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*Solutions Manual for Electromechanical Dynamics*

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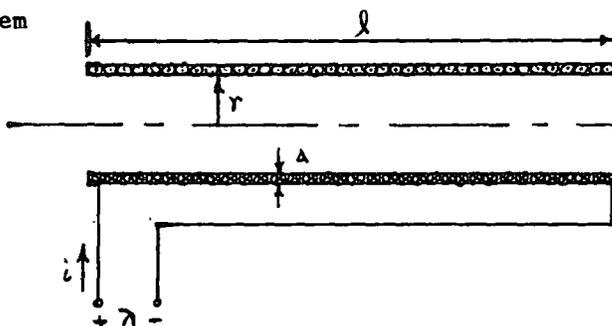
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## MAGNETIC DIFFUSION AND CHARGE RELAXATION

### PROBLEM 7.1

It is the purpose of this problem to illustrate the limitations inherent to common conductors in achieving long magnetic time constants. (Diffusion times.) For convenience in making this point consider the solenoid shown with



$$l = \text{length}$$

$$\Delta = \text{cross-sectional dimensions of single layer of wire (square-cross section).}$$

$$r = \text{radius } (r \gg \Delta) \text{ but } r \ll l.$$

Then there are  $l/\Delta$  turns, each having a length  $2\pi r$ , and the total d-c resistance is directly proportional to the length and inversely proportional to the area and electrical conductivity  $\sigma$ .

$$R = \frac{2\pi r}{\sigma(\Delta^2)} \left(\frac{l}{\Delta}\right)$$

The H field in the axial direction, by Ampere's law, is  $H = \frac{i}{\Delta}$  and the flux linked by one turn is  $\mu_0 H(\pi r^2)$  so that

$$\lambda = \mu_0 H(\pi r^2) \left(\frac{l}{\Delta}\right) = \mu_0 (\pi r^2) \frac{l}{\Delta^2} i$$

and it follows that

$$L = \mu_0 (\pi r^2) \left(\frac{l}{\Delta^2}\right)$$

Finally, the time constant is

$$\frac{L}{R} = \frac{1}{2} \mu_0 r \Delta \sigma$$

Thus, the diffusion time (see Eq. 7.1.28) is based on an equivalent length  $\sqrt{r\Delta}$ . Consider using copper with

$$\sigma = 5.9 \times 10^7 \text{ mhos/m}$$

$$\Delta = 10 \text{ m}$$

and find  $\Delta$  required to give  $L/R = 10^2$

$$\Delta = 2 \left(\frac{L}{R}\right) \frac{1}{\mu_0 r \sigma} = \frac{(200)}{(4\pi \times 10^{-7})(10)(5.9 \times 10^7)}$$

$$= 2.7 \times 10^{-1} \text{ m or } 27 \text{ cm}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.1 (Continued)

Note that to satisfy the condition that  $\ell \gg r$ , the length must be greater than 10 meters also. The coil is larger than the average class-room! Of course, if magnetic materials are used, the dimensions of the coil can be reduced considerably, but long L/R time constants are difficult to obtain on a laboratory scale with ordinary conductors.

PROBLEM 7.2

Part a

Our solution will parallel the one in the text, only now the  $\bar{B}$  field will be trapped in the slab until it diffuses away. The fundamental equations are

$$\begin{aligned} \nabla \times \bar{B} &= \mu \bar{J} = \mu \sigma \bar{E}; & \nabla \times \bar{E} &= - \frac{\partial \bar{B}}{\partial t} \\ \nabla \times \nabla \times \bar{B} &= \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = \mu \sigma \nabla \times \bar{E} = - \mu \sigma \frac{\partial \bar{B}}{\partial t} \end{aligned}$$

Because  $\nabla \cdot \bar{B} = 0$ ,

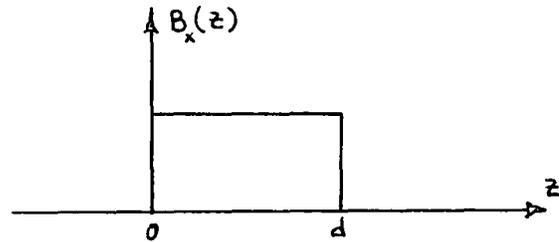
$$\nabla^2 \bar{B} = \mu \sigma \frac{\partial \bar{B}}{\partial t}$$

or in one dimension

$$\frac{1}{\mu \sigma} \frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t} \quad (a)$$

at  $t = 0^+$

$$B_x = \begin{cases} 0 & z < 0 \\ B_0 & 0 < z < d \\ 0 & z > d \end{cases}$$



This suggests that between 0 and d, we can write  $B_x(z)$  as

$$B_x(z) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi z}{d}\right) \quad 0 < z < d$$

To solve for the coefficients  $a_n$ , we take advantage of the orthogonal property of the sine functions.

$$\int_0^d B_x(z) \sin\left(\frac{m\pi z}{d}\right) dz = \sum_{n=1}^{\infty} \int_0^d a_n \sin\left(\frac{n\pi z}{d}\right) \sin\left(\frac{m\pi z}{d}\right) dz$$

