qquad\qquad &= 0\\
A \cos \alpha L - B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L &= 0
\end{aligned}$$

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, D, equal zero. Performing this operation, we obtain

$$\cos \alpha L \cosh \alpha L = +1 \qquad \text{(f)}$$

To solve for the natural frequencies, we must use a graphical procedure.

PROBLEM 11.7 (continued)



The first natural frequency is at about

$$\alpha L = \frac{3\pi}{2}$$

Thus

$$\omega^2 \left(\frac{3\rho}{Eb^2}\right) L^4 = \left(\frac{3\pi}{2}\right)^4$$

or

$$\omega = \frac{\left(\frac{3\pi}{2}\right)^2}{L^2} \left(\frac{Eb^2}{3\rho}\right)^{1/2} \qquad \text{(g)}$$

Part b

We are given that L = .5 m and b = 5×10^{-4} m

From Table 9.1, Appendix G, the parameters for steel are:

$$E \approx 2 \times 10^{11}\ N/m^2$$

$$\rho \approx 7.75 \times 10^3\ kg/m^3$$

PROBLEM 11.7 (continued)

$$\omega \approx 120 \text{ rad/sec.}$$

Then, $f_1 = \frac{\omega}{2\pi} \approx 19$ Hz.

Part c

For the next higher resonance, $\alpha L \approx \frac{5}{2}\pi$

Therefore, $f_2 = \left(\frac{5}{2}\right)^2 f_1 \approx 53$ Hz.

PROBLEM 11.8

Part a

As in Prob. 11.7, the equation of motion for the beam is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4 \xi}{\partial x_1^4} = 0 \qquad \text{(a)}$$

At $x_1 = L$, there is a free end, so the boundary conditions are:

$$T_{11}(x_1{=}L) = 0$$

and

$$T_{21}(x_1{=}L) = 0 \qquad \text{(b)}$$

The boundary conditions at $x_1 = 0$ are

$$M\frac{\partial^2 \xi(0,t)}{\partial t^2} = +\int (T_{21})_{x_1=0}\, D\, dx_2 + \bar{f}_e + \bar{F}_o \qquad \text{(c)}$$

and

$$\delta_1(x_1 = 0) = 0 \qquad \text{(d)}$$

The $\bar{H}$ field in the air gap and in the plunger is

$$\bar{H} = \frac{Ni}{D}\bar{i}_1 \qquad \text{(e)}$$

Using the Maxwell stress tensor

$$\bar{f}^{\,e} = -\frac{(\mu - \mu_o)}{2}\left(\frac{N^2 i^2}{D^2}\right) D^2\, \bar{i}_2 = -\frac{N^2 i^2}{2}(\mu - \mu_o)\bar{i}_2 \qquad \text{(f)}$$

with $i = I_o + i_1 \cos \omega t = I_o + \text{Re}\; i_1 e^{j\omega t}$

PROBLEM 11.8 (continued)

We linearize $\bar{f}^e$ to obtain

$$\bar{f}^e = -\frac{N^2}{2}(\mu - \mu_o)[I_o^2 + 2I_o i_1 \cos \omega t]\bar{i}_2 \tag{g}$$

For equilibrium

$$\bar{F}_o - \frac{N^2}{2}(\mu - \mu_o)I_o^2 \bar{i}_2 = 0$$

Thus

$$\bar{F}_o = \frac{N^2}{2}(\mu - \mu_o)I_o^2 \bar{i}_2 \tag{h}$$

Part b

We write the solution to Eq. (a) in the form

$$\xi(x_1,t) = \text{Re}\ \hat{\xi}(x_1)\ e^{j\omega t}$$

where, from example 11.4.4

$$\hat{\xi}(x_1) = A_1 \sin \alpha x_1 + A_2 \cos \alpha x_1 + A_3 \sinh \alpha x_1 + A_4 \cosh \alpha x_1 \tag{i}$$

with

$$\alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4}$$

Now, from Eqs. 11.4.6 and 11.4.16

$$T_{11}(x=L) = E\frac{\partial \delta_1}{\partial x_1} = -Ex_2\frac{\partial^2\xi}{\partial x_1^2} = 0 \tag{j}$$

Thus

$$\frac{\partial^2\xi}{\partial x_1^2}(x_1 = L) = 0$$

From Eq. 11.4.21

$$T_{21} = \frac{(x_2^2 - b^2)}{2} E\frac{\partial^3\xi}{\partial x_1^3} \tag{k}$$

and from Eq. 11.4.16

$$\delta_1(x_1 = 0) = -x_2\left(\frac{\partial \xi}{\partial x_1}\right)_{x_1=0} = 0 \tag{ℓ}$$

Thus

$$\left(\frac{\partial \xi}{\partial x_1}\right)_{x_1=0} = 0$$

PROBLEM 11.8(continued)

Applying the boundary conditions from Eqs. (b), (c), (d) to our solution of Eq. (i), we obtain the four equations

$$\begin{aligned}
A_1 \qquad\qquad + A_3 \qquad\qquad &= 0 \\
- A_1 \sin \alpha L - A_2 \cos \alpha \ell + A_3 \sinh \alpha L + A_4 \cosh \alpha L &= 0 \\
- A_1 \cos \alpha L + A_2 \sin \alpha L + A_3 \cosh \alpha L + A_4 \sinh \alpha L &= 0 \\
- \frac{2}{3} \alpha^3 b^3 EDA_1 + M \omega^2 A_2 + \frac{2}{3} \alpha^3 b^3 EDA_3 + M\omega^2 A_4 &= + N^2 I_o i_1 (\mu - \mu_o)
\end{aligned} \tag{m}$$

Now

$$v = \frac{d\lambda}{dt} = \frac{d}{dt} \left\{ \frac{N^2 i}{D} D\left[\mu_o \xi(0) + \mu\left(D - \xi(0)\right)\right] \right\} \tag{n}$$

or

$$\hat{v} = - N^2 I_o (\mu - \mu_o) j\omega (A_2 + A_4) + N^2 i_1 \mu D j\omega \tag{o}$$

We solve Eqs. (m) for A_2 and A_4 using Cramer's rule to obtain

$$A_2 = \frac{N^2 I_o i_1 (\mu - \mu_o)(-1 + \sin \alpha L \sinh \alpha L - \cos \alpha L \cosh \alpha L)}{- 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3} (\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \tag{p}$$

and

$$A_4 = \frac{N^2 I_o i_1 (\mu - \mu_o)(- 1 - \cos \alpha L \cosh \alpha L - \sin \alpha L \sinh \alpha L)}{- 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3} (\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \tag{q}$$

Thus

$$Z(j\omega) = \frac{\hat{v}(j\omega)}{i_1} = \frac{+\left[N^2 I_o (\mu - \mu_o)\right]^2 j\omega (+ 2 + 2 \cos \alpha L \cosh \alpha L)}{- 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3} (\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} + N^2 \mu D j\omega \tag{r}$$

Part c

$Z(j\omega)$ has poles when

$$+ 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) = \frac{4}{3} (\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)$$

PROBLEM 11.9

Part a

The flux above and below the beam must remain constant. Therefore, the $\overline{H}$ field above is

$$\overline{H}_a = \frac{H_o(a-b)}{(a-b-\xi)}\overline{i}_1 \tag{a}$$

and the $\overline{H}$ field below is

$$\overline{H}_b = \frac{H_o(a-b)}{(a-b+\xi)}\overline{i}_1 \tag{b}$$

Using the Maxwell stress tensor, the magnetic force on the beam is

$$T_2 = -\frac{\mu_o}{2}(H_a^2 - H_b^2) = -\frac{\mu_o}{2} H_o^2 (a-b)^2\left(+\frac{4\xi}{(a-b)^3}\right)$$
$$= -\frac{2\mu_o H_o^2 \xi}{(a-b)} \tag{c}$$

Thus, from Eq. 11.4.26, the equation of motion on the beam is

$$\frac{\partial^2\xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4\xi}{\partial x_1^4} = -\frac{\mu_o H_o^2\xi}{(a-b)b\rho} \tag{d}$$

Again, we let

$$\xi(x_1,t) = \text{Re}\,\hat{\xi}(x_1)e^{j\omega t} \tag{e}$$

with the boundary conditions

$$\xi(x_1=0) = 0 \qquad \xi(x_1=L) = 0$$
$$\delta_1(x_1=0) \qquad \delta_1(x_1=L) = 0 \tag{f}$$

Since $\delta_1 = -x_2\,\partial\xi/\partial x_1$ from Eq. 11.4.16, this implies that:

$$\frac{\partial\xi}{\partial x_1}(x_1=0) = 0 \text{ and } \frac{\partial\xi}{\partial x_1}(x_1=L) = 0 \tag{g}$$

Substituting our assumed solution into the equation of motion, we have

$$-\omega^2\hat{\xi} + \frac{Eb^2}{3\rho}\frac{\partial^4\hat{\xi}}{\partial x_1^4} + \frac{\mu_o H_o^2\hat{\xi}}{(a-b)b\rho} = 0 \tag{h}$$

Thus we see that our solutions are again of the form

$$\hat{\xi}(x) = A\sin\alpha x + B\cos\alpha x + C\sinh\alpha x + D\cosh\alpha x \tag{i}$$

PROBLEM 11.9 (continued)

where now

$$\alpha = \left[\left(\omega^2 - \frac{\mu_o H_o^2}{(a-b)b\rho}\right)\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4} \tag{j}$$

Since the boundary conditions for this problem are identical to that of problem 11.7, we can take the solutions from that problem, substituting the new value of α. From problem 11.7, the solution must satisfy

$$\cos \alpha L \cosh \alpha L = 1 \tag{k}$$

The first resonance occurs when

$$\alpha L \approx \frac{3\pi}{2}$$

or

$$\omega^2 = \frac{\left(\frac{3\pi}{2}\right)^4\left(\frac{Eb^2}{3\rho}\right)}{L^4} + \frac{\mu_o H_o^2}{(a-b)b\rho} \tag{ℓ}$$

Part c

The resonant frequencies are thus shifted upward due to the stiffening effect of the constant flux constraint.

Part d

We see that, no matter what the values of the system parameters $\omega^2 > 0$, so ω will always be real, and thus stable. This is expected as the constant flux constraint imposes a force which opposes the motion.

PROBLEM 11.10

Part a

We choose a coordinate system as in Fig. 11.4.12, centered at the right end of the rod. Because $\frac{d}{D} = \frac{1}{10}$, we can neglect fringing and consider the right end of the rod as a capacitor plate. Also, since $\frac{D}{\ell} = \frac{1}{10}$, we can assume that the electrical force acts only at $x_1 = 0$. Thus, the boundary conditions at $x_1 = 0$ are

$$-\int_{-b}^{b} T_{21}\, D\, dx_2 + f^e = 0 \tag{a}$$

where $T_{21} = \frac{(x_2^2 - b^2)}{2} E \frac{\partial^3 \xi}{\partial x_1^3}$ (Eq. 11.4.21)

since the electrical force, f^e, must balance the shear stress T_{21} to keep the rod in equilibrium,

PROBLEM 11.10 (continued)

and

$$T_{11}(0) = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2}(0) = 0 \qquad \text{(b)}$$

since the end of the rod is free of normal stresses. At $x_1 = -\ell$, the rod is clamped so

$$\xi(-\ell) = 0 \qquad \text{(c)}$$

and

$$\delta_1(-\ell) = -x_2 \frac{\partial \xi}{\partial x_1}(-\ell) = 0 \qquad \text{(d)}$$

We use the Maxwell stress tensor to calculate the electrical force to be

$$f^e = \frac{\varepsilon A}{2}\left[\frac{(V_o + v_s)^2}{[d - \xi(0)]^2} - \frac{(v_s - V_o)^2}{[d + \xi(0)]^2}\right] \qquad \text{(e)}$$

$$\approx \frac{2\varepsilon A V_o}{d^2}\left[v_s + \frac{V_o\,\xi(0)}{d}\right]$$

The equation of motion of the beam is (example 11.4.4)

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4 \xi}{\partial x_1^4} = 0 \qquad \text{(f)}$$

We write the solution to Eq. (f) in the form

$$\xi(x,t) = \text{Re}\,\hat{\xi}(x)e^{j\omega t} \qquad \text{(g)}$$

where

$$\hat{\xi}(x) = A_1 \sin \alpha x + A_2 \cos \alpha x + A_3 \sinh \alpha x + A_4 \cosh \alpha x$$

with

$$\alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4}$$

Applying the four boundary conditions, Eqs. (a), (b), (c) and (d), we obtain the equations

$$-A_1 \sin \alpha\ell + A_2 \cos \alpha\ell - A_3 \sinh \alpha\ell + A_4 \cosh \alpha L = 0$$

$$A_1 \cos \alpha\ell + A_2 \sin \alpha\ell + A_3 \cosh \alpha\ell - A_4 \sinh \alpha\ell = 0 \qquad \text{(h)}$$

$$-A_2 + A_4 = 0$$

$$-\frac{2}{3} b^3 DE\alpha^3 A_1 + \frac{2\varepsilon_o A V_o^2}{d^3} A_2 + \frac{2}{3} b^3 DE\alpha^3 A_3 + \frac{2\varepsilon_o A V_o^2}{d^3} A_4 = -\frac{2\varepsilon_o A V_o \hat{v}_s}{d^2}$$

PROBLEM 11.10 (continued)

Now $i_s = \dfrac{dq_s}{dt}$ (i)

where
$$q_s = \frac{\varepsilon_o A}{d-\xi(0)}(V_o + v_s) + \frac{\varepsilon_o A(v_s - V_o)}{d + \xi(0)}$$

$$\approx \frac{2\varepsilon_o A v_s}{d} + \frac{2\varepsilon_o A V_o}{d^2}\,\xi(0) \qquad \text{(j)}$$

Therefore

$$\hat{i}_s = j\omega\frac{2\varepsilon_o A}{d}\left[\hat{v}_s + \frac{V_o}{d}\,\hat{\xi}(0)\right] \qquad \text{(k)}$$

where
$$\hat{\xi}(0) = A_2 + A_4$$

We use Cramer's rule to solve Eqs. (h) for A_2 and A_4 to obtain:

$$A_2 = A_4 = \frac{-\dfrac{\varepsilon_o A V_o \hat{v}_s}{d^2}\,[\cos\alpha\ell\sinh\alpha\ell - \sin\alpha\ell\cosh\alpha\ell]}{\dfrac{2}{3}b^3\alpha^3 DE(1 + \cos\alpha\ell\cosh\alpha\ell) + \dfrac{2\varepsilon_o A V_o^2}{d^3}(\cos\alpha\ell\sinh\alpha\ell - \sin\alpha\ell\cosh\alpha\ell)} \qquad (\ell)$$

Thus, from Eq. (k) we obtain

$$Z(j\omega) = \frac{d}{j\omega 2\varepsilon_o A}\left[1 + \frac{3\varepsilon_o A V_o^2}{d^3(\alpha b)^3 ED}\,\frac{(\cos\alpha\ell\sinh\alpha\ell - \sin\alpha\ell\cosh\alpha\ell)}{(1 + \cos\alpha\ell\cosh\alpha\ell)}\right] \qquad \text{(m)}$$

Part b

We define a function $g(\alpha\ell)$ such that Eq. (m) has a zero when

PROBLEM 11.10 (Continued)

$$(\alpha L)^3 g(\alpha L) = \frac{(1 + \cosh \alpha\ell \cos \alpha\ell)(\alpha\ell)^3}{\sin \alpha\ell \cosh \alpha\ell - \cos \alpha\ell \sinh \alpha\ell} = \frac{3\ell^3 V_o^2 A\varepsilon_o}{DEb^3 d^3} \qquad \text{(n)}$$

Substituting numerical values, we obtain

$$\frac{3\ell^3 V_o^2 A\varepsilon_o}{DEb^3 d^3} \approx \frac{3 \times 10^{-3}(10^6)10^{-4}(8.85 \times 10^{-12})}{10^{-2}(2.2 \times 10^{11})10^{-9}\ 10^{-9}} \approx 1.2 \times 10^{-3} \qquad \text{(o)}$$

In Figure 1, we plot $(\alpha\ell)^3 g(\alpha\ell)$ as a function of $\alpha\ell$. We see that the solution to Eq. (n) first occurs when $(\alpha\ell)^3 g(\alpha\ell) \approx 0$. Thus, the solution is approximately

$$\alpha\ell = 1.875$$

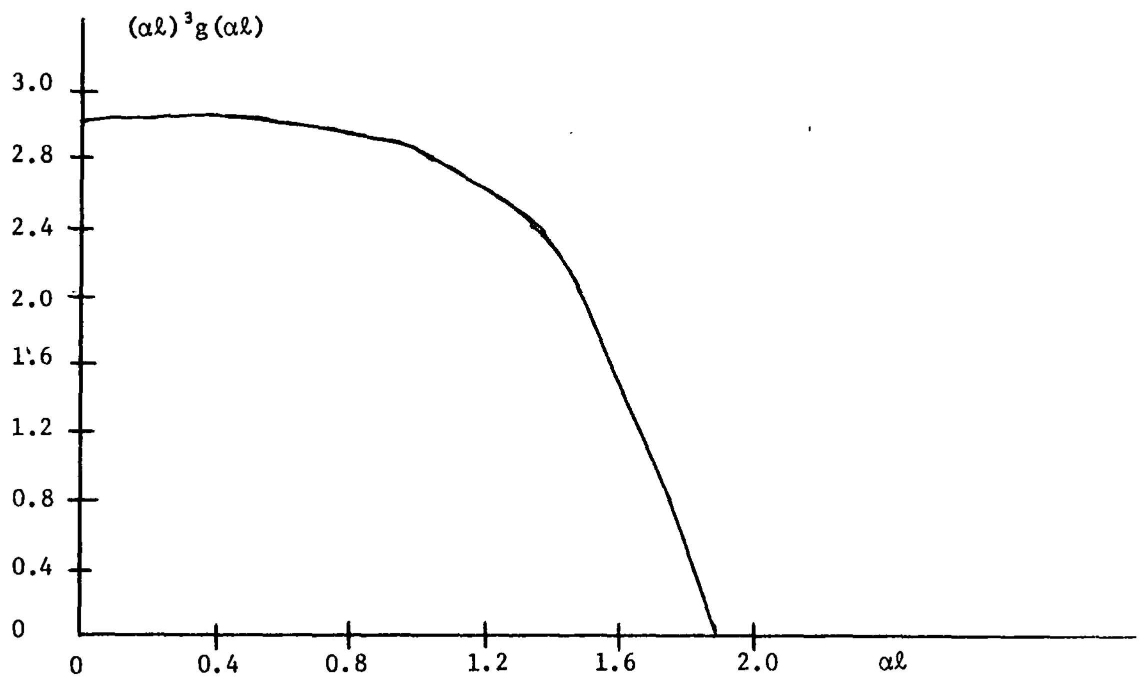


Figure 1

PROBLEM 11.10 (Continued)

From Eq. (g)

$$\alpha\ell = \left[\omega^2 \frac{3\rho}{Eb^2}\right]^{1/4} \ell = 1.875$$

Solving for ω, we obtain

$$\omega \approx 1080 \text{ rad/sec.} \qquad (p)$$

Part c

The input impedance of a series LC circuit is

$$Z(j\omega) = \frac{1 - LC\omega^2}{j\omega C} \qquad (q)$$

Thus the impedance has a zero when

$$\omega_o^2 = \frac{1}{LC} \qquad (r)$$

We let $\omega = \omega_o + \Delta\omega$, and expand (q) in a Taylor series around ω_o to obtain

$$Z(j\omega) \approx + j \frac{2\Delta\omega}{C\omega_o^2} = + 2j L\Delta\omega \qquad (s)$$

(m) can be written in the form

$$Z(j\omega) = \frac{1}{2j\omega C_o}[1 - f(\omega)] \quad \text{where } f(\omega_o) = 1 \qquad (t)$$

$$\text{and } C_o = \frac{\varepsilon_o A}{d}$$

For small deviations around ω_o

$$Z(j\omega) \approx \frac{j}{2\omega C_o} \left.\frac{\partial f}{\partial \omega}\right|_{\omega_o} \Delta\omega$$

Thus, from (q), (r) (s) and (t), we obtain the relations

$$2L = \frac{1}{2\omega C_o} \left.\frac{\partial f}{\partial \omega}\right|_{\omega_o} \qquad (u)$$

and $$C = \frac{1}{\omega_o^2 L} \qquad (v)$$

now $$f(\omega) = \frac{K}{(\alpha\ell)^3 g(\alpha\ell)} \qquad (w)$$

where $$K = \frac{3\ell^3 \varepsilon_o A V_o^2}{d^3 (EDb^3)} = 1.2 \times 10^{-3}$$

PROBLEM 11.10 (Continued)

and $g(\alpha\ell) = \dfrac{1 + \cos\alpha\ell\cosh\alpha\ell}{\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell}$

Thus, we can write

$$\left.\frac{df(\omega)}{d\omega}\right|_{\omega_o} = \left\{\frac{d}{d(\alpha\ell)}\left[\frac{K}{(\alpha\ell)^3 g(\alpha\ell)}\right]\frac{d(\alpha\ell)}{d\omega}\right\}_{\omega_o} \qquad (y)$$

Now from (g),

$$\left.\frac{d(\alpha\ell)}{d\omega}\right|_{\omega_o} = \left(\frac{3\rho}{Eb^2}\right)^{1/4}\frac{\ell}{2\omega_o^{1/2}} \qquad (z)$$

and

$$\frac{d}{d(\alpha\ell)}\left[\frac{K}{(\alpha\ell)^3 g(\alpha\ell)}\right]_{\omega_o} = \frac{-K}{[(\alpha\ell)^3 g(\alpha\ell)]^2}\frac{d}{d(\alpha\ell)}[(\alpha\ell)^3 g(\alpha\ell)]\bigg|_{\omega_o}$$

$$\approx -\frac{1}{K}\frac{d}{d(\alpha\ell)}[(\alpha\ell)^3 g(\alpha\ell)]\bigg|_{\omega_o} \qquad (aa)$$

since at $\omega = \omega_o$

$$(\alpha\ell)^3 g(\alpha\ell) = K. \qquad (bb)$$

Continuing the differentiating in (aa), we finally obtain

$$\frac{d}{d(\alpha\ell)}\left[\frac{(\alpha\ell)^3 g(\alpha\ell)}{-K}\right]\bigg|_{\omega_o} = -\frac{1}{K}\left[g(\alpha\ell)3(\alpha\ell)^2 + (\alpha\ell)^3\frac{d}{d(\alpha\ell)}g(\alpha\ell)\right]_{\omega_o}$$

$$= \frac{-3}{\alpha\ell}\bigg|_{\omega_o} - \frac{(\alpha\ell)^3}{K}\frac{d}{d(\alpha\ell)}g(\alpha\ell)\bigg|_{\omega_o} \qquad (cc)$$

Now

$$\frac{d}{d(\alpha\ell)}g(\alpha\ell) = \frac{-\sin\alpha\ell\cosh\alpha\ell + \cos\alpha\ell\sinh\alpha\ell}{(\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell)}$$

$$-\frac{(1+\cos\alpha\ell\cosh\alpha\ell)(+\cos\alpha\ell\cosh\alpha\ell + \sin\alpha\ell\sinh\alpha\ell + \sin\alpha\ell\sinh\alpha\ell - \cos\alpha\ell\cosh\alpha\ell)}{(\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell)}$$

$$= -1 - \frac{2g(\alpha\ell)(\sin\alpha\ell\sinh\alpha\ell)}{(\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell)} \qquad (dd)$$

PROBLEM 11.10 (Continued)

Substituting numerical values into the second term of (cc), we find it to have value much less than one at $\omega = \omega_o$.

