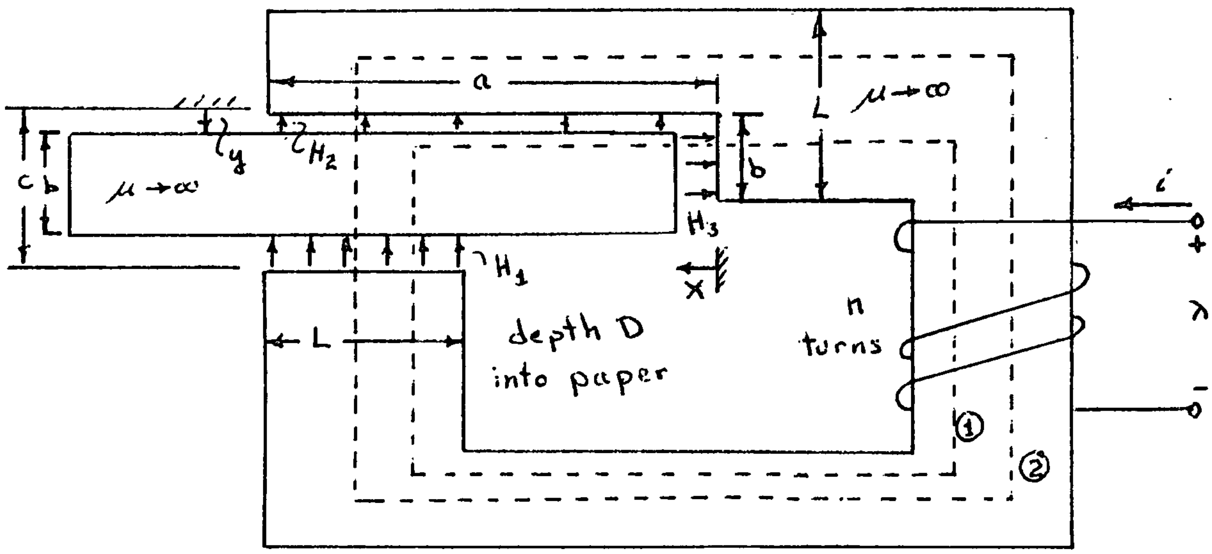
PROBLEM 2.1



We start with Maxwell's equations for a magnetic system in integral form:

$$\oint_C H \cdot d\bar{\ell} = \int_S \bar{J} \cdot d\bar{a}$$

$$\oint_S B \cdot d\bar{a} = 0$$

Using either path 1 or 2 shown in the figure with the first Maxwell equation we find that

$$\int \bar{J} \cdot d\bar{a} = ni$$

To compute the line integral of H we first note that whenever $\mu \to \infty$ we must have $\bar{H} \to 0$ if $\bar{B} = \mu \bar{H}$ is to remain finite. Thus we will only need to know H in the three gaps (H_1, H_2 and H_3) where the fields are assumed uniform because of the shortness of the gaps. Then

path 1
$$\oint_C H \cdot d\bar{\ell} = H_1(c-b-y) + H_3 x = ni \qquad (a)$$

path 2
$$\oint_C H \cdot d\bar{\ell} = H_1(c-b-y) + H_2 y = ni \qquad (b)$$

Using the second Maxwell equation we write that the flux of B into the movable slab equals the flux of B out of the movable slab

$$\mu_o H_1 LD = \mu_o H_2 aD + \mu_o H_3 bD$$

or

$$H_1 L = H_2 a + H_3 b \qquad \text{(c)}$$

Note that in determining the relative strengths of H_1,H_2 and H_3 in this last equation we have let $(a-x) \simeq a$, $(b-y) \simeq b$ to simplify the solution. This means that we are assuming that

$$x/a \ll 1, \; y/b \ll 1 \qquad \text{(d)}$$

Solving for H_1 using (a), (b), and (c)

$$H_1 = \frac{ni(y/a + x/b)}{(c-b-y)(y/a + x/b) + L(y/a \cdot x/b)}$$

The flux of B through the n turns of the coil is then

$$(x,y,i) = nB_1 LD = n\mu_o H_1 LD$$
$$= \frac{\mu_o n^2 (y/a + x/b)LD\, i}{(c-b-y)(y/a+x/b) + L(y/a \cdot x/b)}$$