But

$$\int_0^d B_x(z) \sin\left(\frac{m\pi z}{d}\right) dz = \int_0^d B_0 \sin\left(\frac{m\pi z}{d}\right) dz = \begin{cases} \frac{2dB_0}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.2 (Continued)

Also

$$\int_0^d a_n \sin\left(\frac{n\pi z}{d}\right) \sin\left(\frac{m\pi z}{d}\right) dz = \begin{cases} \frac{a_n d}{2} & n = m \\ 0 & n \neq m \end{cases}$$

Hence,

$$a_m = \begin{cases} \frac{4B_0}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

$$B_x(t=0, z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_0 \sin\left(\frac{n\pi z}{d}\right) \quad 0 < z < d$$

We assume that for  $t > 0$ ,  $0 < z < d$

$$B_x(t, z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_0 \sin\left(\frac{n\pi z}{d}\right) e^{-\alpha_n t}$$

Plugging into (a) we find that  $\frac{1}{\mu\sigma} \left(\frac{n\pi}{d}\right)^2 = \alpha_n$ . Let's define  $\tau = \mu\sigma \left(\frac{d}{\pi}\right)^2$  as the fundamental diffusion time. Then

$$B_x(t, z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} B_0 \sin\left(\frac{n\pi z}{d}\right) e^{-n^2 t/\tau} \quad \begin{matrix} 0 < z < d \\ t > 0 \end{matrix}$$

Part b

$$\bar{J} = \frac{\nabla \times \bar{B}}{\mu} = \frac{1}{\mu} \frac{\partial B_x}{\partial z} \bar{i}_y = \frac{4B_0}{\mu_0 d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \cos\left(\frac{n\pi z}{d}\right) e^{-n^2 t/\tau} \bar{i}_y \quad \begin{matrix} 0 < z < d \\ t > 0 \end{matrix}$$

PROBLEM 7.3

Part a

If we neglect the capacitance of the block, the current we put in at  $t=0$  will have to return by means of the block. This can be seen from the magnetic field system equation

$$\nabla \times \bar{H} = \bar{J} \tag{a}$$

which implies

$$\nabla \cdot \bar{J} = 0 \tag{b}$$

or "what goes in must come out".

If the current penetrated the block at  $t=0^+$  there would be a magnetic field within the block at  $t=0^+$ , a situation we cannot allow since some time must elapse (relative to the diffusion time) before the fields in the block can change significantly.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.3 (Continued)

We conclude that the source current returns as a surface current down the left side of the block. This current must be

$$K_y = -I_o/D \tag{c}$$

where  $y$  is the upwards vertical direction. The current loop between  $x = -L$  and  $x = 0$  thus provides a magnetic field

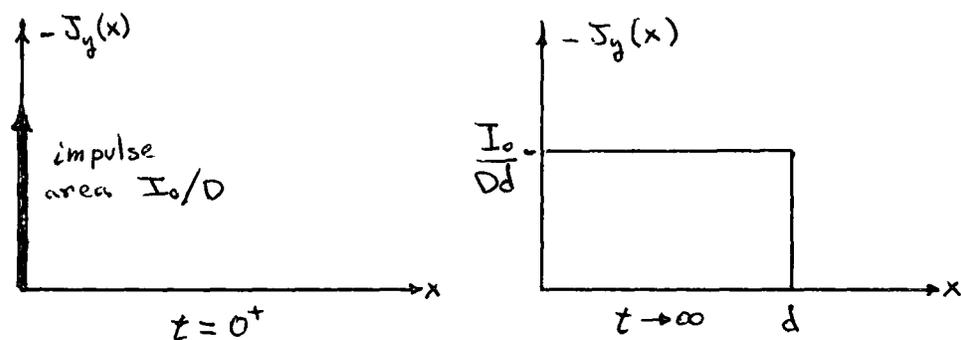
$$H_z(t=0^+) = \begin{cases} -I_o/D & -L < x < 0 \\ 0 & 0 < x \end{cases} \tag{d}$$

where  $z$  points out of sketch.

Part b

As  $t \rightarrow \infty$  the system will reach a static state with input current  $I_o/D$  per unit length. The current will return uniformly through the block. Hence,

$$J_y(x) = -\frac{I_o}{Dd} \tag{e}$$



Part c

As a diffusion problem this system is very much like the system of Fig. 7.1.1 of the text except for the fact that here diffusion occurs on only one side of the block instead of two. This suggests a fundamental diffusion time constant of

$$\tau = \frac{\mu_o \sigma (2d)^2}{\pi^2} \tag{f}$$

where we have replaced the term  $d^2$  by  $(2d)^2$  in Eq. 7.1.28 of the text.