Thus,

$$\frac{d}{d(\alpha\ell)} g(\alpha\ell) \approx -1 \qquad \text{(ee)}$$

Thus, using (y), (z), (aa) (bb) and (dd), we have

$$\left.\frac{df}{d\omega}\right|_{\omega_o} \approx \left(\frac{3\rho}{Eb^2}\right)^{1/4} \frac{\ell}{2\omega_o^{1/2}} \left[-\left.\frac{3}{\alpha\ell}\right|_{\omega_o} + \left.\frac{(\alpha\ell)^3}{K}\right|_{\omega_o} \right] \approx 4.8 \qquad \text{(ff)}$$

Thus, from (v) and (w)

$$L \approx \frac{4.8 \times 10^{-3}}{4(1080)(8.85 \times 10^{-12})(10^{-4})} = 1.25 \times 10^{9} \text{ henries}$$

and

$$C \approx \frac{1}{1.25 \times 10^{9}(1080)^2} = 6.8 \times 10^{-16} \text{ farads.}$$

PROBLEM 11.11

From Eq. (11.4.29), the equation of motion is

$$\rho \frac{\partial^2 \delta_3}{\partial t^2} = G\left(\frac{\partial^2 \delta_3}{\partial x_1^2} + \frac{\partial^2 \delta_3}{\partial x_2^2}\right) \quad (a)$$

We let

$$\delta_3 = \text{Re}\, \hat{\delta}(x_2) e^{j(\omega t - kx_1)} \quad (b)$$

Substituting this assumed solution into the equation of motion, we obtain

$$-\rho\omega^2 \hat{\delta} = G\left(- k^2 \hat{\delta} + \frac{\partial^2 \hat{\delta}}{\partial x_2^2}\right) \quad (c)$$

or

$$\frac{\partial^2 \hat{\delta}}{\partial x_2^2} + \left(\frac{\rho\omega^2}{G} - k^2\right)\hat{\delta} = 0 \quad (d)$$

If we let $\beta^2 = \frac{\rho\omega^2}{G} - k^2$ (e)

the solutions for $\hat{\delta}$ are:

$$\hat{\delta}(x_2) = A \sin \beta x_2 + B \cos \beta x_2 \quad (f)$$

The boundary conditions are

$$\hat{\delta}(0) = 0 \quad \text{and} \quad \hat{\delta}(d) = 0 \quad (g)$$

This implies that $B = 0$

and that $\beta d = n\pi$.

Thus, the dispersion relation is

$$\omega^2 \frac{\rho}{G} - k^2 = \left(\frac{n\pi}{d}\right)^2 \quad (h)$$

Part b

The sketch of the dispersion relation is identical to that of Fig. 11.4.19. However, now the n=0 solution is trivial, as it implies that

$$\hat{\delta}(x_2) = 0$$

Thus, there is no principal mode of propagation.

PROBLEM 11.12

From Eq. (11.4.1), the equation of motion is

$$\rho \frac{\partial^2 \delta}{\partial t^2} = (2G+\lambda)\nabla(\nabla\cdot\delta) - G\nabla\times(\nabla\times\delta) \tag{a}$$

We consider motions

$$\delta = \delta_\theta(r,z,t)\bar{i}_\theta \tag{b}$$

Thus, the equation of motion reduces to

$$\rho \frac{\partial^2 \delta_\theta}{\partial t^2} - G\left[\frac{\partial^2 \delta_\theta}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r\delta_\theta\right)\right] = 0 \tag{c}$$

We assume solutions of the form

$$\delta_\theta(r,z,t) = \mathrm{Re}\, \hat{\delta}(r)e^{j(\omega t - kz)} \tag{d}$$

which, when substituted into the equation of motion, yields

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r} r\hat{\delta}(r)\right] + \left(\frac{\rho\omega^2}{G} - k^2\right)\hat{\delta}(r) = 0 \tag{e}$$

From page 207 of Ramo, Whinnery and Van Duzer, we recognize solutions to this equation as

$$\hat{\delta}(r) = A\, J_1\left[\left(\frac{\rho\omega^2}{G} - k^2\right)^{1/2} r\right] + BN_1\left[\left(\frac{\rho\omega^2}{G} - k^2\right)^{1/2} r\right] \tag{f}$$

On page 209 of this reference there are plots of the Bessel functions J_1 and N_1. We must have B = 0 as at r = 0, N_1 goes to $-\infty$. Now, at r = R

$$\hat{\delta}(R) = 0 \tag{g}$$

This implies that

$$J_1\left[\left(\frac{\rho\omega^2}{G} - k^2\right)^{1/2} R\right] = 0 \tag{h}$$

If we denote α_i as the zeroes of J_1, i.e.

$$J_1(\alpha_i) = 0$$

we have the dispersion relation as

$$\frac{\rho}{G}\omega^2 - k^2 = \frac{\alpha_i^2}{R^2} \tag{i}$$

PROBLEM 12.1

Part a

Since we are in the steady state ($\partial/\partial t = 0$), the total forces on the piston must sum to zero. Thus

$$pLD + (f^e)_x = 0 \tag{a}$$

where $(f^e)_x$ is the upwards vertical component of the electric force

$$(f^e)_x = -\frac{\varepsilon_o V_o^2}{2x^2} LD \tag{b}$$

Solving for the pressure p, we obtain

$$p = \frac{\varepsilon_o V_o^2}{2x^2} \tag{c}$$

Part b

Because $\frac{d}{L} \ll 1$, we approximate the velocity of the piston to be negligibly small. Then, applying Bernoulli's equation, Eq. (12.2.11) right below the piston and at the exit nozzle where the pressure is zero, we obtain

$$\frac{1}{2}\rho V_p^2 = \frac{\varepsilon_o V_o^2}{2x^2} \tag{d}$$

Solving for V_p, we have

$$V_p = \frac{V_o}{x}\sqrt{\frac{\varepsilon_o}{\rho}} \tag{e}$$

Part c

The thrust T on the rocket is then

$$T = V_p \frac{dM}{dt} = V_p^2 \rho d D \tag{f}$$

$$= \frac{\varepsilon_o V_o^2}{x^2} dD$$

PROBLEM 12.2

Part a

The forces on the movable piston must sum to zero. Thus

$$pwD - f^e = 0 \tag{a}$$

where f^e is the component of electrical force normal to the piston in the direction of V, and p is the pressure just to the right of the piston.

$$f^e = \frac{\mu_o}{2}\frac{I^2 D}{w} \tag{b}$$

PROBLEM 12.2 (Continued)

Therefore

$$p = \frac{\mu_o I^2}{2w^2} \qquad \text{(c)}$$

Assuming that the velocity of the piston is negligible, we use Bernoulli's law, Eq. (12.2.11), just to the right of the piston and at the exit orifice where the pressure is zero, to write

$$\frac{1}{2}\rho V^2 = p \qquad \text{(d)}$$

or

$$V = \frac{I}{W}\sqrt{\frac{\mu_o}{\rho}} \qquad \text{(e)}$$

Part b

The thrust T is

$$T = V\frac{dM}{dt} = V^2\rho dW = \frac{\mu_o I^2 d}{W} \qquad \text{(f)}$$

Part c

For $I = 10^3 A$

$d = .1m$

$w = 1m$

$\rho = 10^3\ kg/m^3$

the exit velocity is

$$V = 3.5 \times 10^{-2}\ m/sec.$$

and the thrust is

$$T = .126\ newtons.$$

Within the assumption that the fluid is incompressible, we would prefer a dense material, for although the thrust is independent of the fluid's density, the exhaust velocity would decrease with increasing density, and thus the rocket will work longer. Under these conditions, we would prefer water in our rocket, since it is much more dense than air.

PROBLEM 12.3

Part a

From the results of problem 12.2, we have that the pressure p, acting just to the left of the piston, is

$$p = \frac{\mu_o I^2}{2w^2} \qquad \text{(a)}$$

The exit velocity at each orifice is obtained by using Bernoulli's law just to the left of the piston and at either orifice, from which we obtain

PROBLEM 12.3 (Continued)

$$V = \left(\frac{\mu_o}{\rho}\right)^{1/2} \frac{I}{w} \qquad \text{(b)}$$

at each orifice.

Part b

The thrust is

$$T = 2V \frac{dM}{dt} = 2V^2 \rho d w \qquad \text{(c)}$$

$$T = \frac{2\mu_o I^2 d}{w} \qquad \text{(d)}$$

PROBLEM 12.4

Part a

In the steady state, we choose to integrate the momentum theorem, Eq. (12.1.29), around a rectangular surface, enclosing the system from $-L \leq x_1 \leq + L$.

$$-\rho V_o^{\,2} a + \rho[V(L)]^2 b = P_o a - P(L)b + F \qquad \text{(a)}$$

where F is the x_1 component force per unit length which the walls exert on the fluid. We see that there is no x_1 component of force from the upper wall, therefore F is the force purely from the lower wall.

In the steady state, conservation of mass, (Eq. 12.1.8), yields

$$V(\ell) = V_o \frac{a}{b} \bar{i}_1 \qquad \text{(b)}$$

Bernoulli's equation gives us

$$\frac{1}{2} \rho V_o^2 + P_o = \frac{1}{2} \rho V_o^2 \frac{a^2}{b^2} + P(L) \qquad \text{(c)}$$

Solving (c) for P(L), and then substituting this result and that of (b) into (a), we finally obtain

$$F = P_o(b-a) + \rho V_o^2 \left(-a + \frac{b}{2} + \frac{a^2}{2b}\right) \qquad \text{(d)}$$

The problem asked for the force on the lower wall, which is just the negative of F.

Thus

$$F_{wall} = -P_o(b-a) - \rho V_o^2 \left(-a + \frac{a^2}{2b} + \frac{b}{2}\right) \qquad \text{(e)}$$

PROBLEM 12.5

Part a

We recognize this problem to be analogous to a dielectric or high-permeability cylinder placed in a uniform electric or magnetic field. The solutions are then dipole fields. We expect similar results here. As in Eqs. (12.2.1 - 12.2.3), we

PROBLEM 12.5 (continued)

define

$$\bar{V} = -\nabla\phi$$

and since

$$\nabla\cdot\bar{v} = 0$$

then $\nabla^2\phi = 0.$

Using our experience from the electromagnetic field problems, we guess a solution of the form

$$\phi = \frac{A}{r}\cos\theta + Br\cos\theta$$

Then

$$\bar{V} = \left(\frac{A}{r^2}\cos\theta - B\cos\theta\right)\bar{i}_r + \left(\frac{A}{r^2}\sin\theta + B\sin\theta\right)\bar{i}_\theta$$

Now, as $r \to \infty$

$$V = V_o\bar{i}_1 = V_o(\cos\theta\bar{i}_r - \bar{i}_\theta\sin\theta)$$

Therefore

$$B = -V_o$$

The other boundary condition at r = a is that

$$V_r(r=a) = 0$$

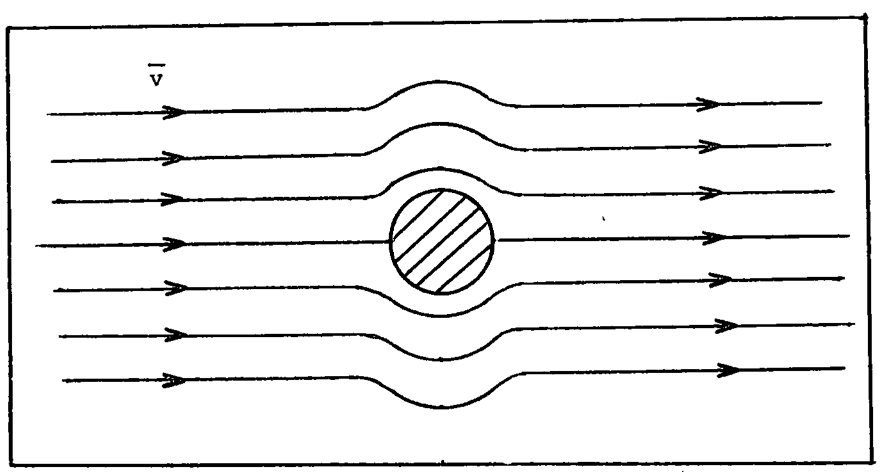
Thus

$$A = B\,a^2 = -V_o a^2$$

Therefore

$$\bar{V} = V_o\cos\theta\left(1 - \frac{a^2}{r^2}\right)\bar{i}_r - V_o\sin\theta\left(1 + \frac{a^2}{r^2}\right)\bar{i}_\theta$$

Part b



PROBLEM 12.5 (continued)

Part c

Using Bernoulli's law, we have

$$\frac{1}{2}\,\rho V_o^2 + p_o = \frac{1}{2}\,\rho V_o^2\,(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\theta) + P$$

Therefore the pressure is

$$P = p_o - \frac{1}{2}\,\rho V_o^2\,(\frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\,\theta)$$

Part d

We choose a large rectangular surface which encloses the cylinder, but the sides of which are far away from the cylinder. We write the momentum theorem as

$$\int_S \rho\bar{v}(\bar{v}\cdot\bar{n})da = -\int_S P d\bar{a} + \bar{F}$$

where $\bar{F}$ is the force which the cylinder exerts on the fluid. However, with our surface far away from the cylinder

$$V = V_o\bar{i}_1$$

and the pressure is constant

$$p = p_o.$$

Thus, integrating over the closed surface

$$\bar{F} = 0$$

The force which is exerted by the fluid on the cylinder is -F, which, however, is still zero.

PROBLEM 12.6

Part a

This problem is analogous to 12.5, only we are now working in spherical coordinates. As in Prob. 12.5,

$$\bar{V} = -\nabla\phi$$

In spherical coordinates, we try the solution to Laplace's equation

$$\phi = Ar\cos\theta + \frac{B}{r^2}\cos\theta \qquad (a)$$

Theta is measured clockwise from the x_1 axis.

Thus

$$\bar{V} = \left(-A\cos\theta + \frac{2B}{r^3}\cos\theta\right)\bar{i}_r + \bar{i}_\theta\left(A + \frac{B}{r^3}\right)\sin\theta \qquad (b)$$

As $r \to \infty$

$$\bar{V} \to V_o(\bar{i}_r\cos\theta - \bar{i}_\theta\sin\theta) \qquad (c)$$

Therefore $A = -V_o$ (d)

At $r = a$

$$V_r(a) = 0 \qquad (e)$$

Thus $\frac{2B}{a^3} = A = -V_o$

or

$$B = -\frac{V_o a^3}{2} \qquad (f)$$

Therefore

$$\bar{V} = V_o\left(1 - \frac{a^3}{r^3}\right)\cos\theta\,\bar{i}_r - V_o\left(1 + \frac{a^3}{2r^3}\right)\sin\theta\,\bar{i}_\theta \qquad (g)$$

with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Part b

At $r = a$, $\theta = \pi$, and $\phi = -\frac{\pi}{2}$

we are given that $p = 0$

At this point

$$\bar{V} = 0$$

Therefore, from Bernoulli's law

$$p = -\frac{1}{2}\rho V_o^2\left[\left(1 - \frac{a^3}{r^3}\right)^2\cos^2\theta + \sin^2\theta\left(1 + \frac{a^3}{2r^3}\right)^2\right] \qquad (h)$$

Part c

We realize that the pressure force acts normal to the sphere in the $-\bar{i}_r$ direction.

PROBLEM 12.6 (continued)

at r = a

$$p = -\frac{9}{8}\rho V_o^2 \sin^2\theta$$

We see that the magnitude of p remains unchanged if, for any value of θ, we look at the pressure at $\theta + \pi$. Thus, by the symmetry, the force in the x_1 direction is zero,

$$\bar{f}_1 = 0.$$

PROBLEM 12.7

Part a

We are given the potential of the velocity field as

$$\phi = \frac{V_o}{a} x_1 x_2. \qquad \bar{v} = -\nabla\phi = -\frac{V_o}{a}(x_2\bar{i}_1 + x_1\bar{i}_2)$$

If we sketch the equipotential lines in the x_1x_2 plane, we know that the velocity distribution will cross these lines at right angles, in the direction of decreasing potential.

Part b

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v}$$

$$= \left(\frac{V_o}{a}\right)^2 (x_1\bar{i}_1 + x_2\bar{i}_2) \tag{a}$$

$$\bar{a} = \left(\frac{V_o}{a}\right)^2 r\,\bar{i}_r \tag{b}$$

where

$r = \sqrt{x_1^2 + x_2^2}$ and $\bar{i}_r$ is a unit vector in the radial direction.

Part c

This flow could represent a fluid impinging normally on a flat plate, located along the line

$$x_1 + x_2 = 0.$$

See sketches on next page.

PROBLEM 12.8

Part a

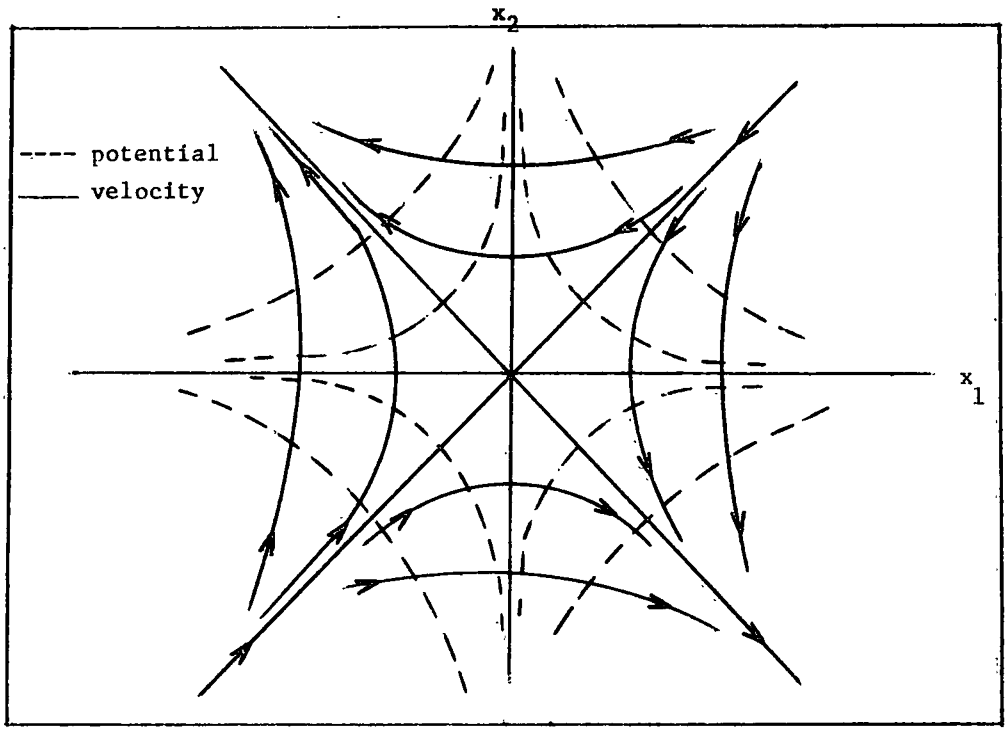
Given that

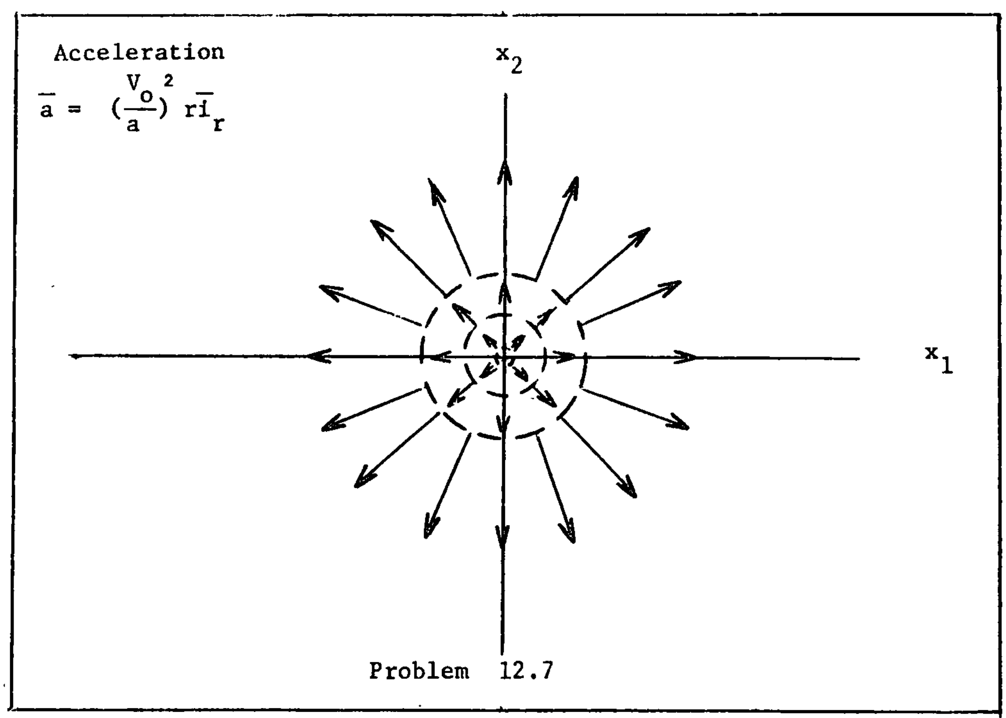
$$\bar{v} = \bar{i}_1 v_o \frac{x_2}{a} + \bar{i}_2 v_o \frac{x_1}{a} \tag{a}$$

we have that

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v}$$

$$= \left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}\right)\bar{v} \tag{b}$$


x_2
---- potential
—— velocity
x_1


Acceleration
$\bar{a} = (\frac{V_o^2}{a}) r\bar{i}_r$
x_2
x_1
Problem 12.7

PROBLEM 12.8 (Continued)

Thus

$$\bar{a} = v_o^2 \frac{x_1}{a^2} \bar{i}_1 + \left(\frac{v_o}{a^2}\right)^2 x_2 \bar{i}_2 \qquad (c)$$

Part b

Using Bernoulli's law, we have

$$p_o = \frac{1}{2} \rho \left(\frac{v_o}{a}\right)^2 (x_2^2 + x_1^2) + p \qquad (d)$$

$$p = p_o - \frac{1}{2} \rho \left(\frac{v_o}{a}\right)^2 (x_2^2 + x_1^2)$$

$$= p_o - \frac{1}{2} \rho v_o^2 \frac{r^2}{a^2} \qquad (e)$$

where

$$r = \sqrt{x_1^2 + x_2^2}$$

PROBLEM 12.9

Part a

The addition of a gravitational force will not change the velocity from that of Problem 12.8. Only the pressure will change. Therefore,

$$\bar{v} = \bar{i}_1 \frac{v_o}{a} x_2 + \bar{i}_2 \frac{v_o}{a} x_1 \qquad (a)$$

Part b

The boundary conditions at the walls are that the normal component of the velocity must be zero at the walls. Consider first the wall

$$x_2 - x_1 = 0 \qquad (b)$$

We take the gradient of this expression to find a normal vector to the curve. (Note that this normal vector does not have unit magnitude.)

$$\bar{n} = \bar{i}_2 - \bar{i}_1 \qquad (c)$$

Then

$$\bar{v} \cdot \bar{n} = \frac{v_o}{a} (x_1 - x_2) = 0 \qquad (d)$$

Thus, the boundary condition is satisfied along this wall.

Similarly, along the wall

$$x_2 + x_1 = 0 \qquad (e)$$

$$\bar{n} = \bar{i}_2 + \bar{i}_1 \qquad (f)$$

and

$$\bar{v} \cdot \bar{n} = \frac{v_o}{a} (x_1 + x_2) = 0 \qquad (g)$$

Thus, the boundary condition is satisfied here. Along the parabolic wall

$$x_2^2 - x_1^2 = a^2 \qquad (h)$$

$$\bar{n} = x_2 \bar{i}_2 - x_1 \bar{i}_1 \qquad (i)$$

PROBLEM 12.9 (Continued)

$$\bar{v}\cdot\bar{n} = \frac{V_o}{a}(x_1x_2 - x_1x_2) = 0 \qquad (j)$$

Thus, we have shown that along all the walls, the fluid flows purely tangential to these walls.

PROBLEM 12.10

Part a

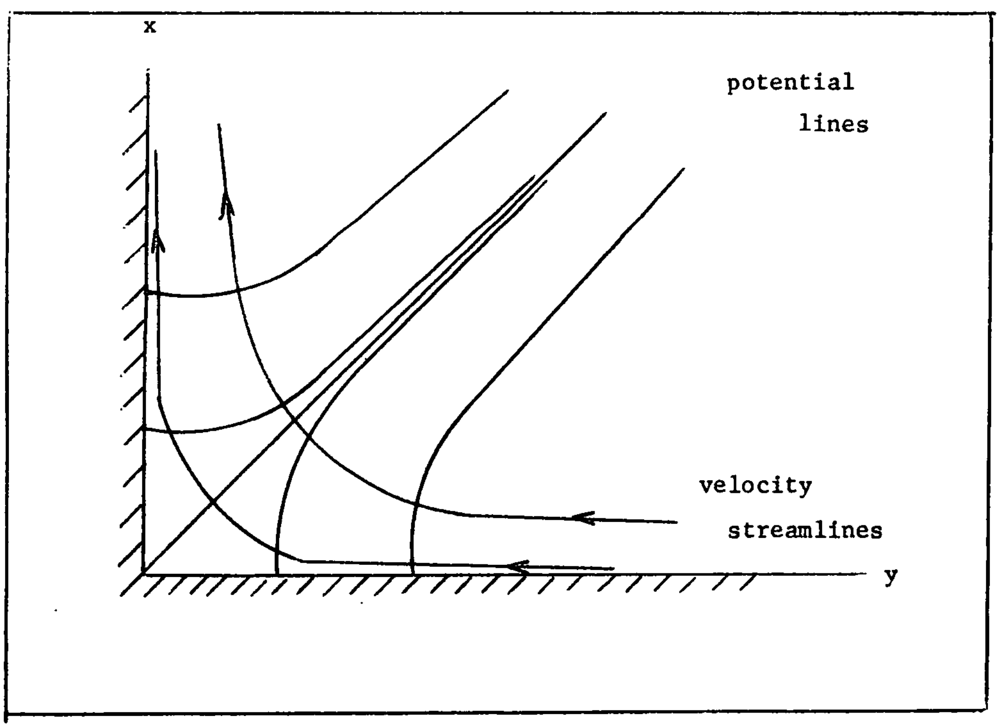
Along the lines x = 0 and y = 0, the normal component of the velocity must be zero. In terms of the potential, we must then have

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0 \qquad (a)$$

and

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=0} = 0 \qquad (b)$$

To aid in the sketch of $\phi(x,y)$, we realize that since at the boundary the velocity must be purely tangential, the potential lines must come in normal to the walls.