Because we have assumed that the air gaps are short compared to their cross-sectional dimensions we must have

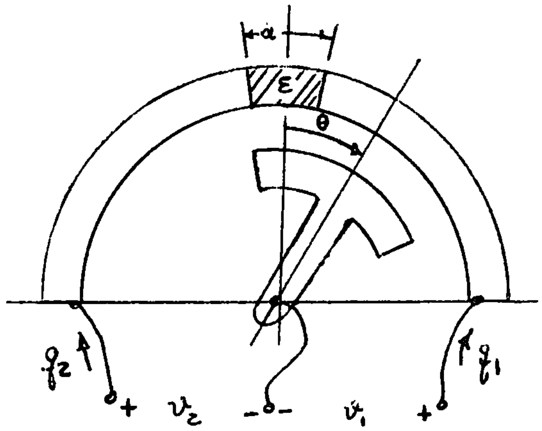
$$\frac{(c-b-y)}{L} \ll 1, \; y/a \ll 1 \text{ and } x/b \ll 1$$

in addition to the constraints of (d) for our expression for λ to be valid. If we assume that $a>L>c>b>(c-b)$ as shown in the diagram, these conditions become

$$x \ll b$$

$$y \ll b$$

PROBLEM 2.2



Because the charge is linearly related to the applied voltages we know that $q_1(V_1,V_2,\theta) = q_1(v_1,0,\theta) + q_1(0,v_2,\theta)$

$$q_1(V_1,0,\theta) = \frac{\varepsilon V_1}{R\alpha}\, w\ell + \varepsilon_o \frac{V_1}{g}\left(\frac{\pi}{4} + \theta\right)R\ell$$

$$q_1(0,V_2,\theta) = -\frac{\varepsilon V_2}{R\alpha}\, w\ell$$

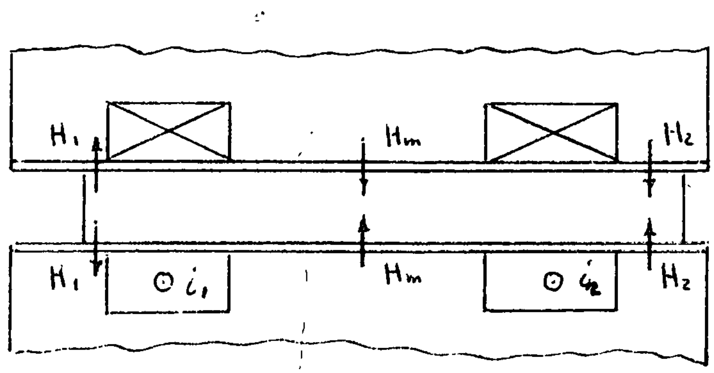
Hence

$$q_1(V_1,V_2,\theta) = V_1\ell\left(\frac{\varepsilon w}{R\alpha} + \frac{\varepsilon_o(\pi/4+\theta)R}{g}\right) - V_2\ell\frac{\varepsilon w}{R\alpha}$$

$$q_2(V_1,V_2,\theta) = -V_1\ell\,\frac{\varepsilon w}{R\alpha} + V_2\ell\left(\frac{\varepsilon w}{R} + \frac{\varepsilon_o(\pi/4-\theta)R}{g}\right)$$

PROBLEM 2.3

The device has cylindrical symmetry so that we assume that the fields in the gaps are essentially radial and denoted as shown in the figure.