PROBLEM 7.4

Part a

This is a magnetic field system characterized by a diffusion equation. With  $B_z = \text{Re} \hat{B}_z(x) e^{j\omega t}$ ,

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.4 (Continued)

$$\frac{1}{\mu\sigma} \frac{d^2 \hat{B}_z}{dx^2} = j\omega \hat{B}_z \quad (a)$$

Let  $\hat{B}_z(x) = \hat{B}_0 e^{\alpha x}$ , then

$$\alpha^2 = j\omega\mu\sigma \quad (b)$$

or

$$\alpha = \pm \frac{1}{\delta} (1 + j), \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (c)$$

The boundary conditions are

$$\hat{B}_z(x=0) = -\mu \hat{i}/D \quad (d)$$

$$\hat{B}_z(x \rightarrow \infty) \rightarrow 0$$

which means that we use only the (-) sign

$$\hat{B}_z(x,t) = -\text{Re} \frac{\mu \hat{i}}{D} e^{-x/\delta} e^{j(\omega t - \frac{x}{\delta})} \quad (e)$$

Part b

$$\nabla \times \bar{B} = \mu \bar{J} \quad (f)$$

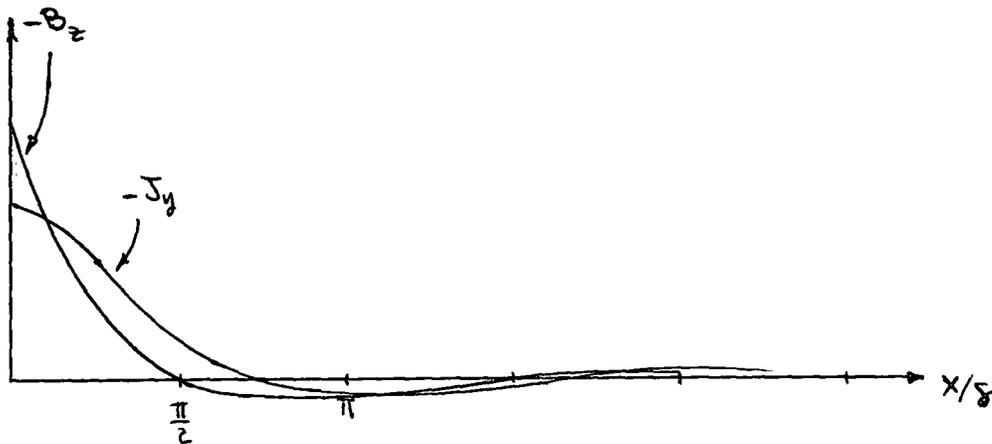
or

$$\frac{\partial B_z}{\partial x} = -\mu J_y \quad (g)$$

so that

$$J_y = -\text{Re} \frac{\hat{i}}{D} \frac{(1+j)}{\delta} e^{-\frac{x}{\delta}} e^{j(\omega t - \frac{x}{\delta})} \quad (h)$$

Part c



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.4 (Continued)

Part d

The electric field is given by

$$\nabla \times \vec{E} = \hat{i}_z \frac{\partial E_y}{\partial x} = -j\omega \hat{i}_z \hat{i}_z \quad (i)$$

$$E_y(x=0) = -\frac{\omega \delta \mu}{2D} (1+j) \hat{i} \quad (j)$$

Faraday's law (Eq. 1.1.23, Table 1.2, Appendix E) written for a counter-clockwise contour through the source and left edge of the block, gives

$$\hat{V} + \hat{E}_y d = \frac{j\omega \mu_o (Ld)}{D} \hat{i} \quad (k)$$

where from (j)

$$\hat{E}_y d = -\frac{1}{\sigma} \left(\frac{d}{\delta}\right) \frac{1}{D} (1+j) \hat{i} \quad (l)$$

Hence, assuming that  $\hat{V} = \hat{i}[R(\omega) + j\omega L(\omega)]$ , (don't confuse the L's)

$$R(\omega) = \frac{1}{\sigma} \left(\frac{d}{D}\right) \frac{1}{\delta} = \frac{d}{D} \sqrt{\frac{\omega \mu}{2\sigma}} \quad (m)$$

$$L(\omega) = \frac{\mu_o Ld}{D} + \frac{d}{D} \sqrt{\frac{\mu}{2\omega\sigma}} \quad (n)$$

Thus, as  $\omega \rightarrow \infty$  the inductance becomes just that due to the free-space portion of the circuit between  $x=0$  and  $x=-L$ . The resistance becomes infinite because the currents crowd to the left edge of the block.

In the opposite extreme, as  $\omega \rightarrow 0$ , the resistance approaches zero because the currents have an infinite  $x$ - $z$  area of the block through which to flow. Similarly, the inductance becomes large because the  $x$ - $y$  area enclosed by the current paths increases without limit. At low frequencies it would be necessary to include the finite extent of the block in the  $x$  direction in the analysis to obtain a realistic estimate of the resistance and inductance.

PROBLEM 7.5

Part a

This is a magnetic field system characterized by a diffusion equation.

Place origin of coordinates at left edge of block,  $x$  to right and  $z$  out of paper.