Part b

For the fluid to be irrotational and incompressible, the potential must obey

PROBLEM 12.10 (Continued)

Laplace's equaiion

$$\nabla^2\phi = 0 \tag{c}$$

From our sketch of part (a), and from the boundary conditions, we guess a solution of the form

$$\phi = -\frac{v_o}{a}(x^2 - y^2) \tag{d}$$

where $\frac{v_o}{a}$ is a scaling constant. By direct substitution, we see that this solution satisfies all the conditions.

Part c

For the potential of part (b), the velocity is

$$\bar{v} = -\nabla\phi = 2\frac{v_o}{a}(x\bar{i}_x - y\bar{i}_y) \tag{e}$$

Using Bernoulli's equation, we obtain

$$p_o = p + 2\left(\frac{v_o}{a}\right)^2 (x^2 + y^2) \tag{f}$$

The net force on the wall between x=c and x=d is

$$\bar{f} = \int_{z=0}^{z=w}\int_{x=c}^{x=d} (p_o - p)dxdz\,\bar{i}_y \tag{g}$$

where w is the depth of the wall.

Thus

$$\bar{f} = +\frac{\left(\frac{v_o}{a}\right)^2}{6} w \int_c^d x^2 dx\,\bar{i}_y$$

$$= +\frac{\left(\frac{v_o}{a}\right)^2}{6} w\,(d^3 - c^3)\bar{i}_y \tag{h}$$

Part d

The acceleration is

$$\bar{a} = (\bar{v}\cdot\nabla)\bar{v} = 2\frac{v_o}{a}x(2\frac{v_o}{a}\bar{i}_x) - 2\frac{v_o}{a}y\,(-2\frac{v_o}{a}y\,\bar{i}_y).$$

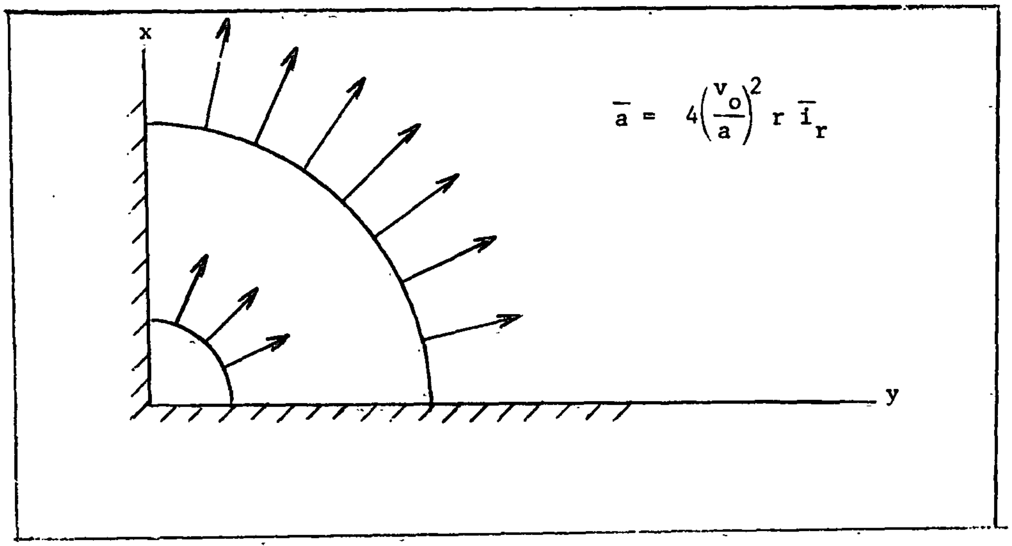
or

$$\bar{a} = 4\left(\frac{v_o}{a}\right)^2 (x\bar{i}_x + y\bar{i}_y) \tag{i}$$

or in cylindrical coordinates

$$\bar{a} = 4\left(\frac{v_o}{a}\right)^2 r\,\bar{i}_r \tag{j}$$

PROBLEM 12.10 (Continued)



PROBLEM 12.11

Part a

Since the $\nabla \cdot \bar{v} = 0$, we must have

$$V_o h = v_x(x)(h - \xi) \tag{a}$$

or

$$v_x(x) = \frac{V_o h}{h-\xi} \approx V_o \left(1 + \frac{\xi}{h}\right) \tag{b}$$

Part b

Using Bernoulli's law, we have

$$\frac{1}{2}\rho V_o + p_o = \frac{1}{2}\rho\,[V_x(x)]^2 + P \tag{c}$$

$$P = P_o + \frac{1}{2}\rho V_o^2 - \frac{1}{2}\rho V_o^2 \left(1 + \frac{\xi}{h}\right)^2 \tag{d}$$

Part c

We linearize P around $\xi = 0$ to obtain

$$P \approx P_o - \rho V_o^{\,2} \frac{\xi}{h} \tag{e}$$

Thus

$$T_z = -P + P_o = \rho V_o^2 \frac{\xi}{h} \tag{f}$$

PROBLEM 12.11 (continued)

Thus $$T_z = C\xi \qquad \text{(g)}$$

with
$$C = \frac{\rho V_o^2}{h}$$

Part d

We can write the equations of motoion of the membrane as

$$\sigma_m \frac{\partial^2\xi}{\partial t^2} = S\frac{\partial^2\xi}{\partial x^2} + T_z \qquad \text{(h)}$$

$$= S\frac{\partial^2\xi}{\partial x^2} + C\xi \qquad \text{(i)}$$

We assume

$$\xi(x,t) = \text{Re}\,\hat{\xi}\, e^{j(\omega t - kx)} \qquad \text{(j)}$$

Solving for the dispersion relation, we obtain

$$-\sigma_m\omega^2 = -Sk^2 + C \qquad \text{(k)}$$

or

$$\omega = \left[\frac{S}{\sigma_m}k^2 - \frac{C}{\sigma_m}\right]^{1/2} \qquad (\ell)$$

Now, since the membrane is fixed at x = 0 and x = L, we know that

$$k = \frac{n\pi}{\ell} \qquad n = 1,2,3,\ \ldots.. \qquad \text{(m)}$$

Now if

$$S\left(\frac{\pi}{\ell}\right)^2 - C < 0 \qquad \text{(n)}$$

we realize that the membrane will become unstable.

So for

$$\frac{\rho V_o^2}{h} < S\left(\frac{\pi}{\ell}\right)^2 \qquad \text{(o)}$$

we have stability.

Part e

As ξ increases, the velocity of the flow above the membrane increases, since the fluid is incompressible. Through Bernoulli's law, the pressure on the membrane must decrease, thereby increasing the net upwards force on the membrane, which tends to make ξ increase even further, thus making the membrane become unstable.

PROBLEM 12.12

Part a

We wish to write the equation of motion for the membrane.

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + p_1(\xi) - p_o + T^e - \sigma_m g \quad \text{(a)}$$

where

$$T^e = \frac{\varepsilon_o}{2}\left(\frac{V_o}{d-\xi}\right)^2 \approx \frac{\varepsilon_o}{2}\frac{V_o^2}{d^2}\left(1 + \frac{2\xi}{d}\right)$$

is the electric force per unit area on the membrane.

In the equilibrium $\xi(x,t) = 0$, we must have

$$p_1(0) = p_o - \frac{\varepsilon_o}{2}\left(\frac{V_o}{d}\right)^2 + \sigma_m g \quad \text{(b)}$$

As in example 12.1.3

$$p_1 = -\rho g y + C$$

and, using the boundary condition of (b), we obtain

$$p_1 = -\rho g y + \sigma_m g + p_o - \frac{\varepsilon_o}{2}\left(\frac{V_o}{d}\right)^2 \quad \text{(c)}$$

Part b

We are interested in calculating the perturbations in p_1 for small deflections of the membrane. From Bernoulli's law, a constant of motion of the fluid is D, where D equals

$$D = \frac{1}{2}\rho U^2 + \sigma_m g + p_o - \frac{\varepsilon_o}{2}\left(\frac{V_o}{d}\right)^2 \quad \text{(d)}$$

For small perturbations $\xi(x,t)$, the velocity in the region $0 \le x \le L$ is

$$v = \frac{Ud}{d+\xi}$$

We use Bernoulli's law to write

$$\frac{1}{2}\rho v^2 + p_1(\xi) + \rho g \xi = D \quad \text{(e)}$$

Since we have already taken care of the equilibrium terms, we are interested only in small changes of p_1, so we omit constant terms in our linearization of p_1.

Thus

$$p_1(\xi) = -\rho g \xi + \frac{\rho U^2 \xi}{d} \quad \text{(f)}$$

Thus, our linearized force equation is

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} + \left(\frac{\rho U^2}{d} - \rho g + \frac{\varepsilon_o V_o^2}{d^3}\right)\xi \quad \text{(g)}$$

We define

$$C = -\rho g + \frac{\rho U^2}{d} + \frac{\varepsilon_o V_o^2}{d^3}$$

and assume solutions of the form

$$\xi(x,t) = \mathrm{Re}\, \hat{\xi}\, e^{j(\omega t - kx)}$$

PROBLEM 12.12 (Continued)

from which we obtain the dispersion relation

$$\omega = \left(\frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m}\right)^{1/2} \tag{h}$$

Since the membrane is fixed at x=0 and at x=L

$$k = \frac{n\pi}{L}, \qquad n = 1, 2, 3, \ldots.. \tag{i}$$

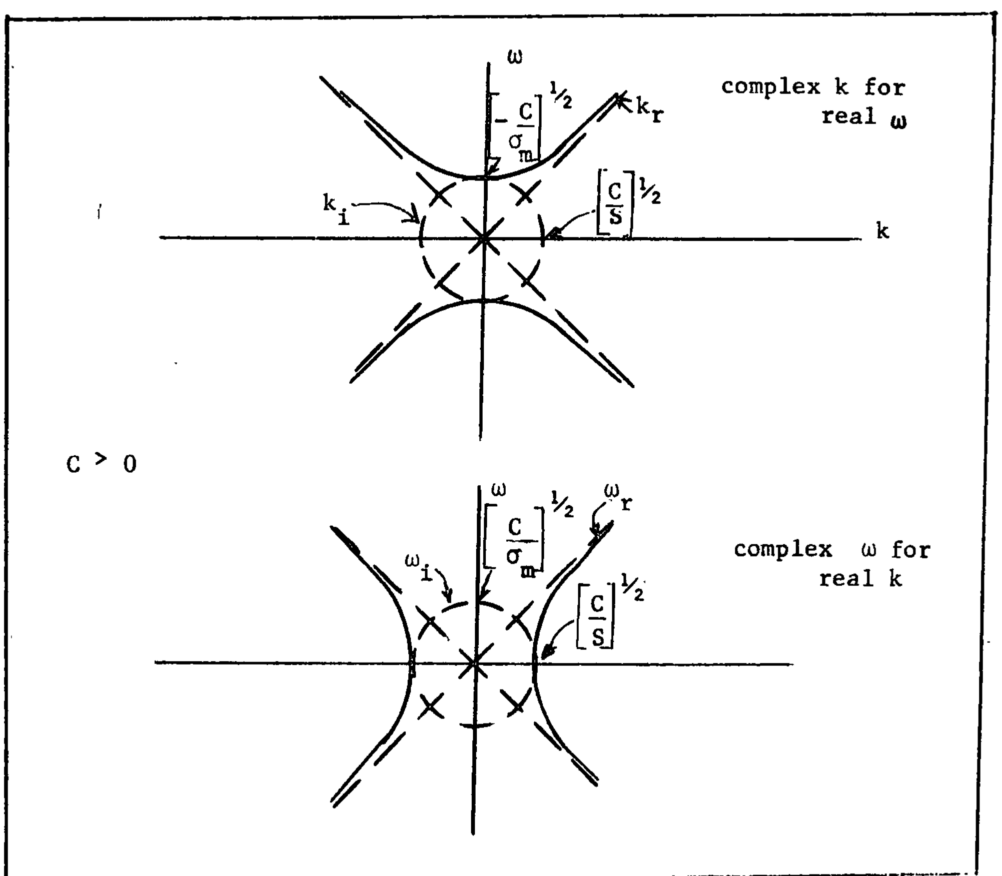
If $C < 0$, then ω is always real, and we can have oscillation about the equilibrium. For $C > S(\frac{\pi}{L})^2$, then ω will be imaginary, and the system is unstable.

Part c

The dispersion relation is thus

$$\omega = \left(\frac{S}{\sigma_m} k^2 - \frac{C}{\sigma_m}\right)^{1/2}$$

Consider first $C < 0$



PROBLEM 12.12 (Continued)

Part d

Since the membrane is not moving, one wave propagates upstream and the other propagates downstream. Thus, to find the solution we need two boundary conditions, one upstream and one downstream. If, however, both waves had propagated downstream, then causality does not allow us to apply a downstream boundary condition. This is not the case here.

PROBLEM 12.13

Part a

Since $\nabla\cdot\bar{v} = 0$, in the region $0 \leq x \leq L$,

$$v_x = \frac{V_o d}{d+\xi_1 - \xi_2} \approx v_o\left[1 - \frac{(\xi_1 - \xi_2)}{d}\right] \tag{a}$$

where d is the spacing between membranes. Using Bernoulli's law, we can find the pressure p_1 right below membrane 1, and pressure p_2 right above membrane 2. Thus

$$\frac{1}{2}\rho V_o^2 + p_o = \frac{1}{2}\rho v_x^2 + p_1 \tag{b}$$

and

$$\frac{1}{2}\rho V_o^2 + p_o = \frac{1}{2}\rho v_x^2 + p_2 \tag{c}$$

Thus

$$p_1 = p_2 \approx p_o + \frac{\rho V_o^2(\xi_1 - \xi_2)}{d} \tag{d}$$

We may now write the equations of motion of the membranes as

$$\sigma_m \frac{\partial^2\xi_1}{\partial t^2} = S\frac{\partial^2\xi_1}{\partial x^2} + (p_1 - p_o) = S\frac{\partial^2\xi_1}{\partial x^2} + \frac{\rho V_o^2(\xi_1 - \xi_2)}{d} \tag{e}$$

$$\sigma_m \frac{\partial^2\xi_2}{\partial t^2} = S\frac{\partial^2\xi_2}{\partial x^2} + p_o - p_2 = S\frac{\partial^2\xi_2}{\partial x^2} - \frac{\rho V_o^2(\xi_1 - \xi_2)}{d} \tag{f}$$

Assume solutions of the form

$$\begin{aligned}\xi_1 &= \text{Re}\ \hat{\xi}_1\, e^{j(\omega t - kx)}\\ \xi_2 &= \text{Re}\ \hat{\xi}_2\, e^{j(\omega t - kx)}\end{aligned} \tag{g}$$

Substitution of these assumed solutions into our equations of motion will yield the dispersion relation

$$\begin{aligned}-\sigma_m\omega^2\hat{\xi}_1 &= -Sk^2\hat{\xi}_1 + \frac{\rho V_o^2}{d}(\hat{\xi}_1 - \hat{\xi}_2)\\ -\sigma_m\omega^2\hat{\xi}_2 &= -Sk^2\hat{\xi}_2 + \frac{\rho V_o^2}{d}(\hat{\xi}_2 - \hat{\xi}_1)\end{aligned} \tag{h}$$

These equations may be rewritten as

PROBLEM 12.13 (Continued)

$$\hat{\xi}_1\left[-\sigma_m\omega^2 + Sk^2 - \frac{\rho V_o^2}{d}\right] + \hat{\xi}_2\left[+\frac{\rho V_o^2}{d}\right] = 0$$

$$\hat{\xi}_1\left[\frac{\rho V_o^2}{d}\right] + \hat{\xi}_2\left[-\sigma_m\omega^2 + Sk^2 - \frac{\rho V_o^2}{d}\right] = 0 \qquad (i)$$

For non-trivial solution, the determinant of coefficients of ξ_1 and ξ_2 must be zero.

Thus
$$\left[-\sigma_m\omega^2 + Sk^2 - \frac{\rho V_o^2}{d}\right]^2 = \left[\frac{\rho V_o^2}{d}\right]^2 \qquad (j)$$

or
$$-\sigma_m\omega^2 + Sk^2 - \frac{\rho V_o^2}{d} = \pm\frac{\rho V_o^2}{d} \qquad (k)$$

If we take the upper sign (+) on the right-hand side of the above equation, we obtain

$$\omega = \left[\frac{S}{\sigma_m}k^2 - \frac{2\rho V_o^2}{\sigma_m d}\right]^{1/2} \qquad (\ell)$$

We see that if V_o is large enough, ω can be imaginary. This can happen when

$$V_o^2 > \frac{Sk^2 d}{2\rho} \qquad (m)$$

Since the membranes are fixed at x=0 and x=L

$$k = \frac{n\pi}{L} \qquad n = 1,2,3,\ldots\ldots \qquad (n)$$

So the membranes first become unstable when

$$V_o^2 > \frac{S(\frac{\pi}{L})^2 d}{2\rho} \qquad (o)$$

For this choice of sign (+), $\xi_1 = -\xi_2$, so we excite the odd mode. If we had taken the negative sign, then the even mode would be excited

$$\xi_1 = \xi_2.$$

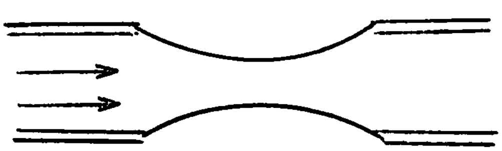
However, the dispersion relation is then

$$\omega = \pm\frac{S}{\sigma_m}k$$

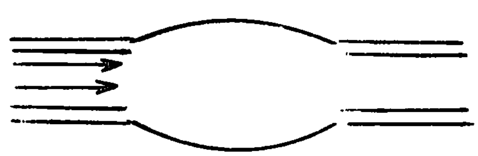
and then we would have no instability.

Part b

The odd mode is unstable.



or



PROBLEM 12.14

Part a

The force equation in the y direction is

$$\frac{\partial p}{\partial y} = -\rho g \tag{a}$$

Thus

$$p = -\rho g(y-\xi) \tag{b}$$

where we have used the fact that at $y=\xi$, the pressure is zero.

Part b

$\nabla\cdot\bar{v} = 0$ implies

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{c}$$

Integrating with respect to y, we obtain

$$v_y = -\frac{\partial v_x}{\partial x}\, y + C \tag{d}$$

where C is a constant of integration to be evaluated by the boundary condition at $y = -a$, that

$$v_y(y=-a) = 0$$

since we have a rigid bottom at $y = -a$.

Thus

$$v_y = -\frac{\partial v_x}{\partial x}(y+a) \tag{e}$$

Part c

The x-component of the force equation is

$$\rho\frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} = -\rho g\frac{\partial \xi}{\partial x} \tag{f}$$

or

$$\frac{\partial v_x}{\partial t} = -g\frac{\partial \xi}{\partial x} \tag{g}$$

Part d

At $y = \xi$,

$$v_y = \frac{\partial \xi}{\partial t} \tag{h}$$

Thus, from part (b), at $y = \xi$

$$\frac{\partial \xi}{\partial t} = -\frac{\partial v_x}{\partial x}(\xi+a) \tag{i}$$

However, since $\xi \ll a$, and v_x and v_y are small perturbation quantities, we can approximately write

$$\frac{\partial \xi}{\partial t} = -a\frac{\partial v_x}{\partial x} \tag{j}$$

Part e

Our equations of motion are now

PROBLEM 12.14 (Continued)

$$\frac{\partial \xi}{\partial t} = - a \frac{\partial v_x}{\partial x} \qquad (k)$$

and

$$\frac{\partial v_x}{\partial t} = - g \frac{\partial \xi}{\partial x} \qquad (\ell)$$

If we take $\partial/\partial x$ of (k) and $\partial/\partial t$ of (ℓ) and then simplify, we obtain

$$\frac{\partial^2 v_x}{\partial t^2} = ag \frac{\partial^2 v_x}{\partial x^2} \qquad (m)$$

We recognize this as the wave equation for gravity waves, with phase velocity

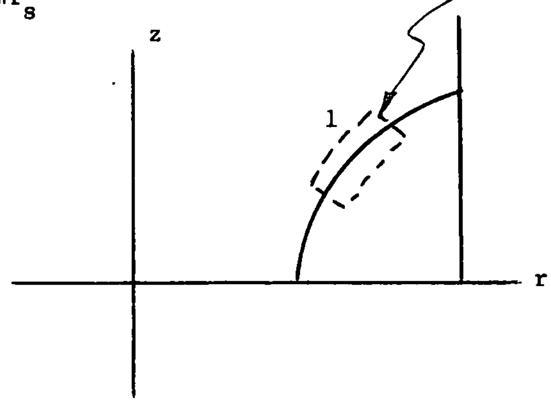
$$v_p = \sqrt{ag} \qquad (n)$$

PROBLEM 12.15

Part a

As shown in Fig. 12P.15b, the H field is in the $-\bar{i}_\theta$ direction with magnitude:

$$|H_s| = \frac{I_o}{2\pi r_s} \qquad (a)$$



If we integrate the MST along the surface defined in the above figure, the only contribution will be along surface (1), so we obtain for the normal traction

$$\tau_n = -\frac{1}{2}\mu_o|H_s|^2 = -\frac{1}{8}\frac{\mu_o I_o}{\pi^2 r_s{}^2} \qquad (b)$$

Part b

Since the net force on the interface must be zero, we must have

$$\tau_n + p_{int} - p_o = 0 \qquad (c)$$

where p_{int} is the hydrostatic pressure on the fluid side of the interface.

PROBLEM 12.15 (continued)

Thus
$$p_{int} = p_o + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 r^2} \qquad \text{(d)}$$

Within the fluid, the pressure p must obey the relation

$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{(e)}$$

or
$$p = -\rho g z + C \qquad \text{(f)}$$

Let us look at the point $z = z_o$, $r = R_o$. There

$$p = -\rho g z_o + C = p_o + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 R_o^2} \qquad \text{(g)}$$

Therefore
$$C = \rho g z_o + p_o + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 R_o^2} \qquad \text{(h)}$$

Now let's look at any point on the interface with coordinates z_s, r_s

Then, by Bernoulli's law,

$$p_o + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 R_o^2} + \rho g z_o = \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 r_s^2} + p_o + \rho g z_s \qquad \text{(i)}$$

Thus, the equation of the surface is

$$\rho g z_s + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 r_s^2} = \rho g z_o + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 R_o^2} \qquad \text{(j)}$$

Part c

The total volume of the fluid is

$$V = \pi [R_o^2 - (\tfrac{b}{2})^2]a. \qquad \text{(k)}$$

We can find the value of z_o by finding the volume of the deformed fluid in terms of z_o, and then equating this volume to V.

Thus
$$V = \pi [R_o^2 - (\tfrac{b}{2})^2]a = 2\pi \int_{r=r_o}^{R_o} \int_{z=0}^{\left[z_o + \frac{1}{8}\frac{\mu_o I_o^2}{\rho g \pi^2}\left(\frac{1}{R_o^2} - \frac{1}{r^2}\right)\right]} r\,dr\,dz \qquad (\ell)$$

where

r_o is that value of r when z = o, or

$$r_o = \left[\frac{\frac{1}{8}\frac{\mu_o I_o^2}{\pi^2}}{\rho g z_o + \frac{1}{8}\frac{\mu_o I_o^2}{\pi^2 R_o^2}}\right]^{1/2} \qquad \text{(m)}$$

Evaluating this integral, and equating to V, will determine z_o.