Ampere's law can be integrated around each of the current loops to obtain the relations

PROBLEM 2.3 (Continued)

$$gH_1 + gH_m = Ni_1 \tag{a}$$

$$gH_2 - gH_m = Ni_2 \tag{b}$$

In addition, the net flux into the plunger must be zero, and so

$$\mu_o(d-x)2\pi rH_1 - 2d(2\pi r)\mu_o H_m - (d+x)(2\pi r)\mu_o H_2 \tag{c}$$

These three equations can be solved for any one of the intensities. In particular we are interested in H_1 and H_2, because the terminal fluxes can be written simply in terms of these quantities. For example, the flux linking the (1) winding is N times the flux through the air gap to the left

$$\lambda_1 = \mu_o N(d-x)(2\pi r)H_1 \tag{d}$$

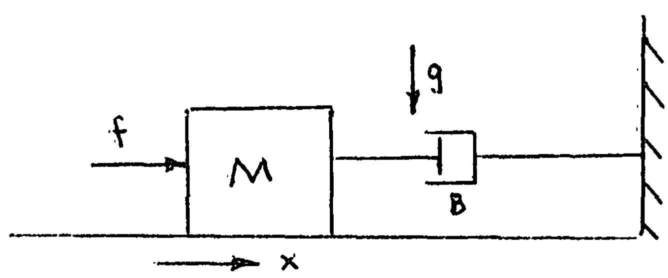
Similarly, to the right,

$$\lambda_2 = \mu_o N(d+x)(2\pi r)H_2 \tag{e}$$

Now, if we use the values of H_1 and H_2 found from (a) - (c), we obtain the terminal relations of Prob. 2.3 with

$$L_o = \frac{\mu_o \pi r N^2 d}{2g}$$

PROBLEM 2.4



PROBLEM 2.4 (Continued)

Part a

$$\sum_i f_i = Ma = M\frac{dx^2}{dt^2}$$

$$f_{DAMPER} = -B\frac{dx}{dt}\ ;\ f_{coul} = -\mu_d\, Mg\,\frac{\frac{dx_1}{dt}}{\left|\frac{dx_1}{dt}\right|}$$

$$M\frac{d^2x}{dt^2} = f(t) - B\frac{dx}{dt} + f_{coul}$$

or

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = f(t) - \mu_d Mg\,\frac{\frac{dx_1}{dt}}{\left|\frac{dx_1}{dt}\right|}$$

Part b

First we recognize that the block will move so that $\frac{dx_1}{dt} > 0$, hence

$$f_{coul} = -\mu_d Mg;\ \frac{dx_1}{dt} > 0$$

Then for $t > 0$

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = -\mu_d Mg$$

which has a solution

$$x(t) = -\frac{\mu_d Mg}{B}\,t + c_1 + c_2\,e^{-(B/M)t}$$

Equating singularities at $t = 0$

$$M\frac{d^2x}{dt^2}(0) = I_o\mu_o(t) \quad \text{or} \quad \frac{d^2x}{dt^2}(0) = \frac{I_o}{M}\,\mu_o(t)$$

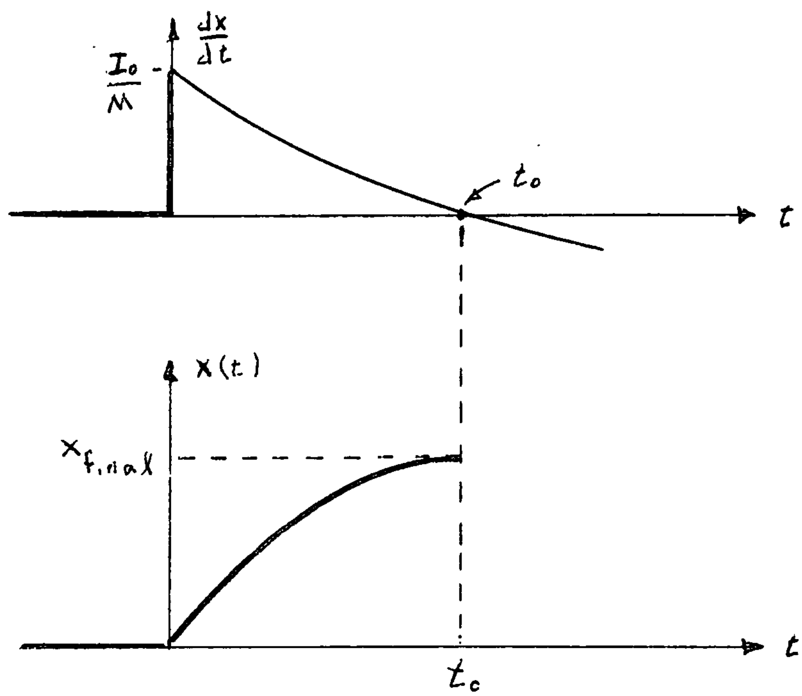
Then since $x(0^-) = \frac{dx}{dt}(0^-) = \frac{d^2x}{dt^2}(0^-) = 0$