With  $B_x = \text{Re} \hat{B}_x(x) e^{j\omega t}$

$$\frac{1}{\mu\sigma} \frac{\partial^2 \hat{B}_x}{\partial x^2} = j\omega \hat{B}_x \quad (a)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.5 (Continued)

Let  $\hat{B}_z(x) = \hat{B}_0 e^{\alpha x}$ , then

$$\alpha^2 = j\omega\mu\sigma \quad (b)$$

$$\alpha = \pm \frac{1}{\delta} (1+j), \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (c)$$

The boundary conditions are

$$\hat{B}_z(x=0) = -\mu \frac{\hat{I}}{D} \quad (d)$$

$$\hat{B}_z(x=l) = 0 \quad (e)$$

because all of the current  $I_0(t)$  is returned through the block. Thus the appropriate linear combination of solutions to satisfy the boundary conditions is

$$B_z(x,t) = \text{Re} \frac{\mu \hat{I}}{D} \frac{\sinh[\alpha(x-l)] e^{j\omega t}}{\sinh(\alpha l)} \quad (f)$$

where  $\alpha$  is a complex quantity, (c). The current is related to  $\hat{B}_z$  by

$$\nabla \times \bar{B} = -\frac{\partial B}{\partial x} \bar{i}_y = \mu \bar{J} = \mu J \bar{i}_y \quad (g)$$

From (f) and (g),

$$J_y = -\frac{\hat{I}\alpha}{D} \frac{\cosh[\alpha(x-l)] e^{j\omega t}}{\sinh \alpha l} \quad (h)$$

Part b

The time average magnetic force on the block is given by

$$f_x = \text{Re} \left[ Dd \int_0^l \frac{\hat{J}_y(x) \hat{B}_z^*(x) dx}{2} \right] \quad (i)$$

where we have taken advantage of the identity

$$\langle \text{Re} \hat{A} e^{j\omega t} \text{Re} \hat{B} e^{j\omega t} \rangle = \frac{1}{2} \text{Re} \hat{A} \hat{B}^*$$

to integrate the force density  $(\bar{J} \times \bar{B})_x$  over the volume of the block. Note that a detailed calculation is required to complete (i), because  $\alpha$  in (f) and (h) is complex.

This example is one where the total force is more easily computed using the Maxwell stress tensor. See Probs. 8.16, 8.17 and 8.22 for this approach.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6

As an example of electromagnetic phenomena that occur in conductors at rest we consider the system of Fig. 7.1.1 with the constant-current source and switch replaced by an alternating current source.

$$i(t) = I \cos \omega t \quad (a)$$

We make all of the assumptions of Sec. 7.1.1 and adopt the coordinate system of Fig. 7.1.2. Interest is now confined to a steady-state problem.

The equation that describes the behavior of the flux density in this system is Eq. 7.1.15

$$\frac{1}{\mu_o \sigma} \frac{\partial^2 B_x}{\partial z^2} = \frac{\partial B_x}{\partial t} \quad (b)$$

and the boundary conditions are now, at  $z = 0$  and  $z = d$ ,

$$B_x = B_o \cos \omega t = \left[ \text{Re } B_o e^{j\omega t} \right] \quad (c)$$

where

$$B_o = \frac{\mu_o NI}{w} \quad (d)$$

The boundary condition of (c) coupled with the linearity of (b) lead us to assume a solution

$$B_x = \text{Re} \left[ \hat{B}(z) e^{j\omega t} \right] \quad (e)$$

We substitute this form of solution into (b), cancel the exponential factor, and drop the Re to obtain

$$\frac{d^2 \hat{B}}{dz^2} = j \mu_o \sigma \hat{B} \quad (f)$$

Solutions to this equation are of the form

$$\hat{B}(z) = e^{rz} \quad (g)$$

where substitution shows that

$$r = \pm \sqrt{j\omega\mu_o\sigma} = \pm \sqrt{\frac{\omega\mu_o\sigma}{2}} (1+j) \quad (h)$$

It is convenient to define the skin depth  $\delta$  as (see Sec. 7.1.3a)

$$\delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} \quad (i)$$

We use this definition and write the solution, (g) as

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6 (Continued)

$$\hat{B}(z) = c_1 e^{(1+j)\frac{z}{\delta}} + c_2 e^{-(1+j)\frac{z}{\delta}} \quad (j)$$

The boundary conditions at  $z = 0$  and  $z = d$  (c) require that

$$\begin{aligned} B_o &= c_1 + c_2 \\ B_o &= c_1 e^{(1+j)d/\delta} + c_2 e^{-(1+j)d/\delta} \end{aligned}$$

Solution of these equations for  $c_1$  and  $c_2$  yields

$$c_1 = \frac{B_o \left[ 1 - e^{-(1+j)\frac{d}{\delta}} \right]}{D} \quad (k)$$

$$c_2 = \frac{-B_o \left[ 1 - e^{(1+j)\frac{d}{\delta}} \right]}{D} \quad (l)$$

where  $D = 2(\cos \frac{d}{\delta} \sinh \frac{d}{\delta} + j \sin \frac{d}{\delta} \cosh \frac{d}{\delta})$