PROBLEM 12.16

We do an analysis similar to that of Sec. 12.2.1a, to obtain

$$E = -\overline{i}_y \frac{V}{w} \quad \text{(a)}$$

and

$$\overline{J} = \overline{i}_y\, \sigma\left(-\frac{V}{w} + vB\right) = \frac{I}{\ell d}\,\overline{i}_y \quad \text{(b)}$$

Here

$$V = IR + V_o \quad \text{(c)}$$

Thus

$$I = \frac{vBw - V_o}{R + \frac{w}{\ell d\sigma}}$$

The electric power out is

$$P_e = VI = (IR + V_o)I$$

$$= \left[V_o + \frac{R(vBw - V_o)}{R + \frac{w}{\ell d\sigma}}\right]\left[\frac{vBw - V_o}{R + \frac{w}{\ell d\sigma}}\right] \quad \text{(e)}$$

From equations (12.2.23 - 12.2.25)

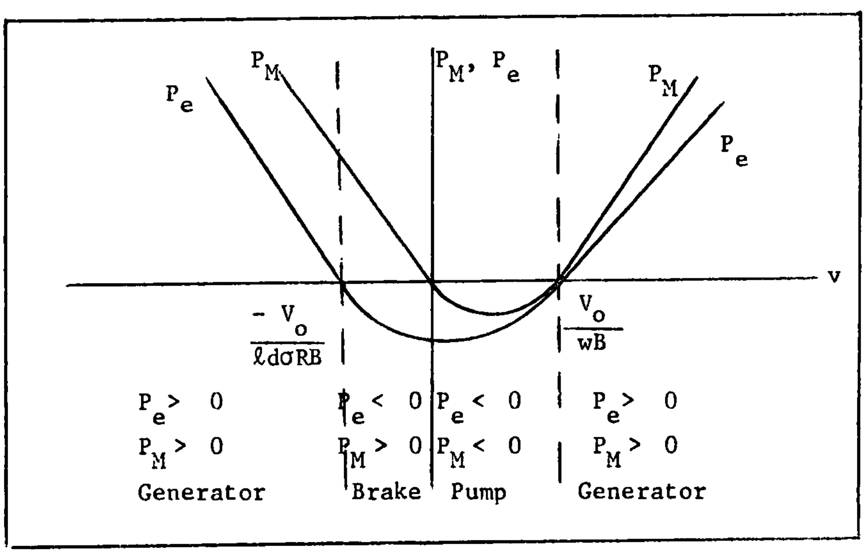
we have

$$\Delta p = p(0) - p(\ell) = \frac{IB}{d} \quad \text{(f)}$$

Thus, the mechanical power in is

$$P_M = (\Delta p w d)v = \frac{Bw(vBw - V_o)v}{R + \frac{w}{\ell d\sigma}} \quad \text{(g)}$$

Plots of P_E and P_M versus v specify the operating regions of the MHD machine.



PROBLEM 12.17

Part a

The mechanical power input is

$$P_M = -\int_{z=0}^{L}\int_{y=0}^{w}\int_{x=0}^{d} \nabla p v_o \, dxdydz \tag{a}$$

The force equation in the steady state is

$$-\nabla p + f^e = 0 \tag{b}$$

where

$$f^e = -J_y B_o \tag{c}$$

Thus

$$P_M = \int_{z=0}^{L}\int_{y=0}^{w}\int_{x=0}^{d} J_y B_o v_o \, dxdydz \tag{d}$$

Now

$$J_y = \sigma(E_y + v_o B_o) = \sigma(-\frac{\partial \phi}{\partial y} + v_o B_o) \tag{e}$$

Integrating, we obtain

$$P_M = \sigma v_o^2 B_o Lwd - \sigma B_o v_o VLd \tag{f}$$

$$= \frac{V_{oc}^2}{Ri} - \frac{VV_{oc}}{R_i} = \frac{1}{R_i}(V_{oc} - V)V_{oc}$$

Part b

Defining $\eta = \frac{P_{out}}{P_M}$

we have

$$\eta = \frac{(V_{oc} - V)V - aV^2}{(V_{oc} - V)V_{oc}} \tag{g}$$

First, we wish to find what terminal voltage maximizes P_{out}. We take

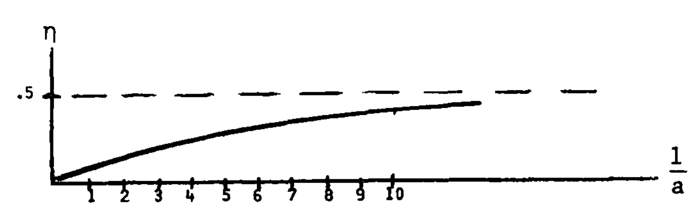
$\frac{\partial P_{out}}{\partial V} = 0$ and find that

$V = \frac{V_{oc}}{2(1+a)}$ maximizes P_{out}.

For this value of V, η equals

$$\eta = \frac{1}{2}\frac{1}{(1+2a)} \tag{h}$$

Plotting η vs. $\frac{1}{a}$ gives



PROBLEM 12.17(Continued)

Now, we wish to find what voltage will give maximum efficiency, so we take

$$\frac{\partial \eta}{\partial V} = 0$$

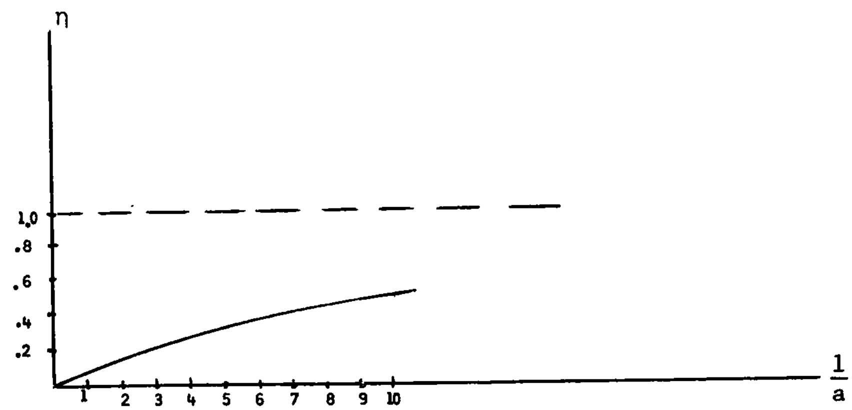
Solving for the maximum, we obtain

$$V = V_{oc}\left[1 \pm \sqrt{\frac{a}{1+a}}\right] \qquad (i)$$

We choose the negative sign, since $V \leq V_{oc}$ for generator operation. We thus obtain

$$\eta = 1 + 2a - 2\sqrt{a(1+a)} \qquad (j)$$

Plotting η vs. $\frac{1}{a}$, we obtain



PROBLEM 12.18

From Fig. 12P.18, we have

$$\bar{E} = \frac{V}{w}\bar{i}_y$$

and

$$\bar{J} = \bar{i}_y\sigma[\frac{V}{w} + vB] = \frac{I}{LD}\bar{i}_y \qquad (b)$$

The z component of the force equation is

$$-\frac{\partial p}{\partial z} - \frac{I}{LD}B = 0 \qquad (c)$$

or

$$\Delta p = p_i - p_o = \frac{IB}{D} = \Delta p_o(1 - \frac{v}{v_o}) \qquad (d)$$

Solving for v, we obtain

$$v = (1 - \frac{IB}{D\Delta p_o})v_o \qquad (e)$$

PROBLEM 12.18 (Continued)

Thus, we have

$$\frac{I}{LD\sigma} = \frac{V}{w} + B\left(1 - \frac{IB}{D\Delta p_o}\right)v_o \tag{f}$$

or

$$V = I\left(\frac{w}{LD\sigma} + \frac{B^2 v_o w}{D\Delta p_o}\right) - v_o Bw \tag{g}$$

Thus, for our equivalent circuit

$$R_i' = \frac{w}{LD\sigma} + \frac{v_o w B^2}{D\Delta p_o} \tag{h}$$

and

$$V_{oc} = -v_o wB \tag{i}$$

We notice that the current I in Fig. 12P.18b is not consistent with that of Fig. 12P.18a. It should be defined flowing in the other direction.

PROBLEM 12.19

Using Ampere's law

$$H_o = \frac{N_o I_o + N_L I_L}{d} \tag{a}$$

Within the fluid

$$\bar{J} = \frac{I_L}{\ell d}\bar{i}_z = \sigma\left(-\frac{V_L}{w} + v\mu_o H_o\right)\bar{i}_z \tag{b}$$

Simplifying, we obtain

$$I_L\left[\frac{1}{\ell d} - \frac{\sigma v \mu_o N_L}{d}\right] = \frac{\sigma v \mu_o N_o I_o}{d} - \frac{\sigma V_L}{w} \tag{c}$$

For V_L to be independent of I_L, we must have

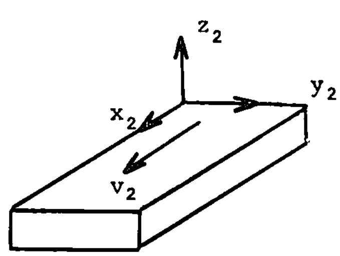
$$\frac{\sigma v \mu_o N_L}{d} = \frac{1}{\ell d} \tag{d}$$

or

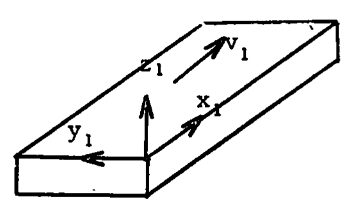
$$N_L = \frac{1}{\ell \sigma v \mu_o} \tag{e}$$

PROBLEM 12.20

We define coordinate systems as shown below.



MHD #2



MHD # 1

PROBLEM 12.20 (Continued)

Now, since $\nabla\cdot\bar{v} = 0$, we have

$$v_1 w_1 d_1 = v_2 w_2 d_2$$

In system (2),

$$\bar{J}_2 = \bar{i}_{y2}\frac{I_2}{\ell_2 d_2} = -\sigma\left(\frac{V_2}{w_2} + v_2 B\right)\bar{i}_{y2} \quad \text{(a)}$$

and

$$\Delta p_2 = p(0_+) - p(\ell_{2-}) = -\frac{I_2 B}{d_2} \quad \text{(b)}$$

In system (1),

$$\bar{J}_1 = \bar{i}_{y1}\frac{I_1}{\ell_1 d_1} = \sigma\left(\frac{V_1}{w_1} - v_1 B\right) \quad \text{(c)}$$

and

$$\Delta p_1 = p(0_+) - p(\ell_{1-}) = -\frac{I_1 B}{d_1} \quad \text{(d)}$$

By applying Bernoulli's law at the points $x_1 = 0_-$ (right before MHD system 1) and at $x_1 = \ell_{1+}$ (right after MHD system 1), we obtain

$$\frac{1}{2}\rho v_1^2 + p_1(0_-) = \frac{1}{2}\rho v_1^2 + p_1(\ell_{1+}) \quad \text{(e)}$$

or

$$p_1(0_-) = p_1(\ell_{1+}) \quad \text{(f)}$$

Similarly on MHD system (2):

$$p_2(0_-) = p_2(\ell_{2+}) \quad \text{(g)}$$

Now,

$$\oint_C \nabla p\cdot d\ell = 0$$

Applying this relation to a closed contour which follows the shape of the channel, we obtain

$$\oint_C \nabla p\cdot d\ell = \int_{x_1=0_+}^{\ell_{1-}} \nabla p\cdot d\ell + \int_{x_1=\ell_{1+}}^{x_2=0_-} \nabla p\cdot d\ell + \int_{x_2=0_+}^{x_2=\ell_{2-}} \nabla p\cdot d\ell + \int_{x_2=\ell_{2+}}^{x_1=0_-} \nabla p\cdot d\ell$$

$$= p_1(\ell_{1-}) - p_1(0_+) + p_2(0_-) - p_1(\ell_{1+}) + p_2(\ell_{2-})$$
$$- p_2(0_+) + p_1(0_-) - p_2(\ell_{2+}) \quad \text{(h)}$$

From (f) and (g) we reduce this to

$$\Delta p_1 + \Delta p_2 = 0 \quad \text{(i)}$$

or

$$\frac{I_1}{d_1} = \frac{-I_2}{d_2} \quad \text{(j)}$$

PROBLEM 12.20 (Continued)

Thus, we may express v_1 as

$$v_1 = \left(+ \frac{I_2}{\ell_1 d_2 \sigma} + \frac{V_1}{w_1} \right) \frac{1}{B} \tag{k}$$

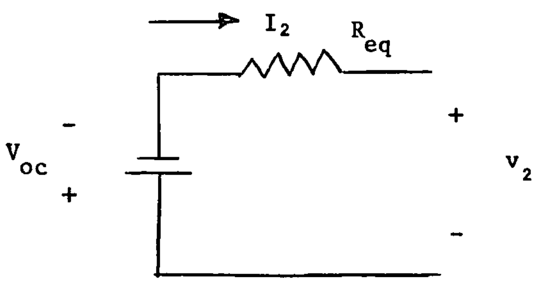
We substitute this into our original equation for J_2 (a), to obtain

$$\frac{I_2}{\ell_2 d_2} = -\sigma \frac{V_2}{w_2} - \sigma \left(\frac{w_1 d_1}{w_2 d_2}\right)\left(\frac{I_2}{\ell_1 d_2 \sigma} + \frac{V_1}{w}\right) \tag{ℓ}$$

This may be rewritten as

$$V_2 = -I_2 \frac{w_2}{\sigma} \left[\frac{1}{\ell_2 d_2} + \frac{w_1 d_1}{w_2 \ell_1 d_2^2} \right] - \frac{d_1}{d_2} V_1 \tag{m}$$

The Thevenin equivalent circuit is:



where $V_{oc} = \frac{d_1}{d_2} V_1$

and

$$R_{eq} = \frac{w_2}{\sigma d_2} \left[\frac{1}{\ell_2} + \frac{w_1 d_1}{w_2 d_2 \ell_1} \right]$$

PROBLEM 12.21

For the MHD system

$$|\bar{J}| = \frac{I}{LW} = \sigma\left(\frac{V_o}{D} - v\mu_o H_o \right) \tag{a}$$

and

$$\Delta p = p_1 - p_2 = + \frac{I\mu_o H_o}{w} \tag{b}$$

Now, since

$$\oint_C \nabla p \cdot d\ell = 0 \tag{c}$$

we must have

$$\Delta p = kv = \mu_o H_o L\sigma\left(\frac{V_o}{D} - v\mu_o H_o \right) \tag{d}$$

Solving for v, we obtain

$$v = \frac{\mu_o H_o L\sigma V_o}{D[k+(\mu_o H_o)^2 L\sigma]} \tag{e}$$

PROBLEM 12.22

Part a

We assume that the fluid flows in the +x direction with velocity v. Thus

$$\bar{J} = \bar{i}_3 \frac{I}{Lw} = \sigma\left(\frac{V}{d} + v\mu_o H_o\right) \bar{i}_3 \qquad \text{(a)}$$

where I is defined as flowing out of the positive terminal of the voltage source V_o. We write the x_1 component of the force equation as

$$-\frac{\partial p}{\partial x_1} - \frac{I\mu_o H_o}{L\,w} - \rho g = 0 \qquad \text{(b)}$$

Thus

$$p = -\left(\frac{I\mu_o H_o}{Lw} + \rho g\right) x_1 \qquad \text{(c)}$$

For $\Delta p = p(0) - p(L) = 0$

Then

$$\frac{I\mu_o H_o}{Lw} = -\rho g \qquad \text{(d)}$$

For the external circuit shown,

$$V = -IR + V_o \qquad \text{(e)}$$

Solving for I we get

$$I = \frac{\frac{V_o}{d} + v\mu_o H_o}{\frac{1}{\sigma Lw} + \frac{R}{d}} = -\frac{\rho g L w}{\mu_o H_o} \qquad \text{(f)}$$

Solving for the velocity, v, we get

$$v = \frac{-\frac{\rho g L w}{\mu_o H_o}\left(\frac{1}{\sigma L w} + \frac{R}{d}\right) - \frac{V_o}{d}}{\mu_o H_o} \qquad \text{(g)}$$

For v > 0, then

$$V_o < \frac{-\rho g}{\mu_o H_o}\left(\frac{d}{\sigma} + RLw\right) \qquad \text{(h)}$$

Part b

If the product $V_o I > 0$, then we are supplying electrical power to the fluid. From part (a), (f) and (h), V_o is always negative, but so is I. So the product $V_o I$ is positive.

PROBLEM 12.23

Since the electrodes are short-circuited,

$$\bar{J} = \bar{i}_z \frac{I}{\ell d} = \sigma v B_o \bar{i}_z \quad (a)$$

In the upper reservoir

$$p_1 = p_o + \rho g(h_1 - y) \quad (b)$$

while in the lower reservoir

$$p_2 = p_o + \rho g(h_2 - y) \quad (c)$$

The pressure drop within the MHD system is

$$\Delta p = p(0) - p(\ell) = \frac{IB}{d} \quad (d)$$

Integrating along the closed contour from $y=h_1$ through the duct to $y=h_2$, and then back to $y=h_1$ we obtain

$$-\oint_C \nabla p \cdot d\ell = 0 = -\rho g(h_1 - h_2) + \frac{IB}{d}\ell \quad (e)$$

Thus

$$I = \frac{\rho g(h_1 - h_2)d}{B\ell} \quad (f)$$

and so

$$v = \frac{I}{\sigma \ell d B_o} = \frac{\rho g(h_1 - h_2)}{\sigma \ell^2 B_o} \quad (g)$$

PROBLEM 12.24

Part a

We define the velocity v_h as the velocity of the fluid at the top interface, where

$$v_h = -\frac{dh}{dt} \quad (a)$$

Since $\nabla \cdot v = 0$, we have

$$v_h A = v_e w D \quad (b)$$

where v_e is the velocity of flow through the MHD generator (assumed constant). We assume that accelerations of the fluid are negligible. When we obtain the solution, we must check that these approximations are reasonable. With these approximations, the pressure in the storage tank is

$$p = -\rho g(y-h) + p_o \quad (c)$$

where p_o is the atmospheric pressure and y the vertical coordinate. The pressure drop in the MHD generator is

$$\Delta p = \frac{I\mu_o H_o}{D} \quad (d)$$

where I is defined positive flowing from right to left within the generator in the end view of Fig. 12P.24.

PROBLEM 12.24 (continued)

We have also assumed that within the generator, v_e does not vary with position. The current within the generator is

$$\frac{I}{L_o D} = \sigma(-\frac{IR}{w} + v_e \mu_o H_o) \qquad (e)$$

Solving for I, we obtain

$$I = \frac{v_e \mu_o H_o}{\frac{1}{\sigma L_o D} + \frac{R}{w}} \qquad (f)$$

Now, since $\oint \nabla p \cdot d\ell = 0$, we have

$$\Delta p - \rho g h = 0 \qquad (g)$$

Thus, using (d), (f) and (g), we obtain

$$-\rho g h + \frac{(\mu_o H_o)^2}{D}\left[\frac{1}{\frac{R}{w} + \frac{1}{\sigma L_o D}}\right] v_e = 0 \qquad (h)$$

Using (b), we finally obtain

$$\frac{dh}{dt} + sh = 0 \qquad (i)$$

where

$$s = \frac{\rho g\, w}{(\mu_o H_o)^2} \frac{D}{A}\left[\frac{RD}{w} + \frac{1}{\sigma L_o}\right]$$

Thus

$$h = 10\, e^{-st} \text{, until time } \tau \text{, when the valve closes at } h = 5. \qquad (j)$$

Numerically

$$s = 7.1 \times 10^{-3} \text{ , thus } \tau \approx 100 \text{ seconds.}$$

For our approximations to be valid, we must have

$$\left|\rho \frac{\partial v_h}{\partial t}\right| << \rho g \qquad (k)$$

or

$$s^2 h << g.$$

Also, we must have

$$\left|\frac{1}{2}\rho v_h^2\right| << |\rho g h|$$

or

$$\frac{1}{2} s^2 h << g \qquad (\ell)$$

Our other approximation was

$$\left|\rho L_o \frac{\partial v_e}{\partial t}\right| << \left|\frac{I \mu_o H_o}{D}\right| \qquad (m)$$

which implies from (f) that

PROBLEM 12.24 (continued)

$$\rho s L_o << \frac{(\mu_o H_o)^2}{D\left[\frac{R}{w} + \frac{1}{\sigma L_o D}\right]} \tag{n}$$

Substituting numerical values, we see that our approximations are all reasonable.

Part b

From (b) and (f)

$$I = \frac{\mu_o H_o A}{wd\left[\frac{1}{\sigma L_o D} \quad \frac{R}{w}\right]} \frac{\partial h}{\partial t}$$

$$= - 650 \times 10^3 \; e^{-st} \text{ amperes.}$$

until $t = 100$ seconds, where $I = -325 \times 10^3$ amperes. Once the valve is closed, $I = 0$.

PROBLEM 12.25

Part a

Within the MHD system

$$\bar{J} = \frac{-i}{L_1 D}\bar{i}_3 = -\sigma\left(\frac{V}{w} - v\mu_o H_o\right)\bar{i}_3 \quad \text{where } V = -iR + V_o \tag{a}$$

and

$$\Delta p = p(0) - p(-L_1) = \frac{i\mu_o H_o}{D} \tag{b}$$

We are considering static conditions (v=0) so the pressure in tank 1 is

$$p_1 = -\rho g(x_2 - h_1) + p_o \tag{c}$$

and in tank 2 is

$$p_2 = -\rho g(x_2 - h_2) + p_o \tag{d}$$

where p_o is the atmospheric pressure,

thus

$$i = \frac{V_o}{w\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]} \tag{e}$$

Now since $\oint_C \nabla p \cdot d\ell = 0$, we must have

$$+\rho g h_1 + \frac{i\mu_o H_o}{D} - \rho g h_2 = 0 \tag{f}$$

Solving in terms of V_o we obtain

$$V_o = \frac{\rho g(h_2 - h_1)wD}{(\mu_o H_o)}\left(\frac{1}{\sigma L_1 D} + \frac{R}{w}\right) \tag{g}$$

For $h_2 = .5$ and $h_1 = .4$ and substituting for the given values of the parameters, we obtain

$$V_o = 6.3 \text{ millivolts}$$

Under these static conditions, the current delivered is

$$i = \frac{\rho g(h_2 - h_1)D}{\mu_o H_o} = 210 \text{ amperes}$$

and the power delivered is

$$P_e = V_o i = \left[\frac{\rho g(h_2 - h_1)D}{\mu_o H_o}\right]^2 w\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right] = 1.33 \quad \text{watts}$$

Part b

We expand h_1 and h_2 around their equilibrium values h_{10} and h_{20} to obtain

$$h_1 = h_{10} + \Delta h_1$$
$$h_2 = h_{20} + \Delta h_2$$

PROBLEM 12.25 (Continued)

Since the total volume of the fluid remains constant

$$\Delta h_2 = -\Delta h_1$$

Since we are neglecting the acceleration in the storage tanks, we may still write

$$\begin{aligned} p_1 &= -\rho g(x_2 - h_1) + p_o \\ p_2 &= -\rho g(x_2 - h_2) + p_o \end{aligned} \tag{h}$$

Within the MHD section, the force equation is

$$\rho \frac{\partial v}{\partial t} = -\nabla p_{MHD} + \frac{i\mu_o H_o}{L_1 D} \tag{i}$$

Integrating with respect to x_1, we obtain

$$\Delta p_{MHD} = p(0) - p(-L) = \frac{i\mu_o H_o}{L_1 D} - \rho L_1 \frac{\partial v}{\partial t} \tag{j}$$

The pressure drop over the rest of the pipe is

$$\Delta p_{pipe} = -L_2 \rho \frac{dv}{dt}$$

Again, since $\oint_C \nabla p \cdot d\ell = 0$, we have

$$\rho g(h_1 - h_2) + \Delta p_{MHD} + \Delta p_{pipe} = 0 \tag{k}$$

For t > 0 we have

$$i = \frac{\frac{2V_o}{w} - v\mu_o H_o}{\frac{1}{\sigma L_1 D} + \frac{R}{w}} \tag{ℓ}$$

and substituting into the above equation, we obtain

$$\rho g(h_1 - h_2) - \rho(L_1 + L_2) \frac{\partial v}{\partial t} + \frac{\left(\frac{2V_o}{w} - v\mu_o H_o\right)}{\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]} \frac{\mu_o H_o}{D} = 0 \tag{m}$$