PROBLEM 2.4 (Continued)

$$\frac{dx}{dt}(0^+) = \frac{I_o}{M} \; ; \; x(0^+) = 0$$

Hence $x(t) = u_{-1}(t)[-\frac{\mu_d Mg}{B} t + (\frac{I_o}{B} + \mu_d g(\frac{M}{B})^2)(1-e^{-(B/M)t})]$

Actually, this solution will only hold until t_o, where $\frac{dx}{dt}(t_o) = 0$, at which point the mass will stop.



PROBLEM 2.5

Part a

Equation of motion

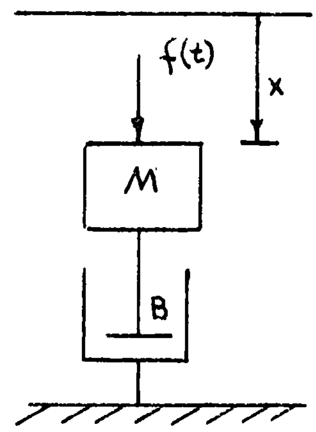
$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} = f(t)$$

(1) $f(t) = I_o u_o(t)$

$$x(t) = u_{-1}(t) \frac{I_o}{B} (1-e^{-(B/M)t})$$

PROBLEM 2.5 (Continued)

as shown in Prob. 2.4 with $\mu_d = 0$.



(2) $f(t) = F_o u_{-1}(t)$

Integrating the answer in (1)

$$x(t) = \frac{F_o}{B}[t + \frac{M}{B}(e^{-(B/M)t} - 1)]u_{-1}(t)$$

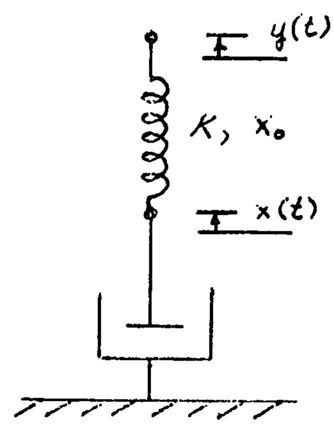
Part b

Consider the node connecting the damper and the spring; there must be no net force on this node or it will suffer infinite acceleration.