We now substitute (k) and (l) into (j); and, after manipulation, obtain

$$\hat{B}(z) = B_o [f(z) + j g(z)] \quad (m)$$

where

$$\begin{aligned} f(z) &= \frac{M}{F} \cos \frac{d}{\delta} \sinh \frac{d}{\delta} + \frac{N}{F} \sin \frac{d}{\delta} \cosh \frac{d}{\delta} \\ g(z) &= \frac{N}{F} \cos \frac{d}{\delta} \sinh \frac{d}{\delta} - \frac{M}{F} \sin \frac{d}{\delta} \cosh \frac{d}{\delta} \\ M &= \cos \frac{z}{\delta} \sinh \frac{z}{\delta} + \cos \left( \frac{d-z}{\delta} \right) \sinh \left( \frac{d-z}{\delta} \right) \\ N &= \sin \frac{z}{\delta} \cosh \frac{z}{\delta} + \sin \left( \frac{d-z}{\delta} \right) \cosh \left( \frac{d-z}{\delta} \right) \\ F &= \cos^2 \frac{d}{\delta} \sinh^2 \frac{d}{\delta} + \sin^2 \frac{d}{\delta} \cosh^2 \frac{d}{\delta} \end{aligned}$$

Substitution of (m) into (e) yields

$$B_x = B_m(z) \cos[\omega t + \theta(z)] \quad (n)$$

where

$$B_m(z) = B_o \sqrt{[f(z)]^2 + [g(z)]^2} \quad (o)$$

$$\theta(z) = \tan^{-1} \frac{g(z)}{f(z)} \quad (p)$$

It is clear from the form of (n) that both the amplitude and phase of the flux density vary as functions of  $z$ .

To illustrate the nature of the distribution of flux density predicted

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6 (Continued)

by this set of equations the maximum flux density is plotted as a function of position for several values of  $d/\delta$  in the figure. Recalling the definition of the skin depth  $\delta$  in (1), we realize that for a system of fixed geometry and fixed properties  $\frac{d}{\delta} \propto \sqrt{\omega}$ , thus, as  $\frac{d}{\delta}$  increases, the frequency of the excitation increases. From the curves of the figure we see that as the frequency increases the flux density penetrates less and less into the specimen until at high frequencies ( $\frac{d}{\delta} \gg 1$ ) the flux density is completely excluded from the conductor. At very low frequencies ( $\frac{d}{\delta} \ll 1$ ) the flux density penetrates completely and is essentially unaffected by the presence of the conducting material.

It is clear that at high frequencies ( $\frac{d}{\delta} \gg 1$ ) when the flux penetrates very little into the slab, the induced (eddy) currents flow near the surfaces. In this case it is often convenient, when considering electromagnetic phenomena external to the slab, to assume  $\sigma \rightarrow \infty$  and treat the induced currents as surface currents.

It is informative to compare the flux distribution of the figure for a steady-state a-c problem with the distribution of Fig. 7.1.4 for a transient problem. We made the statement in Sec. 7.1.1 that when we deal with phenomena having characteristic times that are short compared to the diffusion time constant, the flux will not penetrate appreciably into the slab. We can make this statement quantitative for the steady-state a-c problem by defining a characteristic time as

$$\tau_c = \frac{1}{\omega}$$

We now take the ratio of the diffusion time constant given by Eq. 7.1.28 to this characteristic time and use the definition of skin depth in (1).

$$\frac{\tau}{\tau_c} = \omega\tau = \frac{2}{\pi} \left(\frac{d}{\delta}\right)^2 \quad (q)$$

Thus, for our steady-state a-c problem, this statement that the diffusion time constant is long compared to a characteristic time is the same as saying that the significant dimension  $d$  is much greater than the skin depth  $\delta$ .