We desire an equation just in Δh_2. From the $\nabla \cdot v = 0$, we obtain

$$vwD = \frac{d\Delta h_2}{dt} A \tag{n}$$

Making these substitutions, the resultant equation of motion is

$$\frac{d^2\Delta h_2}{dt^2} + \frac{(\mu_o H_o)^2}{\rho(L_1 + L_2) D\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]} \frac{d\Delta h_2}{dt} + \frac{2gwd\Delta h_2}{(L_1 + L_2)A}$$

$$= \frac{V_o \mu_o H_o}{\rho(L_1 + L_2) A \left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]} \tag{o}$$

PROBLEM 12.25 (continued)

Solving, we obtain

$$\Delta h_2 = \frac{V_o \mu_o H_o}{2\rho g w d\left(\frac{1}{\sigma L_1 D} + \frac{R}{w}\right)} + B_1 e^{s_1 t} + B_2 e^{s_2 t} \quad \text{(p)}$$

where B_1 and B_2 are arbitrary constants to be determined by initial conditions and

$$s_{\frac{1}{2}} = -\frac{[(\mu_o H_o)^2]}{2\rho(L_1+L_2)D\left(\frac{1}{\sigma L_1 D} + \frac{R}{w}\right)} \pm \sqrt{\left(\frac{[\mu_o H_o]^2}{2\rho[L_1+L_2]D\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]}\right)^2 - \frac{2gwd}{(L_1+L_2)A}} \quad \text{(q)}$$

Substituting values, we obtain approximately

$$s_1 = -.025 \ \text{sec.}^{-1}$$
$$s_2 = -.94 \ \text{sec.}^{-1}$$

The initial conditions are

$$\Delta h_2(t=0) = 0$$

and $$\frac{d\Delta h_2(t=0)}{dt} = 0$$

Thus, solving for B_1 and B_2 we have

$$B_1 = \frac{-V_o \mu_o H_o}{2\rho g w D\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]\left(1 - \frac{s_1}{s_2}\right)} = -.051 \quad \text{(r)}$$

$$B_2 = \frac{-V_o \mu_o H_o}{2\rho g w D\left[\frac{1}{\sigma L_1 D} + \frac{R}{w}\right]\left(1 - \frac{s_2}{s_1}\right)} = +1.36 \times 10^{-3}$$

Thus

$$h_2(t) = h_{20} + \Delta h_2(t) = .55 + 1.36 \times 10^{-3} e^{-.94t} - .051 e^{-.025t} \quad \text{(s)}$$

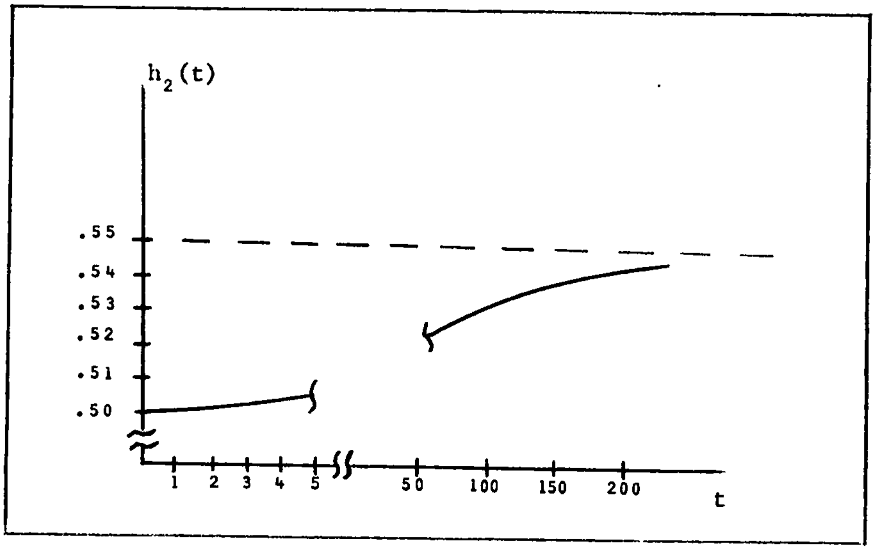
From (ℓ) we have

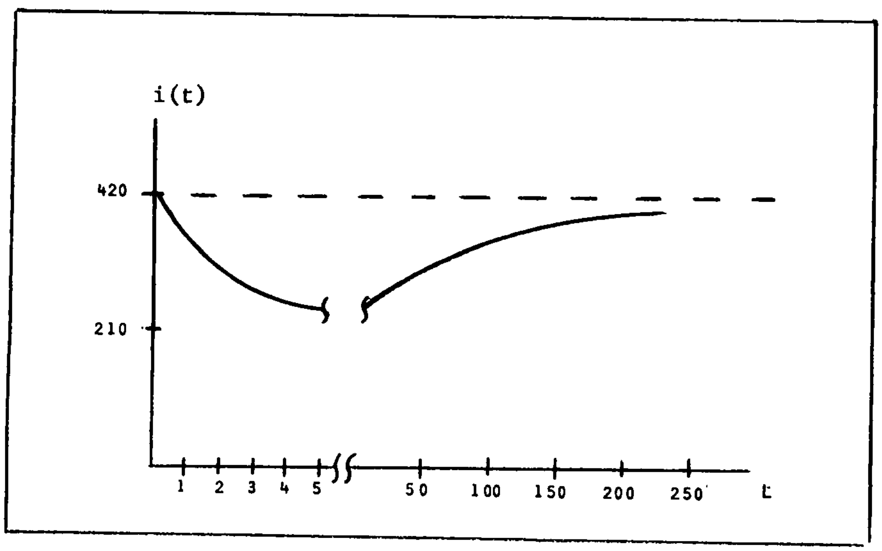
$$i = \frac{\frac{2V_o}{w} - v\mu_o H_o}{\frac{R}{w} + \frac{1}{\sigma L_1 D}} \quad \text{(t)}$$

Substituting numerical values, we obtain

$$i = 420 - 2.08 \times 10^5 (B_1 s_1 e^{s_1 t} + B_2 s_2 e^{s_2 t})$$
$$= 420 - 268 (e^{-.025t} - e^{-.94t}) \quad \text{(u)}$$

PROBLEM 12.25 (continued)





PROBLEM 12.25 (continued)

Our approximations were made in (h) and (k). For them to be valid, the following relations must hold:

$$\frac{\frac{\partial^2 \Delta h_2}{\partial t^2}}{g h_2} << 1$$

and

$$\int_{\substack{\text{transition}\\ \text{region}}} \frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v}ds \approx \frac{\partial \bar{v}}{\partial t}\sqrt{A} << L_2 \frac{\partial \bar{v}}{\partial t}$$

Substituting values, we find the first ratio to be about .001, so there our approximation is good to about .1%. In the second approximation

$$\frac{\sqrt{A}}{L_2} \approx \frac{.3}{2} \approx .15$$

Here, our approximation is good only to about 15%, which provides us with an idea of the error inherent in the approximation.

PROBLEM 12.26

Part a

We use the same coordinate system as defined in Fig. 12P.25. The magnetic field through the pump is

$$\bar{B} = \frac{Ni\mu_o}{d}\,\bar{i}_2 \qquad \text{(a)}$$

We integrate Newton's law across the length ℓ to obtain

$$\rho\ell\frac{\partial v}{\partial t} = p(0) - p(\ell) + JB\ell = -\frac{\Delta p_o}{v_o}v + \frac{i}{d}B \qquad \text{(b)}$$

$$= -\frac{\Delta p_o}{v_o}v + \frac{N\mu_o}{d^2}i^2$$

Thus

$$\frac{\partial v}{\partial t} + \frac{\Delta p_o}{\rho\ell v_o}v = \frac{N\mu_o}{d^2\rho\ell}I^2\sin^2\omega t = \frac{N\mu_o}{2d^2\rho\ell}I^2(1 - \cos\ 2\,\omega t) \qquad \text{(c)}$$

Solving, we obtain

$$v = \frac{N\mu_o I^2}{2d^2\rho\ell}\left[\frac{v_o\rho\ell}{\Delta p_o} - \frac{\left(\frac{\Delta p_o}{\rho\ell\, v_o}\cos 2\omega t + 2\omega\sin 2\omega t\right)}{\left(\frac{\Delta p_o}{\rho\ell v_o}\right)^2 + 4\,\omega^2}\right] \qquad \text{(d)}$$

Part b

The ratio R of ac to dc velocity components is:

$$R = \frac{\Delta p_o/v_o\rho\ell}{\left[\left(\frac{\Delta p_o}{v_o\rho\ell}\right)^2 + 4\omega^2\right]^{1/2}} \qquad \text{(e)}$$

PROBLEM 12.27

Part a

The magnetic field in generator (1) is upward, with magnitude

$$B_1 = \frac{Ni_1\mu_o}{a} - \frac{N_m i_2 \mu_o}{a} \tag{a}$$

and in generator (2) upward with magnitude

$$B_2 = \frac{N_m i_1 \mu_o}{a} + \frac{Ni_2\mu_o}{a} \tag{b}$$

We define the voltages V_1 and V_2 across the terminals of the generators. Applying Kirchoff's voltage law around the loops of wire with currents i_1 and i_2 we have

$$V_1 + N\frac{d\lambda_1}{dt} + N_m \frac{d\lambda_2}{dt} + i_1 R_L = 0 \tag{c}$$

and

$$V_2 + N\frac{d\lambda_2}{dt} - N_m \frac{d\lambda_1}{dt} + i_2 R_L = 0 \tag{d}$$

where

$$\lambda_1 = B_1 wb$$
$$\lambda_2 = B_2 wb \tag{e}$$

From conservation of current we have

$$\frac{i_1}{ab\sigma} = \frac{V_1}{w} + VB_1 \tag{f}$$

and

$$\frac{i_2}{ab\sigma} = \frac{V_2}{w} + VB_2 \tag{g}$$

Combining these relations, we obtain

$$(N^2 + N_m^2)\frac{wb\mu_o}{a}\frac{di_1}{dt} + i_1\left[\frac{w}{ab\sigma} + R_L - \frac{w\mu_o NV}{a}\right] + \frac{\mu_o w}{a} VN_m i_2 = 0 \tag{h}$$

and

$$(N^2 + N_m^2)\frac{wb\mu_o}{a}\frac{di_2}{dt} + i_2\left[\frac{w}{ab\sigma} + R_L - \frac{VN\mu_o w}{a}\right] - \frac{N_m\mu_o}{a} wVi_1 = 0 \tag{i}$$

Part b

We combine these two first-order differential equations to obtain one second-order equation.

$$a_1\frac{d^2 i_2}{dt} + a_2\frac{di_2}{dt} + a_3 i_2 = 0 \tag{j}$$

where

$$a_1 = \frac{\left[(N^2 + N_m^2)\frac{wb\mu_o}{a}\right]^2}{\frac{wN_m V\mu_o}{a}} \tag{k}$$

PROBLEM 12.27 (continued)

$$a_2 = 2\left[\frac{w}{ab\sigma} + R_L - \frac{w\mu_o NV}{a}\right]\left[\frac{(N^2 + N_m^2)b}{N_m V}\right]$$

$$a_3 = \frac{VN_m\mu_o w}{a}$$

If we assume solutions of the form

$$i_2 = Ae^{st} \qquad (\ell)$$

Then we must have

$$a_1 s^2 + a_2 s + a_3 = 0 \qquad (m)$$

or

$$s = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2\,a_1}$$

For the generators to be stable, the real part of s must be negative. Thus

$$a_2 > 0 \quad \text{for stability}$$

which implies the condition for stability is

$$\frac{w}{ab\sigma} + R_L > \frac{w\mu_o NV}{a} \qquad (n)$$

Part c

When $a_2 = 0$

$$\frac{w}{ab\sigma} + R_L = \frac{w\mu_o Nv}{a} \qquad (o)$$

then s is purely imaginary, so the system will operate in the sinusoidal steady state. Then

$$s = \pm j\sqrt{\frac{a_3}{a_1}}$$

$$= \pm j\,\frac{N_m V}{b(N^2 + N_m^2)} \qquad (p)$$

The length b necessary for sinusoidal operation is

$$b = \frac{w}{a\sigma\left[\frac{w\mu_o Nv}{a} - R_L\right]} \qquad (q)$$

Substituting values, we obtain

b = 4 meters.

Part d

Thus, the frequency of operation is

$$\omega = \frac{4000}{8} = 500 \text{ rad/sec.}$$

or

$$f = \frac{\omega}{2\pi} \approx 80 \text{ Hz.}$$

PROBLEM 12.28

Part a

The magnetic field within the generator is

$$\bar{B} = \frac{\mu_o N i}{w} \bar{i}_2 \qquad \text{(a)}$$

The current through the generator is

$$\bar{J} = \bar{i}_3 \frac{i}{\ell w} = \sigma\left(\frac{v}{D} + VB\right)\bar{i}_3 \qquad \text{(b)}$$

Solving for v, the voltage across the channel, we obtain

$$v = \left(\frac{D}{\sigma \ell w} - \frac{V\mu_o N}{w} D\right) i \qquad \text{(c)}$$

We apply Faraday's law around the electrical circuit to obtain

$$v + \frac{1}{C}\int i dt + i R_L = -\frac{\mu_o N^2}{w} \ell d \frac{di}{dt} \qquad \text{(d)}$$

Differentiating and simplifying this equation we finally obtain

$$\frac{d^2 i}{dt^2} + \left(\frac{R_L w}{\mu_o N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_o NDV}{w}\right)\frac{di}{dt} + \frac{w}{\mu_o N^2 \ell d C} i = 0 \qquad \text{(e)}$$

We assume that $i = \text{Re}\, \hat{I} e^{st}$.

Substituting this assumed solution back into the differential equation, we obtain

$$s^2 + \left(\frac{R_L w}{\mu_o N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_o NDV}{w}\right) s + \frac{w}{\mu_o N^2 \ell d C} = 0 \qquad \text{(f)}$$

Solving, we have

$$s = -\frac{\left(\frac{R_L w}{\mu_o N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_o NDV}{w}\right)}{2} \pm \sqrt{\frac{\left(\frac{R_L w}{\mu_o N^2 \ell d} + \frac{D}{\sigma L w} - \frac{\mu_o NDV}{w}\right)^2}{4} - \frac{w}{\mu_o N^2 \ell d C}} \qquad \text{(g)}$$

For the device to be a pure ac generator, we must have that s is purely imaginary, or

$$R_L = \left(\frac{\mu_o NDV}{w} - \frac{D}{\sigma L w}\right) \frac{\mu_o N^2 \ell d}{w} \qquad \text{(h)}$$

Part b

The frequency of operation is then

$$\omega = \frac{w}{\mu_o N^2 \ell d C} \qquad \text{(i)}$$

PROBLEM 12.29

Part a

The current within the MHD generator is

$$\bar{J} = \frac{i}{\ell d} \bar{i}_y = \sigma\left(\frac{V}{w} + vB_o\right)\bar{i}_y \qquad \text{(a)}$$

PROBLEM 12.29 (continued)

where V is the voltage across the channel. The pressure drop along the channel is

$$\Delta p = p_i - p_o = \frac{iB_o}{d} + \rho \frac{\partial v}{\partial t} \ell \tag{b}$$

where we assume that v does not vary with distance along the channel. With the switch open, we apply Faraday's law around the circuit, for which we obtain

$$V + 2iR = 0 \tag{c}$$

Since the pressure drop is maintained constant, we solve for v to obtain

$$\left(\frac{2\sigma R}{w} + \frac{1}{\ell d}\right) \frac{\rho d\ell}{B_o} \frac{\partial v}{\partial t} + \sigma v B_o = \left(\frac{1}{\ell d} + \frac{2\sigma R}{w}\right) \frac{d}{B_o} \Delta p \tag{d}$$

In the steady state

$$v = \left(\frac{1}{\sigma \ell d} + \frac{2R}{w}\right) \frac{d}{B_o^2} \Delta p \tag{e}$$

and

$$i = \frac{d}{B_o} \Delta p \tag{f}$$

Part b

For t > 0, the differential equation for v is

$$\left(\frac{\sigma R}{w} + \frac{1}{\ell d}\right) \frac{\rho \ell d}{B_o} \frac{\partial v}{\partial t} + \sigma v B_o = \left(\frac{1}{\ell d} + \frac{\sigma R}{w}\right) \frac{d}{B_o} \Delta p \tag{g}$$

The general solution for v is

$$v = \left(\frac{1}{\sigma \ell d} + \frac{R}{w}\right) \frac{d}{B_o^2} \Delta p + Ae^{-t/\tau} \tag{h}$$

where $\tau = \left(\frac{\sigma R}{w} + \frac{1}{\ell d}\right) \frac{\ell d}{\sigma B_o^2}$

We evaluate A by realizing that at t = 0, the velocity must be continuous. Therefore

$$v = \left(\frac{1}{\sigma \ell d} + \frac{R}{w}\right) \frac{d}{B_o^2} \Delta p + \frac{R}{w} \frac{d}{B_o^2} \Delta p \; e^{-t/\tau} \tag{i}$$

and

$$i = \Delta p \left(1 + \frac{\rho \ell}{\tau} \frac{R}{w} \frac{d}{B_o^2} e^{-t/\tau}\right) \frac{d}{B_o} \tag{j}$$

$$= \Delta p \left(1 + \frac{R\sigma \; e^{-t/\tau}}{w\left[\frac{\sigma R}{w} + \frac{1}{\ell d}\right]}\right) \frac{d}{B_o}$$

PROBLEM 12.30

Part a

The magnetic field in the generator is

$$B = \frac{\mu_o N i}{d} \tag{a}$$

The current within the generator is

$$\frac{i}{\ell d} = \sigma \left(\frac{V}{w} + vB\right) \tag{b}$$

PROBLEM 12.30 (continued)

where V is the voltage across the channel. The pressure drop in the channel is

$$\Delta p = p_i - p_o = \Delta p_o \left(1 - \frac{v}{v_o}\right) = \frac{iB}{d} \tag{c}$$

Applying Faraday's law around the external circuit, we obtain

$$V + i(R_L + R_C) = -\frac{d(NB\ell w)}{dt} = -\frac{\ell w}{d}\mu_o N^2 \frac{di}{dt} \tag{d}$$

Using (a), (b), (c) and (d), the differential equation for i is then

$$\frac{\ell\mu_o N^2}{d}\frac{di}{dt} + i\left[\frac{R_L + R_C}{w} + \frac{1}{\sigma\ell d} - \frac{\mu_o N}{d} v_o\right] + \frac{\left(\frac{\mu_o N}{d}\right)^2}{d\,\Delta p_o} v_o i^3 = 0 \tag{e}$$

In the steady state, we have

$$i^2 = -\frac{\left[\frac{R_L + R_C}{w} + \frac{1}{\sigma\ell d} - \frac{\mu_o N v_o}{d}\right] d\Delta p_o}{\left[\frac{\mu_o N}{d}\right]^2 v_o} \tag{f}$$

The power dissipated in R_L is

$$P = i^2 R_L$$

For $P = 1.5 \times 10^6$, then

$$i^2 = .6 \times 10^8 \text{ (amperes)}^2$$

Substituting in values for the parameters in (f), we obtain

$$i^2 = .6 \times 10^8 = -\frac{(.125 + 2.5 \times 10^{-6} N^2 - 6.3 \times 10^{-4} N) 40 \times 10^3}{N^2 (4 \times 10^{-8})} \tag{g}$$

Rearranging (g), we obtain

$$N^2 - 102N + 2.04 \times 10^3 = 0$$

or

$$N = 75,\ 27$$

The most efficient solution is that one which dissipates the least power in the coil's resistance. Thus, we choose

$$N = 27$$

Part b

Substituting numerical values into (e), using N = 27, we obtain

$$(1.27 \times 10^7)\frac{di}{dt} - (6 \times 10^7) i + i^3 = 0 \tag{h}$$

or, rewriting, we have

$$\frac{dt}{1.27 \times 10^7} = \frac{di}{i(6 \times 10^7 - i^2)} \tag{i}$$

Integrating, we obtain

$$9.4t + C = \log\left(\frac{i^2}{6 \times 10^7 - i^2}\right) \tag{j}$$

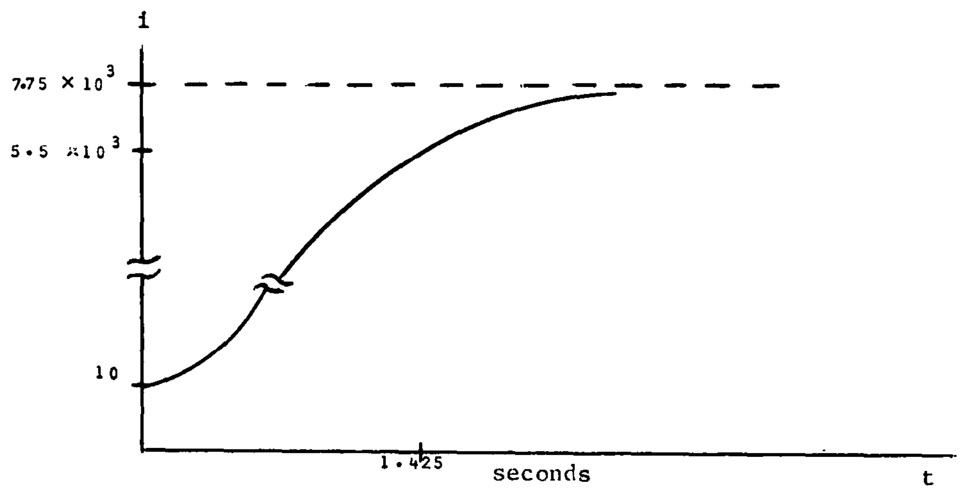
We evaluate the arbitrary constant C by realizing that at t=0, i = 10 amps

PROBLEM 12.30 (continued)

Thus $C = -13.3$

We take the anti-log of both sides of (j), and solve for i^2 to obtain

$$i^2 = \frac{6 \times 10^7}{1 + e^{(13.3 - 9.4t)}} \tag{k}$$



Part c

For N = 27, in the steady state, we use (f) to write

$$P = i^2 R_L = \frac{-\left[\frac{R_L + R_C}{w} + \frac{1}{\sigma \ell d} - \frac{\mu_o N v_o}{d}\right] d\Delta p_o R_L}{\left(\frac{\mu_o N}{d}\right)^2 v_o}$$

or

$$P = a_1 R_L - a_2 R_L^2$$

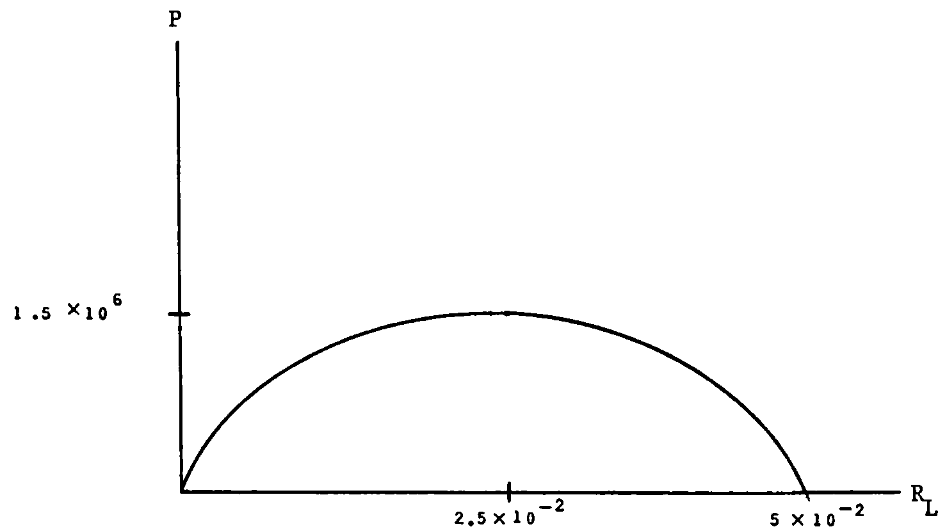
where

$$a_1 = -\frac{d\Delta p_o \left(\frac{R_C}{w} + \frac{1}{\sigma \ell d} - \frac{\mu_o N v_o}{d}\right)}{\left(\frac{\mu_o N}{d}\right)^2 v_o} \approx 1.47 \times 10^8$$

and

$$a_2 = \frac{d\Delta p_o}{\left(\frac{\mu_o N}{d}\right)^2 v_o} \approx \frac{1}{2.85 \times 10^{-10}}$$

PROBLEM 12.30 (continued)



PROBLEM 12.31

Part a

With the switch open, the current through the generator is

$$\overline{J} = 0 = \frac{i}{\ell d}\,\overline{i}_y = \sigma\left(-\frac{V}{w} + vB_o\right)\overline{i}_y \tag{a}$$

where V is the voltage across the channel. In the steady state, the pressure drop in the channel is

$$\Delta p = p_i - p_o = \frac{iB}{d} = 0 = \Delta p_o\left(1 - \frac{v}{v_o}\right) \tag{b}$$