$$-B\frac{dx}{dt} + K(y-x) = 0$$

or

$$B/K\frac{dx}{dt} + x = y(t)$$



1. Let $y(t) = Au_o(t)$

$$\frac{B}{K}\frac{dx}{dt} + x = 0 \qquad t > 0$$

$$x(t) = C\,e^{-K/Bt} \qquad t > 0$$

But at $t = 0$

$$\frac{B}{K}\frac{dx}{dt}(0) = Au_o(0)$$

Now since $x(t)$ and $\frac{dx}{dt}(t)$ are zero for $t < 0$

$$x(0^+) = \frac{AK}{B} = C$$

$$x(t) = u_{-1}(t)\frac{AK}{B}e^{-(K/B)t} \quad \text{all } t$$

2. Let $y(t) = Au_{-1}(t)$

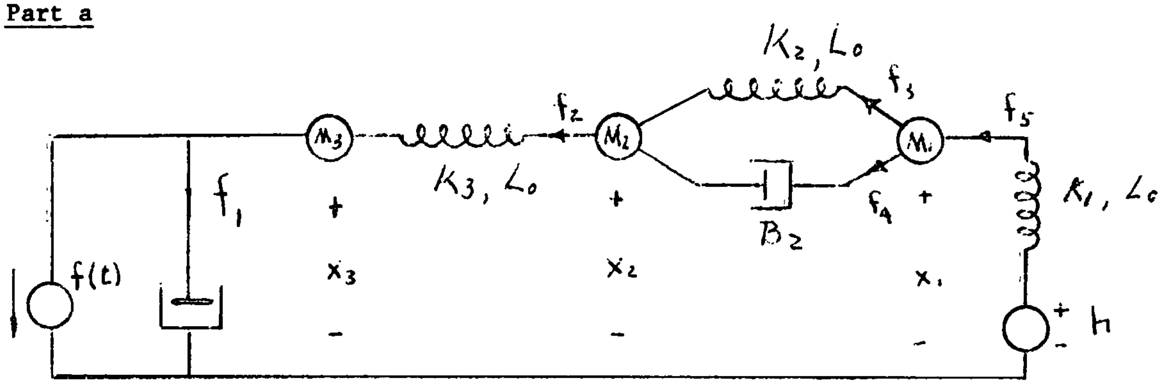
PROBLEM 2.5 (Continued)

Integrating the answer in (1)

$$x(t) = u_{-1}(t)\ Y_o(1-e^{-(K/B)t} \qquad \text{all } t$$

PROBLEM 2.6

Part a



$$f_1 = B_3 \frac{dx_3}{dt}\ ;\ f_2 = K_3(x_2-x_3-t-L_o)$$

$$f_3 = K_2(x_1-x_2-t-L_o);\ f_4 = B_2 \frac{d}{dt}(x_1-x_2)$$

$$f_5 = K_1(h-x_1-L_o)$$

Part b

Summing forces at the nodes and using Newton's law

$$K_1(h-x_1-L_o) = K_2(x_1-x_2-t-L_o) + B_2 \frac{d\ (x_1-x_2)}{dt} + M_1 \frac{d^2x_1}{dt^2}$$

$$K_2(x_1-x_2-t-L_o) + B_2 \frac{d\ (x_1-x_2)}{dt} = K_3(x_2-x_3-t-L_o) + M_2 \frac{d^2x_2}{dt^2}$$

$$K_3(x_2-x_3-t-L_o) = f(t) + B_3 \frac{dx_3}{dt} + M \frac{d^2x_3}{dt^2}$$

PROBLEM 2.6 (Continued)

Let's solve these equations for the special case

$$M_1 = M_2 = M_3 = B_2 = B_3 = L_o = 0$$

Now nothing is left except three springs pulled by force f(t). The three equations are now

$$K_1(h-x_1) = K_2(x_1-x_2) \qquad (a)$$

$$K_2(x_1-x_2) = K_3(x_2-x_3) \qquad (b)$$

$$K_3(x_2-x_3) = f(t) \qquad (c)$$

We write the equation of geometric constraint

$$x_3 + (x_2-x_3) + (x_1-x_2) + (h-x_1)-h = 0$$

or

$$(h-x_3) = (x_2-x_3) + (x_1-x_2) + (h-x_1) \qquad (d)$$

which is really a useful identity rather than a new independent equation. Substituting in (a) and (b) into (d)

$$(h-x_3) = \frac{K_3(x_2-x_3)}{K_3} + \frac{K_3(x_2-x_3)}{K_2} + \frac{K_3(x_2-x_3)}{K_1}$$

$$= K_3(x_2-x_3)\left(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1}\right)$$

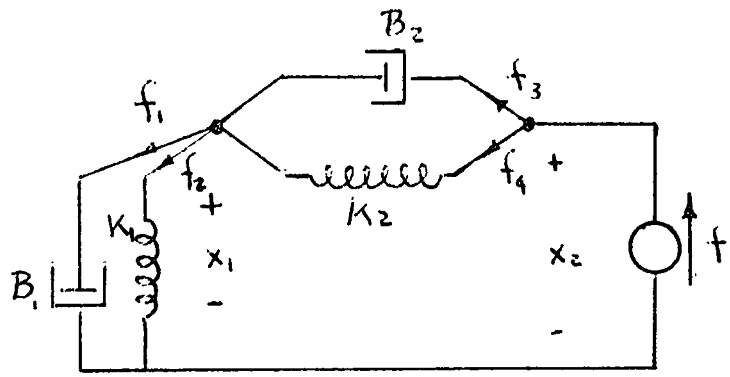
which can be plugged into (c)