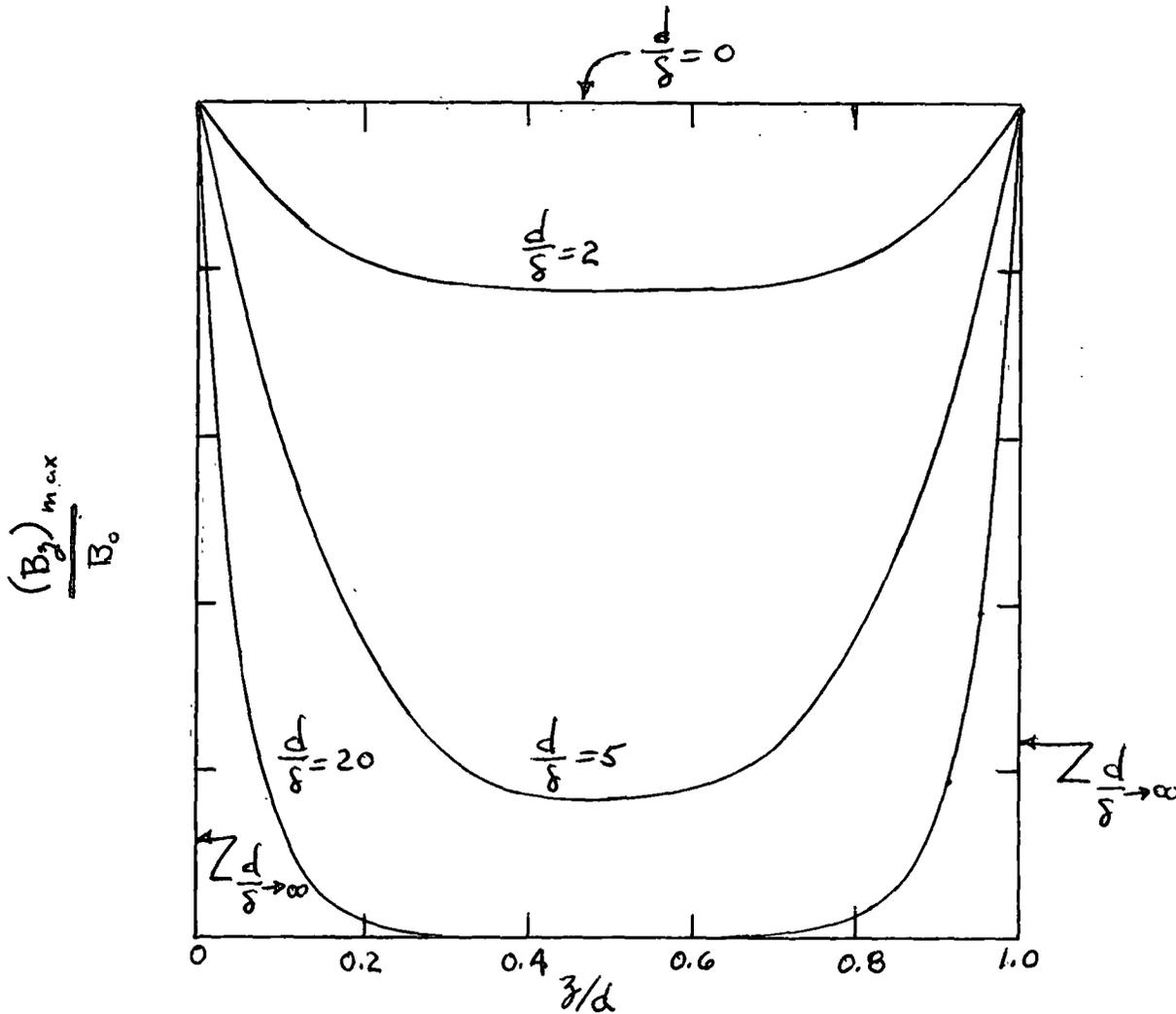
The current distribution follows from the magnetic flux density by using Ampere's law;

$$J_y = \frac{1}{\mu_0} \frac{\partial B_x}{\partial z} \quad (r)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.6 (Continued)

Thus the distribution of  $|J_y|$  is somewhat as shown in the figure for  $B_x$ .  
 The instantaneous  $J_y$  has odd symmetry about  $z = 0.5 d$ .



DISTRIBUTION OF FLUX DENSITY WITH SKIN EFFECT



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.7 (Continued)

Since diffusion in the slabs takes negligible time compared to the main problem, each slab could be modeled as a conducting sheet with

$$\vec{K} = (\sigma\Delta)\vec{E} \quad (d)$$

In region 2.

$$\nabla \times \vec{H} = 0 \quad \text{or} \quad \vec{H} = H_0(t)\vec{i}_z = -K_2(t)\vec{i}_z \quad (e)$$

From

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \vec{n} \, da \quad (f)$$

we learn that

$$\frac{a}{\sigma\Delta} [K_1(t) - K_2(t)] = +\frac{d}{dt} [\mu_0 a\ell K_2(t)] \quad (g)$$

Since  $K_0(t) = K_1(t) + K_2(t)$  we know that

$$K_0(t) = \frac{I}{D} u_{-1}(t) = 2K_2(t) + \sigma\mu_0 \Delta\ell \frac{dK_2(t)}{dt} \quad (h)$$

The solution is therefore

$$K_2(t) = \frac{I}{2D} (1 - e^{-t/\tau}) u_{-1}(t); \quad \tau = \frac{\sigma\mu_0 \Delta\ell}{2} \quad (i)$$

and, because  $K_2 = -H_0$ , the magnetic field fills region (2) with the time constant  $\tau$ .