Thus, $v = v_o$ and the voltage across the channel is

$$V = v_o B_o w. \tag{c}$$

Part b

With the switch closed, applying Faraday's law around the circuit we obtain

$$V = i\,R_L \tag{d}$$

Thus

$$\frac{i}{\ell d} = -\frac{\sigma R_L}{w}\,i + \sigma v B_o \tag{e}$$

and

$$\Delta p = \frac{iB}{d} + \rho\,\frac{\partial v}{\partial t}\,\ell = \Delta p_o\left(1 - \frac{v}{v_o}\right) \tag{f}$$

Obtaining an equation in v, we have

$$\rho\ell\,\frac{\partial v}{\partial t} + v\left[\frac{\Delta p_o}{v_o} + \frac{\sigma B_o}{\frac{1}{\ell d} + \frac{\sigma R_L}{w}}\right] = \Delta p_o \tag{g}$$

PROBLEM 12.31 (continued)

Solving for v we obtain

$$v = Ae^{-t/\tau} + \frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{B_o w}{R_L + R_i}\right)} \quad \text{where } R_i = \frac{w}{\sigma \ell d} \tag{h}$$

and where

$$\tau = \frac{\rho \ell}{\left[\frac{\Delta p_o}{v_o} + \frac{wB_o}{R_L + R_i}\right]} \tag{i}$$

at t = 0, the velocity must be continuous. Therefore,

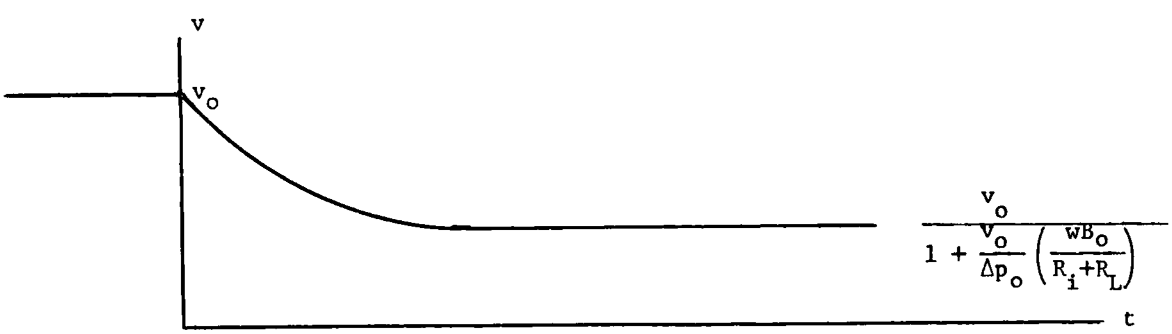
$$A = v_o - \frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{wB_o}{R_L + R_i}\right)}$$

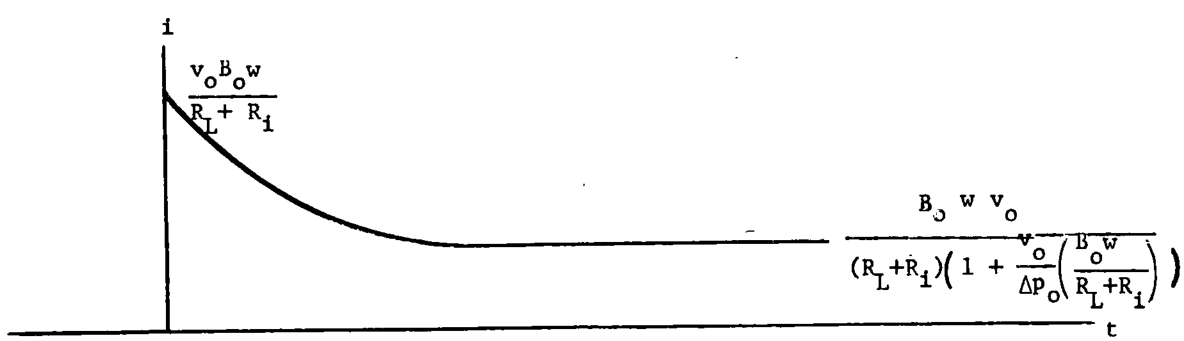
Now, the current is

$$i = \frac{wB_o v}{R_L + R_i} \tag{k}$$

Thus

$$i = \left(\frac{wB_o}{R_L \quad R_i}\right)\left[\frac{\Delta p_o}{\left(\frac{\Delta p_o}{v_o} + \frac{wB_o}{R_L + R_i}\right)}(1 - e^{-t/\tau}) + v_o e^{-t/\tau}\right] \tag{ℓ}$$





PROBLEM 12.32

The current in the generator is

$$\frac{i}{\ell_1 d} = \sigma\left(\frac{V}{w} - vB\right) \qquad \text{(a)}$$

where we assume that the $\overline{B}$ field is up and that the fluid flows counter-clockwise. We integrate Newton's law around the channel to obtain

$$\rho\ell \frac{\partial v}{\partial t} = JB\ell_1 = \frac{i}{d} B \qquad \text{(b)}$$

or, using (a),

$$\frac{\partial V}{\partial t} = \frac{w}{d\ell_1\sigma}\frac{\partial i}{\partial t} + \frac{B^2 w}{d\rho\ell} i \qquad \text{(c)}$$

Integrating, we have

$$V = \frac{w}{d\ell_1\sigma} i + \frac{B^2 w}{d\rho\ell}\int_0^\infty i dt \qquad \text{(d)}$$

Defining $R_i = \frac{w}{\sigma\ell_1 d}$

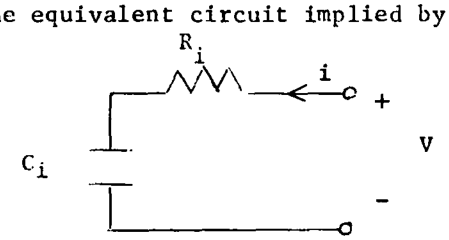
and

$$C_i = \frac{\rho\ell d}{wB^2}$$

we rewrite (d) as

$$V = i R_i + \frac{1}{C_i}\int_0^\infty i dt \qquad \text{(e)}$$

The equivalent circuit implied by (e) is



PROBLEM 12.33

Part a

We assume that the capacitor is initially uncharged when the switch is closed at t = 0. The current through the capacitor is

$$i = C\frac{dV_C}{dt} = \sigma\ell d\left(-\frac{V_C}{w} + v_o B_o\right) \qquad \text{(a)}$$

or

$$\frac{dV_C}{dt} + \frac{\sigma\ell d}{wC} V_C = \frac{\sigma\ell d v_o}{C} B_o \qquad \text{(b)}$$

PROBLEM 12.33 (Continued)

The solution for V_C is

$$V_C = v_o B_o w(1 - e^{-t/\tau}) \tag{c}$$

with $\tau = \frac{wC}{\sigma \ell d}$, where we have used the initial condition that at t = 0, the voltage cannot change instantaneously across the capacitor. The energy stored as $t \to \infty$, is

$$W_e = \frac{1}{2} C V_C^2 = \frac{1}{2} C(v_o B_o w)^2 \tag{d}$$

Part b

The pressure drop along the fluid is

$$\Delta p = \frac{iB_o}{d} = B_o^2 v_o \sigma \ell \, e^{-t/\tau} \tag{e}$$

The total energy supplied by the fluid source is

$$W_f = \int_0^\infty \Delta p \, v_o dwdt$$

$$= \int_0^\infty (v_o B_o)^2 \, \sigma \ell w d e^{-t/\tau} \, dt \tag{f}$$

$$= - \sigma \ell (v_o B_o)^2 \, \tau w d e^{-t/\tau} \Big|_0^\infty$$

$$W_f = C(w v_o B_o)^2 \tag{g}$$

Part c

We see that the energy supplied by the fluid source is twice that stored in the capacitor. The rest of the energy has been dissipated by the conducting fluid. This dissipated energy is

$$W_d = \int_0^\infty V_C \, idt \tag{h}$$

$$= \int_0^\infty + (v_o B_o)^2 w(1 - e^{-t/\tau}) \sigma \ell d e^{-t/\tau} \, dt$$

$$= \sigma \ell d w (v_o B_o)^2 \left[-\tau e^{-t/\tau} + \frac{\tau}{2} e^{-2t/\tau} \right] \Big|_0^\infty$$

$$= \sigma \ell d w (v_o B_o)^2 \frac{\tau}{2} \tag{i}$$

Therefore

$$W_d = \frac{1}{2} C(v_o B_o w)^2 \tag{j}$$

Thus

$$W_{fluid} = W_{elec} + W_{dissipated} \tag{k}$$

As we would expect from conservation of energy.

PROBLEM 12.34

The current through the generator is

$$\frac{i}{\ell_1 d} = \sigma\left(\frac{V}{w} - vB_o\right) \tag{a}$$

Since the fluid is incompressible, and the channel has constant cross-sectional area, the velocity of the fluid does not change with position. Thus, we write Newton's law as in Eq. (12.2.41) as

$$\rho \frac{\partial \bar{v}}{\partial t} = -\nabla(p + U) + \bar{J} \times \bar{B} \tag{b}$$

where U is the potential energy due to gravity. We integrate this expression along the length of the tube to obtain

$$\rho \frac{\partial v}{\partial t} \ell = \frac{iB_o}{d} - \rho g(x_a + x_b) \tag{c}$$

Realizing that $x_a = x_b$

and

$$v = \frac{dx_a}{dt} \tag{d}$$

We finally obtain

$$\frac{d^2 x_a}{dt^2} + \frac{\sigma B_o^2 \ell_1}{\rho \ell} \frac{dx_a}{dt} + \frac{2g}{\ell} x_a = \frac{\sigma B_o V}{w\rho} \frac{\ell_1}{\ell} \tag{e}$$

We assume the transient solution to be of the form

$$x_a = \hat{x} e^{st} \tag{f}$$

Substituting into the differential equation, we obtain

$$s^2 + \frac{\sigma B_o^2 \ell_1 s}{\rho \ell} + \frac{2g}{\ell} = 0 \tag{g}$$

Solving for s, we obtain

$$s = -\frac{\sigma B_o^2 \ell_1}{2\rho \ell} \pm \sqrt{\left(\frac{\sigma B_o^2 \ell_1}{2\rho\ell}\right)^2 - \frac{2g}{\ell}} \tag{h}$$

Substituting the given numerical values, we obtain

$$s_1 = -29.4$$

$$s_2 = -.665 \tag{i}$$

In the steady state

$$x_a = \frac{\sigma B_o V \ell_1}{w\rho\, 2g} \approx .075 \text{ meters} \tag{j}$$

Thus the general solution is of the form

$$x_a = .075 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{k}$$

where the initial conditions to solve for A_1 and A_2 are

PROBLEM 12.34 (continued)

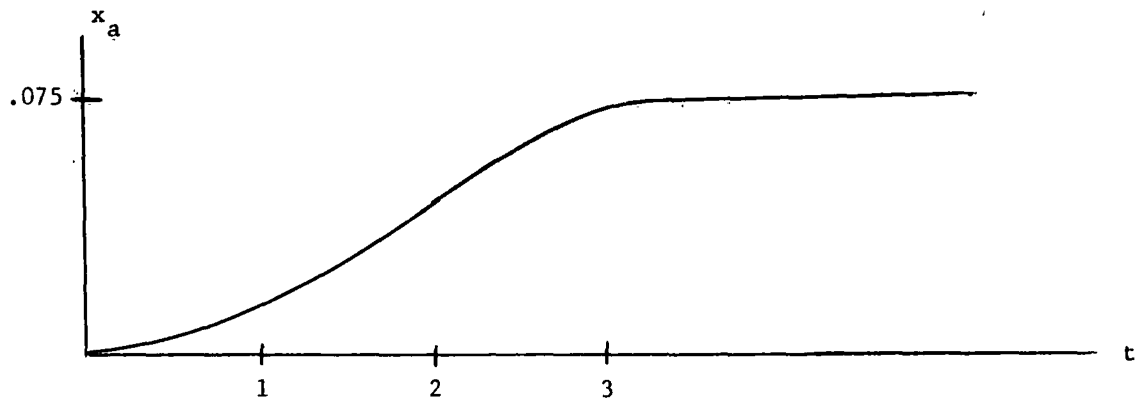
$$x_a(t=0) = 0$$

$$\frac{dx_a}{dt}(t=0) = 0 \qquad (\ell)$$

Thus, $A_2 = \dfrac{.075\ s_1}{s_2 - s_1} = -\ .0765$ and $A_1 = -\dfrac{.075\ s_2}{s_2 - s_1} = .00174$

Thus, we have:

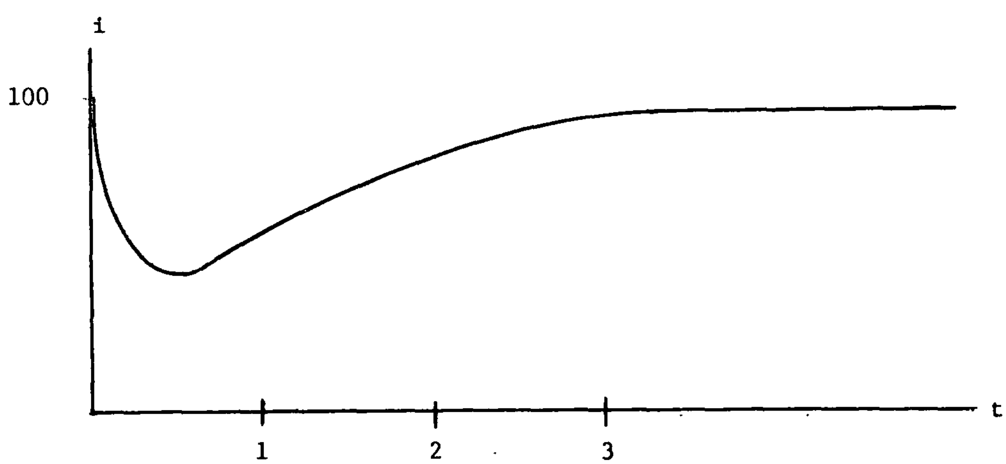
$$x_a = .075 + .00174e^{-29.4t} - .0765e^{-.665t}$$



Now the current is

$$\begin{aligned} i &= \ell_1 d\sigma\left(\frac{V}{w} - B_o\frac{dx}{dt}\right) \\ &= \ell_1 d\sigma\left[\frac{V}{w} - B_o(s_1A_1e^{s_1t} + s_2A_2e^{s_2t})\right] \qquad (m) \\ &= 100 - 2\times 10^3(s_1A_1e^{s_1t} + s_2A_2e^{s_2t}) \text{ amperes} \\ &= 100(1 + e^{-29.4t} - e^{-.665t}) \end{aligned}$$

Sketching, we have



PROBLEM 12.35

The currents I_1 and I_2 are determined by the resistance of the fluid between the electrodes. Thus

$$I_1 = \frac{V_o \sigma D x}{w} \tag{a}$$

and

$$I_2 = \frac{V_o \sigma D y}{w} \tag{b}$$

The magnetic field produced by the circuit is

$$\bar{B} = \frac{\mu_o N}{w}(I_2 - I_1)\bar{i}_2 \tag{c}$$

or

$$\bar{B} = \frac{\mu_o N}{w^2} V_o \sigma D (y - x)\bar{i}_2 \tag{d}$$

From conservation of mass,

$$y = (L - x) \tag{e}$$

Thus

$$\bar{B} = \frac{\mu_o N V_o \sigma D}{w^2}(L - 2x)\bar{i}_2 \tag{f}$$

The momentum equation is

$$\rho \frac{\partial \bar{v}}{\partial t} = -\nabla(p + U) + \bar{J} \times \bar{B} \tag{g}$$

Integrating the equation along the conduit's length, we obtain

$$\rho \frac{\partial v}{\partial t}(2L + 2a) = -\rho g(y - x) - J_o BL \tag{h}$$

Now

$$v = -\frac{\partial x}{\partial t} \tag{i}$$

so we write:

$$2\rho(L + a)\frac{\partial^2 x}{\partial t^2} + \left(\rho g + J_o \frac{\mu_o N V_o \sigma D L}{w^2}\right)(2x - L) = 0 \tag{j}$$

We assume solutions of the form

$$x = \text{Re}\, \hat{x}\, e^{st} + \frac{L}{2} \tag{k}$$

Thus

$$s^2 + \frac{g}{(L + a)} + \frac{\mu_o N V_o \sigma D}{\rho w^2 (L + a)} J_o L = 0 \tag{ℓ}$$

Defining

$$\omega_o^2 = \frac{g}{(L + a)} + \frac{\mu_o N V_o \sigma D J_o L}{\rho w^2 (L + a)} \tag{m}$$

we have our solution in the form

$$x = A \sin \omega_o t + B \cos \omega_o t + \frac{L}{2} \tag{n}$$

Applying the initial conditions

$$x(0) = L \quad \text{and} \quad \frac{dx(0)}{dt} = 0 \tag{o}$$

we obtain

$$x = \frac{L}{2}(1 + \cos \omega_o t) \tag{p}$$

PROBLEM 12.36

As from Eqs. (12.2.88 - 12.2.91), we assume that

$$\begin{aligned} \bar{v} &= \bar{i}_\theta v_\theta \\ \bar{B} &= B_o\bar{i}_z + \bar{i}_\theta B_\theta \\ \bar{J} &= \bar{i}_r J_r + \bar{i}_z J_z \\ \bar{E} &= \bar{i}_r E_r + \bar{i}_z E_z \end{aligned} \tag{a}$$

As derived in Sec. 12.2.3, Eq. (12.2.102), we know that the equation governing Alfvén waves is

$$\frac{\partial^2 v_\theta}{\partial t^2} = \frac{B_o^2}{\mu_o\rho}\frac{\partial^2 v_\theta}{\partial z^2} \tag{b}$$

For our problem, the boundary conditions are:

$$\begin{aligned} &\text{at } z = 0 \qquad && E_r = 0 \\ &\text{at } z = \ell \qquad && v_\theta = \text{Re}[\Omega r e^{j\omega t}] \end{aligned} \tag{c}$$

As in section 12.2.3, we assume

$$v_\theta = \text{Re}[A(r)\hat{v}_\theta(z)e^{j\omega t}] \tag{d}$$

Thus, the pertinent differential equation reduces to

$$\frac{d^2\hat{v}_\theta}{dz^2} + k^2\hat{v}_\theta = 0 \tag{e}$$

where $k = \omega\sqrt{\dfrac{\mu_o\rho}{B_o^2}}$

The solution is

$$\hat{v}_\theta = C_1 \cos kz + c_2 \sin kz \tag{f}$$

Imposing the boundary condition at $z = \ell$, we obtain

$$A(r)[C_1 \cos k\ell + C_2 \sin k\ell\,] = \Omega r \tag{g}$$

We let

$$A(r) = \frac{r}{R} \tag{h}$$

and thus

$$\Omega R = C_1 \cos k\ell + C_2 \sin k\ell \tag{i}$$

Now

$$E_r = -v_\theta B_o \tag{j}$$

Thus, applying the second boundary condition, we obtain

$$v_\theta(z{=}0) = 0$$

or

$$C_1 = 0 \tag{k}$$

Thus

$$C_2 = \frac{\Omega R}{\sin k\ell} \tag{ℓ}$$

Now, using the relations

$$E_r = -v_\theta B_o \tag{m}$$

PROBLEM 12.36 (continued)

$$E_z = 0 \tag{n}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t} \tag{o}$$

$$-\frac{1}{\mu_o}\frac{\partial B_\theta}{\partial z} = J_r \tag{p}$$

$$\frac{1}{\mu_o r}\frac{\partial (rB_\theta)}{\partial r} = J_z \tag{q}$$

we obtain

$$v_\theta = \text{Re}\left[\frac{\Omega\ r}{\sin k\ell}\ \sin kz\ e^{j\omega t}\right] \tag{r}$$

$$B_\theta = \text{Re}\left[\frac{\Omega r B_o k}{j\ \omega \sin k\ell}\ \cos kz\ e^{j\omega t}\right] \tag{s}$$

$$J_r = \text{Re}\left[\frac{\Omega r B_o\ k^2}{\mu_o j\ \omega \sin k\ell}\ \sin kz\ e^{j\omega t}\right] \tag{t}$$

$$J_z = \text{Re}\left[\frac{2\ \Omega B_o\ k}{\mu_o j\ \omega \sin k\ell}\ \cos kz\ e^{j\omega t}\right] \tag{u}$$

PROBLEM 12.37

Part a

We perform a similar analysis as in section 12.2.3, Eqs. (12.2.84 - 12.2.88). From Maxwell's equation

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{a}$$

which yields

$$\frac{\partial E_y}{\partial z} = \frac{\partial}{\partial t} B_x \tag{b}$$

Now, since the fluid is perfectly conducting,

$$\overline{E}' = \overline{E} + \overline{v} \times \overline{B} = 0 \tag{c}$$

or $$E_y = v_x B_o \tag{d}$$

Substituting, we obtain

$$B_o \frac{\partial v_x}{\partial z} = \frac{\partial B_x}{\partial t} \tag{e}$$

The x component of the force equation is

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial T_{xz}}{\partial z} \tag{f}$$

where

$$T_{xz} = \frac{B_o}{\mu_o} B_x \tag{g}$$

PROBLEM 12.37 (continued)

Thus

$$\rho \frac{\partial v_x}{\partial t} = \frac{B_o}{\mu_o} \frac{\partial B_x}{\partial z} \tag{h}$$

Eliminating B_x and solving for v_x, we obtain

$$\frac{\partial^2 v_x}{\partial t^2} = \frac{B_o^2}{\mu_o \rho} \frac{\partial^2 v_x}{\partial z^2} \tag{i}$$

or eliminating and solving for H_x, we have

$$\frac{\partial^2 H_x}{\partial t^2} = \frac{B_o^2}{\mu_o \rho} \frac{\partial^2 H_x}{\partial z^2} \tag{j}$$

where

$$B_x = \mu_o H_x \tag{k}$$

Part b

The boundary conditions are

$$v_x(-\ell,t) = \text{Re } Ve^{j\omega t} \tag{ℓ}$$

$$E_y(0,t) = 0 \rightarrow \quad v_x(0,t) = 0 \tag{m}$$

We write the solution in the form

$$v_x = A\, e^{j(\omega t-kz)} + B\, e^{j(\omega t+kz)} \tag{n}$$

where

$$k = \omega\sqrt{\frac{\mu_o \rho}{B_o^2}}$$

Applying the boundary conditions, we obtain

$$v_x(\ell,t) = \text{Re}\left[-\frac{V \sin kz}{\sin k\ell}\right] e^{j\omega t} \tag{o}$$

Now

$$B_o \frac{\partial v_x}{\partial z} = \frac{\partial B_x}{\partial t} \tag{p}$$

or

$$\frac{-B_o Vk \cos kz}{\sin k\ell} = j\omega\mu_o \hat{H}_x \tag{q}$$

Thus

$$H_x = \text{Re}\left[\frac{-B_o Vk \cos kz}{j\omega\mu_o \sin k\ell} e^{j\omega t}\right] \tag{r}$$

Part c

From Maxwell's equations

$$\nabla \times \bar{H} = \bar{i}_y \frac{\partial H_x}{\partial z} = \bar{J} \tag{s}$$

Thus

$$\bar{J} = \bar{i}_y \text{ Re}\left[\frac{B_o Vk^2 \sin kz}{j\omega\, \mu_o \sin k\ell} e^{j\omega t}\right] \tag{t}$$

PROBLEM 12.37 (continued)

Since $\nabla\cdot\bar{J} = 0$, the current must have a return path, so the walls in the x-z plane must be perfectly conducting.

Even though the fluid has no viscosity, since it is perfectly conducting, it interacts with the magnetic field such that for any motion of the fluid, currents are induced such that the magnetic force tends to restore the fluid to its original position. This shearing motion sets the neighboring fluid elements into motion, whereupon this process continues throughout the fluid.