$$\left(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1}\right)^{-1}(h-x_3)=f(t)$$

which tells us that three springs in series act like a spring with

$$K' = \left(\frac{1}{K_3} + \frac{1}{K_2} + \frac{1}{K_1}\right)^{-1}$$

PROBLEM 2.7



$$f_1 = B_1 \frac{dx_1}{dt} \qquad f_2 = K_1 x_1$$

$$f_3 = B_2 \frac{d(x_2 - x_1)}{dt} \quad f_4 = K_2(x_2 - x_1)$$

Node equations:

Node 1 $$B_1 \frac{dx_1}{dt} + K_1 x_1 = B_2 \frac{d(x_2 - x_1)}{dt} + K_2(x_2 - x_1)$$

Node 2 $$B_2 \frac{d(x_2 - x_1)}{dt} + K_2(x_2 - x_1) = f$$

To find natural frequencies let $f = 0$

$$B_1 \frac{dx_1}{dt} + K_1 x_1 = 0 \quad \text{Let} \quad x_1 = e^{st}$$

$$B_1 s + K_1 = 0 \qquad s_1 = -K_1/B_1$$

$$B_2 \frac{d(x_2 - x_1)}{dt} + K_2(x_2 - x_1) = 0 \quad \text{Let} \quad (x_2 - x_1) = e^{st}$$

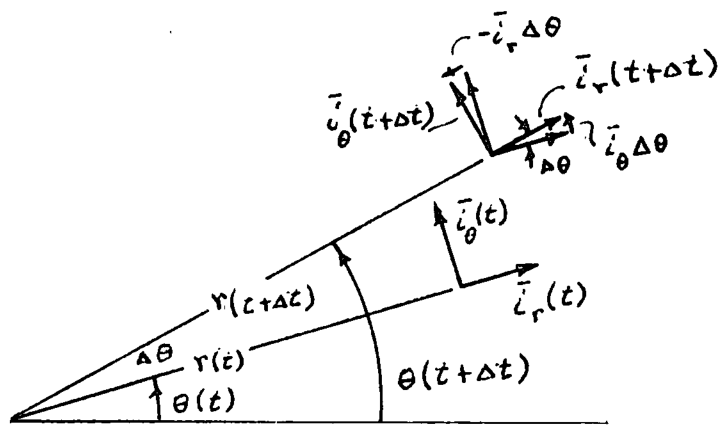
$$B_2 s + K_2 = 0 \qquad s_2 = -K_2/B$$

The general solution when $f = 0$ is then

$$x_1 = c_1\, e^{-(K_1/B_1)t}$$

$$x_2 = (x_2 - x_1) + x_1 = c_1 e^{-(K_1/B_1)t} + c_2 e^{-(K_2/B_2)t}$$

PROBLEM 2.8



From the diagram, the change in $\bar{i}_r$ in the time Δt is $\bar{i}_\theta \Delta\theta$. Hence

$$\frac{d\bar{i}_r}{dt} = \lim_{\Delta t \to 0} \bar{i}_\theta \frac{\Delta\theta}{\Delta t} = \bar{i}_\theta \frac{d\theta}{dt} \tag{a}$$

Similarly,

$$\frac{d\bar{i}_\theta}{dt} = \lim_{\Delta t \to 0} - \bar{i}_r \frac{\Delta\theta}{\Delta t} = - \bar{i}_r \frac{d\theta}{dt} \tag{b}$$

Then, the product rule of differentiation on $\bar{v}$ gives

$$\frac{d\bar{v}}{dt} = \frac{d\bar{i}_r}{dt}\frac{dr}{dt} + \bar{i}_r \frac{d^2 r}{dt^2} + \frac{di_\theta}{dt}\left(r \frac{d\theta}{dt}\right) + i_\theta \frac{d}{dt}\left(r \frac{d\theta}{dt}\right) \tag{c}$$

and the required acceleration follows by combining these equations.