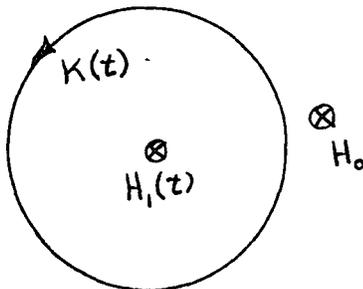
PROBLEM 7.8

As in Prob. 7.7, the diffusion time associated with the thin conducting shell is small compared to the time required for the field to fill the region  $r < R$ . Modeling the thin shell as having the property

$$\vec{K} = \Delta\sigma \vec{E} \quad (a)$$

and assuming that

$$H_1(t)\vec{i}_z = [H_0 - K(t)]\vec{i}_z \quad (b)$$



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.8 (Continued)

We can use the induction equation

$$\oint \bar{E} \cdot d\bar{\ell} = - \frac{d}{dt} \int \bar{B} \cdot \bar{n} da \quad (c)$$

to learn that, because  $H_o = \text{constant}$  for  $t > 0$

$$\frac{2\pi R}{\Delta\sigma} K(t) = - \pi R^2 \mu_o \frac{dK(t)}{dt} \quad (d)$$

The solution to (d) is

$$K(t) = H_o e^{-t/\tau} u_{-1}(t); \quad \tau = \frac{\mu_o \sigma R \Delta}{2} \quad (e)$$

and from (b), it follows that

$$H_1(t) = H_o - K(t) = H_o (1 - e^{-t/\tau}) u_{-1}(t) \quad (f)$$

The  $\bar{H}$  field is finally distributed uniformly for  $r < a$ , with a diffusion time based on the length  $\sqrt{R\Delta}$ .

PROBLEM 7.9

Part a

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad (a)$$

$$\nabla \times \bar{B} = \mu\sigma E \quad (b)$$

So

$$\nabla \times \nabla \times \bar{B} = - \mu\sigma \frac{\partial \bar{B}}{\partial t} \quad (c)$$

But

$$\nabla \times (\nabla \times \bar{B}) = \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = - \nabla^2 \bar{B} \quad (d)$$

So

$$\nabla^2 \bar{B} = \mu\sigma \frac{\partial \bar{B}}{\partial t}$$

Part b

Since  $\bar{B}$  only has a  $z$  component

$$\nabla^2 B_z = \mu\sigma \frac{\partial B_z}{\partial t} \quad (e)$$

In cylindrical coordinates

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (f)$$

Here  $B_z = B_z(r, t)$  so

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.9 (Continued)

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \hat{B}}{\partial r} \right) + \mu_0 \alpha \hat{B} = 0 \quad (g)$$

Part c

We want the magnetic field to remain finite at  $r = 0$ , hence  $C_2 = 0$ .

Part d

At  $r = a$

$$B(a, t) = \mu_0 H_0 - C_1 J_0(\sqrt{\mu_0 \sigma \alpha} a) = \mu_0 H_0 \quad (h)$$

Hence if  $C_1 \neq 0$

$$J_0(\sqrt{\mu_0 \sigma \alpha} a) = 0 \quad (i)$$

Part e

Multiply both sides of expression for  $B(r, t=0) = 0$  by  $r J_0(v_j r/a)$  and integrate from 0 to  $a$ . Then,

$$\int_0^a \mu_0 H_0 r J_0(v_j r/a) dr = \mu_0 H_0 \frac{a^2}{v_j} J_1(v_j) \quad (j)$$

$$\int_0^a \sum_{i=1}^{\infty} C_i J_0(v_i r/a) r J_0(v_j r/a) dr = C_j \frac{a^2}{2} J_1^2(v_j) \quad (k)$$

from which it follows that

$$C_j = \frac{2\mu_0 H_0}{v_j J_1(v_j)} \quad (l)$$

The values of  $v_j$  and  $J_1(v_j)$  given in the table lead to the coefficients

$$\frac{C_1}{2\mu_0 H_0} = .802; \frac{C_2}{2\mu_0 H_0} = -.535; \frac{C_3}{2\mu_0 H_0} = 0.425 \quad (m)$$

Part f

$$\alpha_1 = \frac{1}{\mu_0 \sigma} \left( \frac{v_1}{a} \right)^2$$

$$\tau_1 = \frac{\mu_0 \sigma a^2}{v_1^2} = 0.174 \mu_0 \sigma a^2 \quad (n)$$

$$\begin{aligned} \tau_1 &= (0.174) (4\pi \times 10^{-7}) \frac{10^4}{4\pi} (25) \times 10^{-4} \\ &\approx 4.35 \times 10^{-7} \text{ seconds} \quad (o) \end{aligned}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.10

Part a

$$\nabla \times \bar{E} = \hat{i}_z \frac{\partial E_y}{\partial x} = - \frac{\partial}{\partial t} B_z \hat{i}_z = 0 \quad (a)$$

$$\nabla \times \bar{B} = - \hat{i}_y \frac{\partial B_z}{\partial x} = \mu_0 \bar{J} = \mu_0 \sigma (E_y - UB_z) \hat{i}_y \quad (b)$$

$$\begin{aligned} \nabla \times \nabla \times \bar{B} &= \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} \\ &= - \frac{\partial^2 B_z}{\partial x^2} \hat{i}_z = \mu_0 \sigma \left( \frac{\partial E_y}{\partial x} - U \frac{\partial B_z}{\partial x} \right) \hat{i}_z \end{aligned} \quad (c)$$

But  $\frac{\partial E_y}{\partial x} = 0$  from (a), so

$$\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \sigma U \frac{\partial B_z}{\partial x} \quad (d)$$