PROBLEM 13.1

In static equilibrium, we have

$$-\nabla p - \rho g \overline{i}_1 = 0 \tag{a}$$

Since $p = \rho RT$, (a) may be rewritten as

$$RT \frac{d\rho}{dx_1} + \rho g = 0 \tag{b}$$

Solving, we obtain

$$\rho = \rho_o e^{-\frac{g}{RT} x_1} \tag{c}$$

PROBLEM 13.2

Since the pressure is a constant, Eq. (13.2.25) reduces to

$$\rho v \frac{dv}{dz} = -J_y B \tag{a}$$

where we use the coordinate system defined in Fig. 13P.4. Now, from Eq. (13.2.21) we obtain

$$J_y = \sigma(E_y + vB) \tag{b}$$

If the loading factor K, defined by Eq. (13.2.32) is constant, then

$$-KvB = +E \tag{c}$$

Thus, $J_y = \sigma vB(1-K)$ (d)

Then

$$\rho v \frac{dv}{dz} = -\sigma v B^2 (1-K) \tag{e}$$

or

$$\rho \frac{dv}{dz} = -\sigma B^2 (1-K) = -\sigma(1-K) \frac{B_i^2 A_i}{A(z)} \tag{f}$$

From conservation of mass, Eq. (13.2.24), we have

$$\rho_i v_i A_i = \rho A(z) v \tag{g}$$

Thus

$$\frac{\rho_i v_i A_i}{v} \frac{dv}{dz} = -\sigma(1-K) B_i^2 A_i \tag{h}$$

Integrating, we obtain

$$\ln v = \frac{-\sigma(1-K)B_i^2}{\rho_i v_i} z + C \tag{i}$$

or

$$v = v_i e^{-\frac{z}{\ell_d}} \tag{j}$$

where $\ell_d = \dfrac{\rho_i v_i}{\sigma(1-K)B_i^2}$ and we evaluate the arbitrary constant by realizing that

$v = v_i$ at $z = 0$.

PROBLEM 13.3

Part a

We assume T, B_o, w, σ, c_p and c_v are constant. Since the electrodes are short-circuited, E = 0, and so

$$J_y = v B_o. \tag{a}$$

We use the coordinate system defined in Fig. 13P.4. Applying conservation of energy, Eq. (13.2.26), we have

$$\rho v \frac{d}{dz}\left(\frac{1}{2} v^2\right) = 0, \text{ where we have set h = constant.} \tag{b}$$

Thus, v is a constant, $v = v_i$. Conservation of momentum, Eq. (13.2.25), implies

$$\frac{dp}{dz} = - v_i B_o^2 \tag{c}$$

Thus, $p = - v_i B_o^2 z + p_i$ (d)

The mechanical equation of state, Eq. (13.1.10) then implies

$$\rho = \frac{p}{RT} = - \frac{v_i B_o^2 z + p_i}{R\ T} = \rho_i - \frac{v_i B_o^2 z}{RT} \tag{e}$$

From conservation of mass, we then obtain

$$\rho_i v_i w d_i = \left(- \frac{v_i B_o^2 z}{RT} + \rho_i\right) v_i\ w d(z) \tag{f}$$

Thus

$$d(z) = \frac{\rho_i d_i}{\left(\rho_i - \frac{v_i B_o^2 z}{RT}\right)} \tag{g}$$

Part b

Then

$$\rho(z) = \rho_i - \frac{v_i B_o^2\ z}{RT} \tag{h}$$

PROBLEM 13.4

Note:

There are errors in Eqs. (13.2.16) and (13.2.31). They should read:

$$\frac{1}{M^2}\ \frac{d(M^2)}{dx_1} = \frac{\{(\gamma-1)(1+\gamma M^2)E_3 + \gamma[2 + (\gamma-1)M^2]v_1 B_2\}\ J_3}{(1 - M^2)\gamma\ p v_1} \tag{13.2.16}$$

and

$$\frac{1}{M^2}\frac{d(M^2)}{dx_1} = \frac{1}{(1-M^2)}\left\{[(\gamma-1)(1+\gamma M^2)E_3 + \gamma\{2 + (\gamma-1)M^2\}\ v_1 B_2]\ \frac{J_3}{\gamma p v_1} - \frac{[2 + (\gamma-1)M^2]}{A}\frac{dA}{dx_1}\right\} \tag{13.2.31}$$

Part a

We assume that σ, γ, B_o, K and M are constant along the channel. Then, from the corrected form of Eq. (13.2.31), we must have

PROBLEM 13.4 (continued)

$$0 = \frac{1}{1-M^2}\left\{\left[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)\right]\frac{vB_o^2\sigma(1-K)}{\gamma p} - \frac{[2+(\gamma-1)M^2]}{A}\frac{dA}{dz}\right\} \quad (a)$$

Now, using the relations

$$v^2 = M^2\gamma RT$$

and

$$p = \rho RT$$

we write

$$\frac{v}{\gamma p} = \frac{M^2}{\rho v} \quad (b)$$

Thus, we obtain

$$\frac{1}{A^2}\frac{dA}{dz} = \frac{\left[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)\right]\frac{B_o^2\sigma(1-K)M^2}{\rho VA}}{2+(\gamma-1)M^2} \quad (c)$$

From conservation of mass,

$$\rho vA = \rho_i v_i A_i \quad (d)$$

Using (d), we integrate (c) and solve for $\frac{A(z)}{A_i}$

to obtain

$$\frac{A(z)}{A_i} = \frac{1}{1-\beta_1 z} \quad (e)$$

where

$$\beta_1 = \frac{\left[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)\right]\sigma B_o^2 M^2(1-K)}{\rho_i v_i[2+(\gamma-1)M^2]}$$

We now substitute into Eq. (13.2.27) to obtain

$$\frac{1}{v}\frac{dv}{dz} = \frac{1}{(1-M^2)}\left[(\gamma-1)(-K)+\gamma\right]\frac{vB_o^2(1-K)\sigma}{\gamma p} - \frac{1}{A}\frac{dA}{dz} \quad (f)$$

Thus may be rewritten as

$$\frac{1}{v}\frac{dv}{dz} = \frac{1}{(1-M^2)}\left[\left[(\gamma-1)(-K)+\gamma\right]\frac{\sigma B_o^2(1-K)M^2}{\rho_i v_i A_i} - \frac{\beta_1}{A_i}\right] A \quad (g)$$

Solving, we obtain

$$\ln v = \frac{-\beta_2}{\beta_1}\ln(1-\beta_1 z) + \ln v_i \quad (h)$$

or

$$\frac{v(z)}{v_i} = (1-\beta_1 z)^{-\beta_2/\beta_1} \quad (i)$$

where $\beta_2 = \frac{1}{(1-M^2)} \quad \frac{[(\gamma-1)(-K)+\gamma]\sigma B_o^2(1-K)M^2 - \beta_1}{\rho_i v_i}$

Now the temperature is related through Eq. (13.2.12), as

PROBLEM 13.4 (continued)

$$M^2\gamma RT = v^2 \qquad \text{(j)}$$

Thus

$$\frac{T(z)}{T_i} = \left(\frac{v}{v_i}\right)^2 \qquad \text{(k)}$$

From (d), we have

$$\frac{\rho(z)}{\rho_i} = \frac{v_i}{v}\frac{A_i}{A} \qquad (\ell)$$

Thus, from Eq. (13.1.10)

$$\frac{p(z)}{p_i} = \frac{v_i}{v}\frac{A_i}{A}\frac{T}{T_i} \qquad \text{(m)}$$

Since the voltage across the electrodes is constant,

$$E = -\frac{V}{w(z)} = -Kv(z)B_o \qquad \text{(n)}$$

or

$$w(z) = \frac{Kv_iB_ow_i}{Kv(z)B_o} = \frac{v_i}{v(z)}w_i \qquad \text{(o)}$$

Thus,

$$\frac{w(z)}{w_i} = \frac{v_i}{v(z)} \qquad \text{(p)}$$

Then

$$\frac{d(z)}{d_i} = \frac{A(z)}{A_i}\frac{w_i}{w(z)} \qquad \text{(q)}$$

Part b

We now assume that σ, γ, B_o, K and v are constant along the channel. Then, from Eq. (13.2.27) we have

$$0 = \frac{1}{(1-M^2)}\left\{[(\gamma-1)(-K)+\gamma]v_iB_o^2\frac{(1-K)\sigma}{\gamma p} - \frac{1}{A}\frac{dA}{dz}\right\} \qquad \text{(r)}$$

But, from Eq. (13.2.25) we know that

$$\frac{p}{p_i} = 1 - \frac{(1-K)\sigma v_iB_o^2z}{p_i} = 1 - \beta_3 z \qquad \text{(s)}$$

where $\beta_3 = (1-K)\dfrac{\sigma v_iB_o^2}{p_i}$

Substituting the results of (b), into (a) and solving for $\frac{A(z)}{A_i}$, we obtain

$$\frac{A(z)}{A_i} = \left(\frac{p}{p_i}\right)^{-\beta_4/\beta_3} \qquad \text{(t)}$$

where $\beta_4 = [(\gamma-1)(-K)+\gamma]\dfrac{v_iB_o^2}{\gamma p_i}(1-K)\sigma$

From conservation of mass,

$$\frac{\rho(z)}{\rho_i} = \frac{A_i}{A(z)} \qquad \text{(u)}$$

PROBLEM 13.4 (continued)

and so, from Eq. (13.1.10)

$$\frac{T(z)}{T_i} = \frac{p(z)}{p_i}\frac{\rho_i}{\rho(z)} \quad . \tag{v}$$

As in (p)

$$\frac{w(z)}{w_i} = \frac{v_i}{v(z)} = 1 \tag{w}$$

Thus

$$\frac{d(z)}{d_i} = \frac{A(z)}{A_i} \tag{x}$$

Part c

We wish to find the length ℓ such that

$$\frac{C_pT(\ell) + \frac{1}{2}[v(\ell)]^2}{C_pT(o) + \frac{1}{2}[v(o)]^2} = .9 \tag{y}$$

For the constant M generator of part (a), we obtain from (i) and (k)

$$\frac{C_p\left[\frac{v(\ell)}{v_i}\right]^2 T_i + \frac{1}{2}[v(\ell)]^2}{C_p\left[\frac{v(o)}{v_i}\right]^2 T_i + \frac{1}{2}[v(o)]^2} = \frac{C_p(1-\beta_1\ell)^{-2\beta_2/\beta_1}T_i + \frac{1}{2}[v_i(1-\beta_1\ell)]^{-2\beta_2/\beta_1}}{C_p T_i + \frac{1}{2}v_i^2} = .9 \tag{z}$$

Reducing, we obtain

$$(1-\beta_1\ell)^{-2\beta_2/\beta_1} = .9 \tag{aa}$$

Substituting the given numerical values, we have

$$\beta_1 = .396 \quad \text{and} \quad \beta_2/\beta_1 = -7.3\times 10^{-2}$$

We then solve (aa) for ℓ, to obtain

$$\ell \approx 1.3 \text{ meters}$$

For the constant v generator of part (b), we obtain from (s), (t), (u) and (v)

$$\frac{C_pT_i\left[\frac{p(\ell)}{p_i}\frac{\rho_i}{\rho(\ell)}\right] + \frac{1}{2}v_i^2}{C_p T_i + \frac{1}{2}v_i^2} = .9 \tag{bb}$$

or

$$\frac{C_pT_i\ (1-\beta_3\ell)^{(1-\beta_4/\beta_3)} + \frac{1}{2}v_i^2}{C_p T_i + \frac{1}{2}v_i^2} = .9 \tag{cc}$$

Substituting the given numerical values, we have

PROBLEM 13.4 (continued)

$$\beta_3 = .45 \quad \text{and} \quad \beta_4/\beta_3 = .857$$

Solving for ℓ, we obtain

$$\ell \approx 1.3 \text{ meters.}$$

PROBLEM 13.5

We are given the following relations:

$$\frac{B(z)}{B_i} = \frac{E(z)}{E_i} = \frac{w_i}{w(z)} = \frac{d_i}{d(z)} = \left(\frac{A_i}{A(z)}\right)^{1/2}$$

and that v, σ, γ, and K are constant.

Part a

From Eq. (13.2.33),

$$J = (1-K)\sigma vB \tag{a}$$

For constant velocity, conservation of momentum yields

$$\frac{dp}{dz} = -(1-K)\sigma vB^2 \tag{b}$$

Conservation of energy yields

$$\rho vC_p\frac{dT}{dz} = -K(1-K)\sigma(vB)^2 \tag{c}$$

Using the equation of state,

$$p = \rho RT \tag{d}$$

we obtain

$$T\frac{d\rho}{dz} + \rho\frac{dT}{dz} = -\frac{(1-K)}{R}\sigma vB^2 \tag{e}$$

or

$$T\frac{d\rho}{dz} + \frac{(-K)(1-K)\sigma vB^2}{C_p} = -\frac{(1-K)\sigma vB^2}{R} \tag{f}$$

Thus,

$$T\frac{d\rho}{dz} = \sigma vB^2(1-K)\left(-\frac{1}{R} + \frac{K}{c_p}\right) \tag{g}$$

Also

$$B^2 = \frac{B_i^{\,2}(A_i)}{A(z)}$$

and

$$\rho_i A_i = \rho(z)A(z)$$

Therefore

$$T\frac{d\rho}{dz} = \frac{\sigma vB_i^2\ (1-K)(-\frac{1}{R} + \frac{K}{c_p})}{\rho_i}\rho(z) \tag{h}$$

and

$$\rho\, c_p\frac{dT}{dz} = -K(1-K)\sigma v\frac{B_i^2\rho}{\rho_i} \tag{i}$$

PROBLEM 13.5 (continued)

and so

$$\frac{dT}{dz} = -\frac{K(1-K)\sigma v B_i^2}{\rho_i c_p} \qquad (j)$$

Therefore

$$T = -K(1-K)\frac{\sigma v B_i^2}{\rho_i c_p} z + T_i \qquad (k)$$

Let

$$\alpha = \frac{-K(1-K)\sigma v B_i^2}{\rho_i c_p} \qquad (\ell)$$

Then

$$T = T_i\left(\frac{\alpha z}{T_i} + 1\right) \qquad (m)$$

$$\frac{d\rho}{\rho} = \frac{+\sigma v B_i^2(1-K)\left(\frac{K}{c_p} - \frac{1}{R}\right)}{\rho_i\,(\alpha z + T_i)} dz \qquad (n)$$

We let

$$\beta = \frac{+\sigma v B_i^2(1-K)\left(\frac{K}{c_p} - \frac{1}{R}\right)}{\rho_i\ \alpha}$$

$$= \frac{c_p}{KR} - 1$$

Integrating (n), we then obtain

$$\ell n\ \rho = \beta\ \ell n(\alpha z + T_i) + \text{constant}$$

or

$$\rho = \rho_i\left(\frac{\alpha z}{T_i} + 1\right)^\beta \qquad (o)$$

Therefore

$$A(z) = \frac{A_i}{\left(\frac{\alpha z}{T_i} + 1\right)^\beta} \qquad (p)$$

Part b

From (m),

$$\frac{T(\ell)}{T_i} = \frac{\alpha\ell + T_i}{T_i} = .8$$

or

$$\frac{\alpha\ell}{T_i} = -.2$$

Now

$$\frac{\alpha}{T_i} = -\frac{K(1-K)\sigma v_i B_i^2}{\rho_i c_p T_i}$$

But

$$c_p T_i = \frac{R\ T_i}{(1-\frac{1}{\gamma})} = \frac{p_i}{\rho_i(1-\frac{1}{\gamma})} = 2.5 \times 10^6$$

PROBLEM 13.5 (Continued)

Thus

$$\frac{\alpha}{T_i} = \frac{-.5(.5)50(700)16}{.7(2.5\times 10^6)} = -8.0\times 10^{-2}$$

Solving for ℓ, we obtain

$$\ell = \frac{.2}{8}\times 10^2 = 1.25 \text{ meters}$$

Part c

$$\rho = \rho_i\left(\frac{\alpha z}{T_i}+1\right)^{\beta}$$

Numerically

$$\beta = \frac{c_p}{KR} - 1 = \frac{1}{(1-\frac{1}{\gamma})K} - 1 \approx 6.$$

Thus

$$\rho(z) = .7(1-.08z)^6$$

Then it follows:

$$p(z) = \rho RT = p_i(1-.08z)^7 = 5\times 10^5(1-.08z)^7$$

$$T(z) = T_i(1-.08z)$$

From the given information, we cannot solve for T_i, only for

$$RT_i = \frac{p_i}{\rho_i} = \frac{v_i^2}{\gamma M_i^2} \approx 7\times 10^5$$

Now

$$M^2(z) = \frac{v_i^2}{\gamma RT(z)} = \frac{v_i^2}{\gamma p(z)}\rho(z) = \frac{v_i^2}{\gamma}\frac{\rho_i\left(\frac{\alpha z}{T_i}+1\right)^{\beta}}{p_i\left(\frac{\alpha z}{T_i}+1\right)^{(\beta+1)}}$$

$$= \frac{.5}{1-.08z}$$

Part d

The total electric power drawn from this generator is

$$p^e = VI = -E(z)w(z)J(z)\ell d(z)$$

$$= -E(z)(1-K)\sigma vB(z)\ell d(z)w(z)$$

$$= -E_i w_i(1-K)\sigma vB_i d_i \ell$$

But

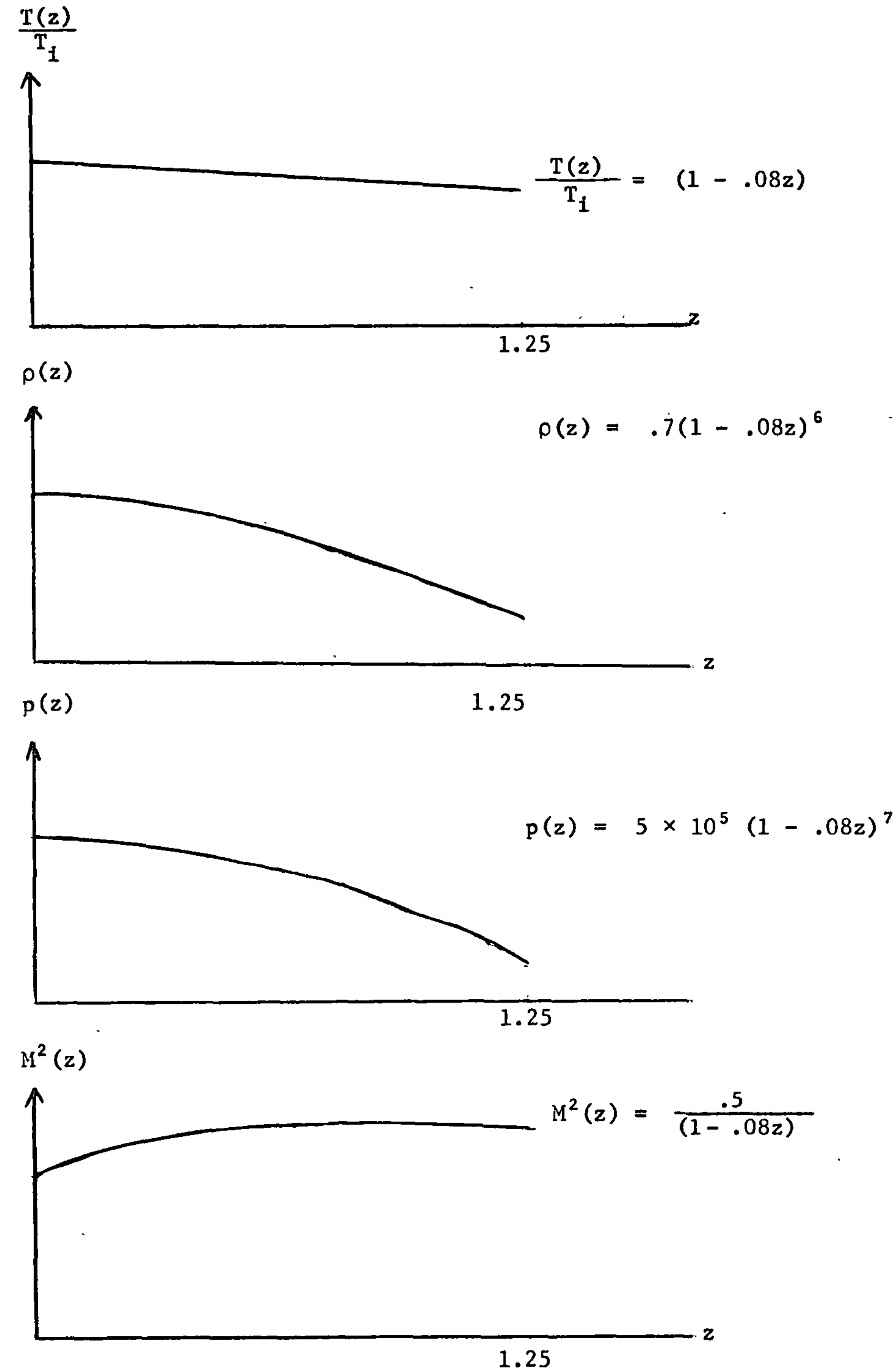
$$E_i = -KvB_i$$

Thus

$$p^e = K(vB_i)^2 w_i d_i \sigma(1-K)\ell$$

$$= .5(700)^2 16(.5)50(.5)1.25$$

$$= 61.3\times 10^6 \text{ watts} = 61.3 \text{ megawatts}$$



T(z)/T_i
T(z)/T_i = (1 - .08z)
z
1.25
ρ(z)
ρ(z) = .7(1 - .08z)^6
z
1.25
p(z)
p(z) = 5 × 10^5 (1 - .08z)^7
1.25
M^2(z)
M^2(z) = .5/(1 - .08z)
z
1.25

PROBLEM 13.6

Part a

We are given that

$$\bar{E} = \bar{i}_x \frac{4}{3} \frac{V_o}{L^{4/3}} x^{1/3} \tag{a}$$

and

$$\rho_e = \frac{4}{9} \frac{\varepsilon_o V_o}{L^{4/3} x^{2/3}} \tag{b}$$

The force equation in the steady state is

$$\rho_m v_x \frac{dv_x}{dx} \bar{i}_x = \rho_e \bar{E} \tag{c}$$

Since $\rho_e/\rho_m = q/m =$ constant, we can write

$$\frac{d}{dx}\left(\frac{1}{2} v_x^2\right) = \frac{q}{m} \frac{4}{3} \frac{V_o}{L^{4/3}} x^{1/3} \tag{d}$$

Solving for v_x we obtain

$$v_x = \sqrt{\frac{2q}{m} V_o \left(\frac{x}{L}\right)^{2/3}} \tag{e}$$

Part b

The total force per unit volume acting on the accelerator system is

$$\bar{F} = \rho_e \bar{E} \tag{f}$$

Thus, the total force which the fixed support must exert is

$$\bar{f}_{total} = -\int F dV \bar{i}_x$$

$$= -\int_0^{\ell} \frac{16}{27} \frac{\varepsilon_o V_o^2}{L^{8/3}} x^{-1/3} A dx\, \bar{i}_x$$

$$\bar{f}_{total} = -\frac{8}{9} \frac{\varepsilon_o V_o^2}{L^2} A\, \bar{i}_x$$

PROBLEM 13.7

Part a

We refer to the analysis performed in section 13.2.3a. The equation of motion for the velocity is, Eq. (13.2.76),

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x_1^2} \tag{a}$$

The boundary conditions are

$$v(-L) = V_o \cos \omega t$$

$$v(0) = 0$$

We write the solution in the form

PROBLEM 13.7 (continued)

$$v(x_1t) = \text{Re}[A\, e^{j(\omega t - kx_1)} + B\, e^{j(\omega t + kx_1)}] \qquad (b)$$

where $k = \frac{\omega}{a}$

Using the boundary condition at $x_1 = 0$, we can alternately write the solution as

$$v = \text{Re}[A \sin kx_1 e^{j\omega t}]$$

Applying the other boundary condition at $x_1 = -L$, we finally obtain

$$v(x_1,t) = -\frac{V_o}{\sin kL} \sin kx_1 \cos \omega t. \qquad (d)$$

The perturbation pressure is related to the velocity through Eq. (13.2.74)

$$\rho_o \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial x_1} \qquad (e)$$

Solving, we obtain

$$\frac{\rho_o V_o \omega}{\sin kL} \sin kx_1 \sin \omega t = -\frac{\partial p'}{\partial x_1} \qquad (f)$$

or

$$p' = \frac{\rho_o V_o \omega}{k \sin kL} \cos kx_1 \sin \omega t \qquad (g)$$

where ρ_o is the equilibrium density, related to the speed of sound a, through Eq. (13.2.83).

Thus, the total pressure is

$$p = p_o + p' = p_o + \frac{\rho_o V_o \omega}{k \sin kL} \cos kx_1 \sin \omega t \qquad (h)$$

and the perturbation pressure at $x_1 = -L$ is

$$p'(-L, t) = \frac{\rho_o V_o a}{\sin kL} \cos kL \sin \omega t \qquad (i)$$

Part b

There will be resonances in the pressure if

$$\sin kL = 0 \qquad (j)$$

or

$$kL = n\pi \qquad n = 1,2,3.... \qquad (k)$$

Thus

$$\omega = \frac{n\pi}{L} a \qquad (\ell)$$

PROBLEM 13.8

Part a

We carry through an analysis similar to that performed in section 13.2.3b. We assume that

$$\bar{E} = \bar{i}_2 E_2(x_1,t)$$
$$\bar{J} = \bar{i}_2 J_2(x_1,t)$$

PROBLEM 13.8

$$\bar{B} = \bar{i}_3 \, [\mu_o H_o + \mu_o H_3' \, (x_1,t)]$$

Conservation of momentum yields

$$\rho \frac{Dv_1}{Dt} = -\frac{\partial p}{\partial x_1} + J_2 \mu_o (H_o + H_3') \tag{a}$$

Conservation of energy gives us

$$\rho \frac{D}{Dt} \left(u + \frac{1}{2} v_1^2\right) = -\frac{\partial}{\partial x_1} (pv_1) + J_2 E_2 \tag{b}$$

We use Ampere's and Faraday's laws to obtain

$$\frac{\partial H_3'}{\partial x_1} = -J_2 \tag{c}$$

and

$$\frac{\partial E_2}{\partial x_1} = -\frac{\mu_o \partial H_3'}{\partial t} \tag{d}$$

while

Ohm's law yields

$$J_2 = \sigma[E_2 - v_1 B_3] \tag{e}$$

Since $\sigma \to \infty$

$$E_2 = v_1 B_3 \tag{f}$$

We linearize, as in Eq. (13.2.91), so $E_2 \approx v_1 \mu_o H_o$

Substituting into Faraday's law

$$\mu_o H_o \frac{\partial v_1}{\partial x_1} = -\mu_o \frac{\partial H_3'}{\partial t} \tag{g}$$

Linearization of the conservation of mass yields

$$\frac{\partial \rho'}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \tag{h}$$

Thus, from (g)

$$\frac{\mu_o H_o}{\rho_o} \frac{\partial \rho'}{\partial t} = \mu_o \frac{\partial H_3'}{\partial t} \tag{i}$$

Then

$$\frac{H_o}{H_3'} = \frac{\rho_o}{\rho'}$$

Linearizing Eq. (13.2.71), we obtain

$$\frac{Dp'}{Dt} = \frac{\gamma p_o}{\rho_o} \frac{D\rho'}{Dt} \tag{k}$$

PROBLEM 13.8 (continued)

Defining the acoustic speed

$$a_s = \left(\frac{\gamma p_o}{\rho_o}\right)^{1/2} \text{ where } p_o \text{ is the equilibrium pressure,}$$

$$p_o = p_1 - \frac{\mu_o H_o^2}{2}$$

we have

$$p' = a_s^2 \rho' \qquad (\ell)$$

Linearization of convervation of momentum (a) yields

$$\rho_o \frac{\partial v_1}{\partial t} = -\frac{\partial p'}{\partial x_1} - \frac{\partial H_3'}{\partial x_1} \mu_o H_o \qquad (m)$$

or, from (j) and (ℓ),

$$\rho_o \frac{\partial v_1}{\partial t} = \frac{\partial \rho'}{\partial x_1}\left(- a_s^2 - \frac{\mu_o H_o^2}{\rho_o}\right) \qquad (n)$$

Differentiating (n) with respect to time, and using conservation of mass (h), we finally obtain

$$\frac{\partial^2 v_1}{\partial t^2} = \left(a_s^2 + \frac{\mu_o H_o^2}{\rho_o}\right)\frac{\partial^2 v_1}{\partial x_1^2} \qquad (o)$$

Defining

$$a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \qquad (p)$$

we have

$$\frac{\partial^2 v_1}{\partial t^2} = a^2 \frac{\partial^2 v_1}{\partial x_1^2} \qquad (q)$$

Part b

We assume solutions of the form

$$V_1 = \text{Re } [A_1 e^{j(\omega t - kx_1)} + A_2 e^{j(\omega t + kx_1)}] \qquad (r)$$

where $k = \frac{\omega}{a}$

The boundary condition at $x_1 = -L$ is

$$V(-L,t) = V_s \cos \omega t = V_s \text{Re } e^{j\omega t} \qquad (s)$$

and at $x_1 = 0$

$$M \frac{dv_1(0,t)}{dt} = p'A\Big|_{x_1=0} + \mu_o H_o H_3' A\Big|_{x_1=0} \qquad (t)$$

From (h), (j) and (ℓ),

$$\frac{1}{a_s^2}\frac{\partial p'}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \qquad (u)$$

PROBLEM 13.8 (continued)

$$\frac{H_3'}{H_o} = \frac{p'}{a_s^2\rho_o} \tag{v}$$

Thus

$$M\frac{dv_1(0,t)}{dt} = A\left(\frac{\mu_o H_o^2}{a_s^2\rho_o} + 1\right)p' = A\frac{a^2}{a_s^2}p' \tag{w}$$

From (u), we solve for $p'\Big|_{x_1=0}$ to obtain:

$$p'\Big|_{x_1=0} = -\frac{\rho_o a_s^2 k}{w}(A_2 - A_1)e^{j\omega t} \tag{x}$$

Substituting into (s) and (t), we have

$$Mjw(A_1 + A_2) = A\left(\frac{a}{a_s}\right)^2\left(\frac{\rho_o a_s^2 k}{w}\right)(A_1 - A_2)$$

and

$$A_1 e^{+jk\ell} + A_2 e^{-jk\ell} = V_s \tag{y}$$

Solving for A_1 and A_2, we obtain

$$A_1 = \frac{(Mjw + Aa\rho_o)V_s}{2(-Mw\sin k\ell + Aa\rho_o\cos k\ell)}$$

$$A_2 = \frac{(Aa\rho_o - Mjw)V_s}{2(-Mw\sin k\ell + Aa\rho_o\cos k\ell)} \tag{z}$$

Thus, the velocity of the piston is

$$v_1(0,t) = \text{Re}\,[A_1 + A_2]e^{j\omega t}$$

$$v_1(0,t) = \frac{Aa\rho_o V_s}{-Mw\sin k\ell + Aa\rho_o\cos k\ell}\cos\omega t \tag{aa}$$

PROBLEM 13.9

Part a

The differential equation for the velocity as derived in problem 13.8 is

$$\frac{\partial^2 v_1}{\partial t^2} = a^2\frac{\partial^2 v_1}{\partial x_1^2} \tag{a}$$

where

$$a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o}$$

with

$$a_s^2 = \left(\frac{\gamma p_o}{\rho_o}\right)^{1/2} \quad \text{where } p_o = p_1 - \frac{\mu_o H_o^2}{2}$$

Part b

We assume a solution of the form

PROBLEM 13.9 (continued)

$$V(x_1,t) = \text{Re}\,[De^{j(\omega t-kx_1)}] \quad \text{where } k = \frac{w}{a}$$

We do not consider the negatively traveling wave, as we want to use this system as a delay line without distortion. The boundary condition at $x_1 = -L$ is

$$V(-L,t) = \text{Re}\,V_s e^{j\omega t}$$

and at $x_1 = 0$ is

$$M\frac{dV_1(0,t)}{dt} \to 0 = p'(0,t)A - BV_1(0,t) + \mu_o H_o H_3' A \qquad \text{(b)}$$

From problem 13.8, (h), (j) and (ℓ)

$$p' = a_s^2\rho' \quad , \quad \frac{\partial\rho'}{\partial t} = -\rho_o\frac{\partial v_1}{\partial x_1} \quad \text{and} \quad \frac{H_3'}{H_o} = \frac{p'}{a_s^2\rho_o}$$

Thus, (b) becomes

$$-BDe^{j\omega t} + \left(\frac{a}{a_s}\right)^2 p'A = 0 \qquad \text{(c)}$$

where

$$p'\Big|_{x_1=0} = -\frac{\rho_o D(-jk)}{jw}a_s^2 e^{j\omega t} \qquad \text{(d)}$$

Thus, for no reflections

$$-B + \left(\frac{a}{a_s}\right)^2\frac{A\rho_o a_s^2}{a} = 0 \qquad \text{(e)}$$

or

$$B = Aa\rho_o \qquad \text{(f)}$$

PROBLEM 13.10

The equilibrium boundary conditions are

$$T[-(L_1 + L_2 + \Delta),t] = T_o$$

$$T[-(L_1 + \Delta),t]A_s = -p_o A_c$$

Boundary conditions for incremental motions are

1) $T[-(L_1 + L_2 + \Delta),t] = T_s(t)$
2) $-T[-(L_1 + \Delta),t]A_s - p(-L_1,t)A_c = M\frac{d}{dt}v_\ell(-L_1,t)$
3) $v_\ell(-L_1,t) = v_e[-(L_1 + \Delta),t]$ since the mass is rigid

and 4) $v_\ell(0,t) = 0$ since the wall at x=0 is fixed.

PROBLEM 13.11

Part a

We can immediately write down the equation for perturbation velocity, using equations (13.2.76) and (13.2.77) and the results of chapters 6 and 10.

PROBLEM 13.11 (continued)

We replace $\partial/\partial t$ by $\partial/\partial t + v\cdot\nabla$ to obtain

$$\left(\frac{\partial}{\partial t} + V_o \frac{\partial}{\partial x}\right)^2 v' = a_s^2 \frac{\partial^2 v'}{\partial x^2}$$

Letting $v' = \text{Re}\ \hat{V}\, e^{j(\omega t - kx)}$

we have

$$(\omega - kV_o)^2 = a_s^2 k^2$$

Solving for w, we obtain

$$\omega = k(V_o \pm a_s)$$

Part b

Solving for k, we have

$$k = \frac{\omega}{V_o \pm a_s}$$

For $V_o > a_s$, both waves propagate in the positive x- direction.

PROBLEM 13.12

Part a

We assume that

$$\bar{E} = \bar{i}_z\, E_z(x,t)$$
$$\bar{J} = \bar{i}_z\, J_z(x,t)$$
$$\bar{B} = \bar{i}_y\, \mu_o[H_o + H_y'(x,t)]$$

We also assume that all quantities can be written in the form of Eq. (13.2.91).

$$\rho_o \frac{\partial v_x}{\partial t} = -\frac{\partial p'}{\partial x} - J_z \mu_o H_o \qquad \text{(conservation of momentum linearized)} \qquad (a)$$

The relevant electromagnetic equations are

$$\frac{\partial H_y'}{\partial x} = J_z \qquad (b)$$

and

$$\frac{\partial E_z}{\partial x} = \mu_o \frac{\partial H_y'}{\partial t} \qquad (c)$$

and the constitutive law is

$$J_z = \sigma(E_z + v_x \mu_o H_o) \qquad (d)$$

We recognize that Eqs. (13.2.94), (13.2.96) and (13.2.97) are still valid, so

$$\frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} = -\frac{\partial v_x}{\partial x} \qquad (e)$$

PROBLEM 13.12 (continued)

and

$$p' = a_s^2 \rho' \tag{f}$$

Part b

We assume all perturbation quantities are of the form

$$v_x = \text{Re}[\hat{v}\, e^{j(\omega t - kx)}]$$

Using (b), (a) may be rewritten as

$$\rho_o j\omega\hat{v} = +jk\hat{p} + \mu_o H_o jk\hat{H} \tag{g}$$

and (c) may now be written as

$$-jk\hat{E} = \mu_o j\omega\hat{H} \tag{h}$$

Then, from (b) and (d)

$$-jk\hat{H} = \sigma(\hat{E} + \hat{v}\mu_o H_o) \tag{i}$$

Solving (g) and (h) for $\hat{H}$ in terms of $\hat{v}$, we have

$$\hat{H} = \frac{\hat{v}\sigma\mu_o H_o}{\left(-jk + \sigma\mu_o \frac{\omega}{k}\right)} \tag{j}$$

From (e) and (f), we solve for $\hat{p}$ in terms of $\hat{v}$ to be

$$\hat{p} = \frac{k}{\omega}\rho_o a_s^2 \hat{v} \tag{k}$$

Substituting (j) and (k) back into (g), we find

$$\hat{v}\left[\rho_o j\omega - \frac{jk^2}{\omega}\rho_o a_s^2 - \frac{jk(\mu_o H_o)^2\sigma}{\left[-jk + \frac{\sigma\mu_o\omega}{k}\right]}\right] = 0 \tag{ℓ}$$

Thus, the dispersion relation is

$$(\omega^2 - k^2 a_s^2) - \frac{j(\mu_o H_o)^2 \omega\, k^2}{(+\frac{k^2}{\sigma} + j\mu_o\omega)\rho_o} = 0 \tag{m}$$

We see that in the limit as $\sigma \to \infty$, this dispersion relation reduces to the lossless dispersion relation

$$\omega^2 - k^2\left(a_s^2 + \frac{\mu_o H_o^2}{\rho_o}\right) = 0 \tag{n}$$

Part c

If σ is very small, we can approximate (m) as

$$\omega^2 - k^2 a_s^2 - j(\mu_o H_o)^2 \frac{\omega\sigma}{\rho_o}\left(1 - \frac{j\omega\mu_o\sigma}{k^2}\right) = 0 \tag{o}$$

for which we can rewrite (o) as

PROBLEM 13.12 (continued)

$$k^4 a_s^2 - k^2\left[\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}\right] + \left(\frac{\mu_o H_o}{\rho_o}\right)^2 \omega^2\sigma^2\mu_o = 0 \tag{p}$$

Solving for k^2, we obtain

$$k^2 = \frac{\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}}{2\,a_s^2} \pm \sqrt{\left[\frac{\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}}{2a_s^2}\right]^2 - \frac{\left(\frac{\mu_o\omega^2\sigma^2}{\rho_o}\right)(\mu_o H_o)^2}{a_s^2}} \tag{q}$$

Since σ is very small, we expand the radical in (q) to obtain

$$k^2 = \frac{\left[\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}\right]}{2\,a_s^2} \pm \left[\frac{\omega^2 - \frac{j\omega\sigma}{\rho_o}(\mu_o H_o)^2}{2\,a_s^2} - \frac{\left(\frac{\mu_o\omega^2\sigma^2}{\rho_o}\right)(\mu_o H_o)^2}{\left[\omega^2 - \frac{j\omega\sigma}{\rho_o}(\mu_o H_o)^2\right]}\right] \tag{r}$$

Thus, our approximate solutions for k^2 are

$$k^2 \approx \frac{\left[\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}\right]}{a_s^2} \tag{s}$$

and

$$k^2 \approx \frac{\left(\frac{\mu_o\omega^2\sigma^2}{\rho_o}\right)(\mu_o H_o)^2}{\left[\omega^2 - \frac{j\omega\sigma}{\rho_o}(\mu_o H_o)^2\right]} \approx \left(\frac{\mu_o\sigma^2}{\rho_o}\right)(\mu_o H_o)^2 \tag{t}$$

The wavenumbers for the first pair of waves are approximately:

$$k \approx \pm\left(\frac{\omega - j\frac{\sigma}{2\rho_o}(\mu_o H_o)^2}{a_s}\right) \tag{u}$$

while for the second pair, we obtain

$$k \approx \pm\ \sigma(\mu_o H_o)\sqrt{\frac{\mu_o}{\rho_o}} \tag{v}$$

The wavenumbers from (u) represent a forward and backward traveling wave, both with amplitudes exponentially decreasing. Such waves are called 'diffusion waves'. The wavenumbers from (v) represent pure propagating waves in the forward and reverse directions.

PROBLEM 13.12 (continued)

Part d

If σ is very large, then (m) reduces to

$$\omega^2 - k^2 a^2 - j\,\frac{H_o^2}{\rho_o}\,\frac{k^4}{\sigma\omega} = 0\ ;\quad a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \tag{w}$$

This can be put in the form

$$k^2 = \frac{\omega^2}{a^2} - j\,\frac{f(\omega,k)}{\sigma} \tag{x}$$

where

$$f(\omega,k) = \frac{H_o^2\ k^4}{\rho_o\ \omega a^2}$$

As σ becomes very large, the second term in (x) becomes negligible, and so

$$k^2 \approx \frac{\omega^2}{a^2} \tag{y}$$

However, it is this second term which represents the damping in space; that is,

$$k \approx \pm\left[\frac{\omega}{a} - j\,\frac{f(\omega,k)}{2\sigma\ \omega}\,a\right] \tag{z}$$

Thus, the approximate decay rate, k_i, is

$$k_i \approx \frac{f(\omega,k)\,a}{2\sigma\ \omega} = \frac{H_o^2\ k^4}{2\rho_o\,\omega a^2\sigma}\,\frac{a}{\omega} \tag{aa}$$

or

$$k_i \approx \frac{H_o^2}{2\rho_o\ a\sigma}\ \frac{k^4}{\omega^2} = \frac{H_o^2}{2\rho_o a^5\sigma}\ \omega^2$$

PROBLEM 14.1

Part a

We can specify the relevant variables as

$$\begin{aligned} \bar{v} &= \bar{i}_1\, v_1(x_2) \\ \bar{E} &= \bar{i}_2 E_2(x_2) + i_3 E_3(x_2) \\ \bar{J} &= \bar{i}_2\, J_o \\ \bar{B} &= \bar{i}_2 B_o + \bar{i}_1 B_1(x_3) \end{aligned} \qquad (a)$$

The x_1 component of the momentum equation is

$$0 = \mu \frac{\partial^2 v_1}{\partial x_2^{\,2}} \qquad (b)$$

with solution

$$v_1 = C_1 x_2 + C_2$$

Applying the boundary conditions

$$\begin{aligned} v_1 &= 0 \quad @ \quad x_2 = 0 \\ v_1 &= v_o \quad @ \quad x_2 = d \end{aligned} \qquad (c)$$

We obtain

$$v_1 = \frac{v_o x_2}{d} \qquad (d)$$

We note that there is no magnetic force density since the imposed current and magnetic field are colinear. We apply Ohm's law for a moving fluid

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B}) \qquad (e)$$

in the x_2 and x_3 directions to obtain

$$J_o = \sigma E_2 \qquad (f)$$

and

$$0 = \sigma(E_3 + v_1 B_o) \qquad (g)$$

since no current can flow in the x_3 direction.

Thus

$$E_2 = \frac{J_o}{\sigma} \qquad (h)$$

and

$$E_3 = -\frac{v_o x_2 B_o}{d} \qquad (i)$$

As from Eq. (14.2.5),

$$V = \int_0^d E_2 dx_2 = \frac{J_o}{\sigma} d \qquad (j)$$

Thus, the electrical input p_e per unit area in an $x_1 - x_3$ plane is

$$p_e = J_o V = \frac{J_o^2 d}{\sigma} \qquad (k)$$

PROBLEM 14.1 (continued)

We see that this power is dissipated as Ohmic loss. The moving fluid looks just like a resistor from the electrical terminals. The traction that must be applied to the upper plate to maintain the steady motion is

$$\tau = \mu \frac{\partial v_1}{\partial x_2}\bigg|_{x_2=d} = \frac{\mu v_o}{d} \qquad (\ell)$$

Again we note no contribution from the magnetic forces.

The mechanical input power per unit area is then

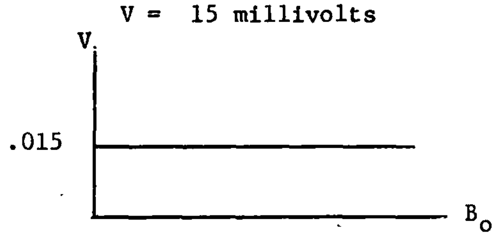
$$p_m = \tau_1 v_o = \frac{\mu v_o^2}{d} \qquad (m)$$

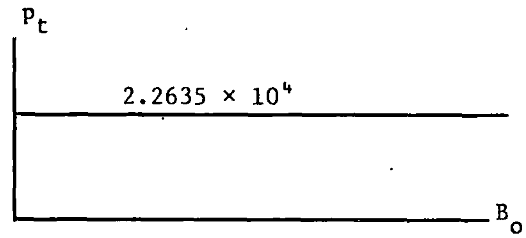
The total input power per unit area is thus

$$p_t = p_e + p_m = \frac{\mu v_o^2}{d} + \frac{J_o^2 d}{\sigma} \qquad (n)$$

The first term is due to viscous loss that results from simple shear flow, while the second term is simply the Joule loss associated with Ohmic heating. There is no electromechanical coupling. Using the parameters from Table 14.2.1, we obtain

$V = 15$ millivolts





and

$$p_t = 2.2635 \times 10^4 \text{ watts/m}^2, \text{ independent of } B_o.$$

These results correspond to the plots of Fig. 14.2.3 in the limit as

$$B_o \rightarrow 0.$$

We see that the brush losses and brush voltage are much less for this configuration than for that analyzed in Sec. 14.2.1. This is because the electrical and mechanical equations were uncoupled when the applied flux density was in the x_2 direction. This configuration is better, because low voltages at the brush eliminate arcing, and because the net power input per unit area is less no matter the field strength B_o.

The only effect of applying a flux density in the x_2 direction was to cause an electric field in the x_3 direction. However, since there was no current flow in the x_3 direction, there was no additional dissipated power. However, if E_3 became too large, the fluid might experience electrical breakdown, resulting in corona arcs.

PROBLEM 14.2

The momentum equation for the fluid is

$$\rho\frac{\partial\overline{v}}{\partial t} + \rho(\overline{v}\cdot\nabla)\overline{v} = -\nabla p + \mu\nabla^2\overline{v} \tag{a}$$

We consider solutions of the form

$$\overline{v} = \overline{i}_z\, v_z(r)$$

and

$$p = p(z).$$

Then in the steady state, we write the z component of (a) in cylindrical coordinates as

$$\frac{\partial p}{\partial z} = \mu\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} \tag{b}$$

Now, the left side of (b) is only a function of z, while the right side is only a function of r. Thus, from the given information

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L} \tag{c}$$

Using the results of (c) in (b), we solve for $v_z(r)$ in the form

$$v_z(r) = \frac{p_2 - p_1}{4L\mu} r^2 + A\,\ell n\, r + B \tag{d}$$

where A and B are arbitrary constants to be evaluated by the boundary conditions

$$v_z(r = 0) \text{ is finite}$$

and

$$v_z(r = R) = 0$$

Thus the solution is

$$v_z(r) = \frac{(p_2 - p_1)}{4\,\mu L}(r^2 - R^2) \tag{e}$$

We can also find relations between the flow rate and the pressure difference, since

$$\int_0^R v_z\, 2\pi r dr = Q$$

PROBLEM 14.3

Part a

We are given the pressure drop Δp, the magnetic field B_o, the conductivity σ, and the dimensions of the system.

Now

$$i = \int_{-d}^{+d} J\ell dx_2 = \sigma\ell\int_{-d}^{+d}(E_3 + v_1 B_o)dx_2 = \frac{V}{R} \tag{a}$$

where

$V = -\frac{E}{w}$ is defined as the voltage across the resistor.

From Eq. (14.2.29), we have the solution for the velocity v_1. We then perform the integrations of (a) and solve for the voltage V to obtain

PROBLEM 14.3(continued)

$$V = \frac{\frac{(\Delta p)2d}{B_o}\left(1 - \frac{\tanh M}{M}\right)}{\frac{1}{R} + \frac{2\sigma \ell d}{w} \frac{\tanh M}{M}} \qquad \text{(b)}$$

where

$$M = B_o d\left(\frac{\sigma}{\mu}\right)^{1/2}$$

Then, the power p^e dissipated in the resistor is

$$p^e = \frac{V^2}{R} = \frac{\left(\frac{\Delta p\ 2d}{B_o}\right)^2 \left(1 - \frac{\tanh M}{M}\right)^2}{\left(\frac{1}{R} + \frac{1}{R_i} \frac{\tanh M}{M}\right)^2 R} \qquad \text{(c)}$$

where we have defined the internal resistance R_i as

$$R_i = \frac{w}{2\sigma \ell d}$$

Part b

To maximize p^e, we differentiate (c) with respect to R, solve for that value of R which makes this quantity zero, and then check that this value does indeed maximize p^e. Performing these operations, we obtain

$$R_{max} = \frac{M R_i}{\tanh M} \qquad \text{(d)}$$

Part c

We must convert the given numerical values to MKS units, using the conversions

10,000 gauss = 1 Weber/meter^2

and 100 cm = 1 meter

For mercury

$$\sigma = 10^6 \text{ mhos/m}$$

and $\mu = 1.5 \times 10^{-3}$ kg/m-sec.

Thus

$$M = B_o d\left(\frac{\sigma}{\mu}\right)^{1/2} = 2 \times 10^{-2}\left(\frac{1}{1.5} \times 10^9\right)^{1/2}$$

$$M = 520$$

Then $\tanh M \approx 1$

and so

$$R_{max} = 520\left(\frac{10^{-1}}{2 \times 10^6 \times 10^{-2}}\right) \approx 2.60 \times 10^{-3} \text{ ohms.}$$