Part b

At

$$x = 0 \quad B_z = - \mu_0 K \quad (e)$$

$$\text{at } x = L \quad B_z = 0 \quad (f)$$

Part c

Let

$$B_z(x) = C e^{\alpha x}, \text{ then}$$

$$\alpha(\alpha - \mu_0 \sigma U) = 0 \quad (g)$$

$$\alpha = 0, \alpha = \mu_0 \sigma U \quad (h)$$

Using the boundary conditions

$$B_z(x) = - \mu_0 K \frac{(1 - e^{\mu_0 \sigma U(x-L)})}{1 - e^{-\mu_0 \sigma UL}} \quad (i)$$

Note that as  $U \rightarrow 0$

$$B_z(x) = - \mu_0 K \left( \frac{L-x}{L} \right) \quad (j)$$

as expected.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.11

Part a

$$\begin{aligned}\bar{F} &= \bar{J} \times \bar{B} = - J_y B_x \bar{i}_z \\ &= - \frac{\mu_0 I^2}{w^2 \ell} \frac{R_m e^{R_m z/\ell} (e^{R_m z/\ell} - 1)}{(e^{R_m} - 1)^2} \bar{i}_z\end{aligned}\quad (a)$$

Part b

$$f_z = \int -F_z w dz = - \frac{\mu_0 I^2 d}{2w} \quad (b)$$

This result can be found more simply by using the Maxwell Stress Tensor by methods similar to those used with Probs. 8.16 and 8.17.

Part c

The power supplied by the velocity source is

$$P_U = - f_z U = \frac{\mu_0 I^2 d U}{2w} = \frac{I^2 d R_m}{\sigma w \ell 2} \quad (c)$$

The electric field at the current source is

$$E_y(z=\ell) = \frac{J_y(z=\ell)}{\sigma} - U B_x(z=\ell) \quad (d)$$

$$= \frac{I}{\sigma \ell w} \frac{R_m}{(e^{R_m} - 1)} \quad (e)$$

Power supplied by the current source is then

$$-V_g I = + E_y dI = + \frac{I^2 d}{\sigma \ell w} \left( \frac{R_m}{e^{R_m} - 1} \right) \quad (f)$$

Power dissipated in the moving conductor is then

$$P_d = P_U + V_g I = \frac{I^2 d}{\sigma \ell w} \frac{R_m}{2} \left( \frac{e^{R_m} + 1}{e^{R_m} - 1} \right) \quad (g)$$

which is just what is obtained from

$$P_d = w d \int_0^\ell \frac{J^2}{\sigma} dx \quad (h)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.12

If a point in the reference frame is outside the block it must satisfy

$$\nabla_{\mathbf{x}} \bar{\mathbf{E}} = - \frac{\partial \bar{\mathbf{B}}}{\partial t} \quad (\text{a})$$

$$\bar{\mathbf{J}} = 0 \quad \text{and} \quad \nabla_{\mathbf{x}} \bar{\mathbf{B}} = 0 \quad (\text{b})$$

Since the points outside the block have  $\bar{\mathbf{J}}=0$ , and uniform static fields (for differential changes in time), (a) and (b) are satisfied.

Points inside the block must satisfy

$$\frac{1}{\mu_0} \frac{\partial B_x}{\partial z} = J_y \quad (\text{c})$$

$$- \frac{1}{\mu_0 \sigma} \frac{\partial^2 B_x}{\partial z^2} + \frac{\partial B_x}{\partial t} = - v \frac{\partial B_x}{\partial z} \quad (\text{d})$$

Since these points see

$$J_y = \sigma v B_0, \quad \frac{\partial B_x}{\partial z} = \frac{B_0}{\ell}, \quad \frac{\partial^2 B_x}{\partial z^2} = 0 \quad (\text{e})$$

$$\frac{\partial B_x}{\partial t} = - v \frac{B_0}{\ell} \quad \text{and} \quad v \mu_0 \sigma \ell = 1$$

these conditions are satisfied.

Points on the block boundaries are satisfied because the field quantities  $\bar{\mathbf{E}}$  and  $\bar{\mathbf{B}}$  are continuous.

PROBLEM 7.13

Part a

Because  $\nabla \cdot \bar{\mathbf{B}}=0$  the magnetic flux lines run in closed loops. The field lines prefer to run through the high  $\mu$  material near the source, hence very few lines will close beyond the edge of the material at  $z=0$ . Currents in the slab will tend to remain between the pole pieces.

Part b

$$\frac{1}{\mu \sigma} \frac{\partial^2 B_y}{\partial z^2} = \frac{\partial B_y}{\partial t} + v \frac{\partial B_y}{\partial z} \quad (\text{a})$$

Let  $B_y(z,t) = C e^{j(\omega t - kz)}$ , then

$$k^2 - j\mu\sigma v k + j\mu\sigma\omega = 0; \quad (\text{b})$$

A quadratic equation with roots

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.13 (Continued)

$$k = \mu\sigma \left[ j \frac{V}{2} \pm j \sqrt{\left(\frac{V}{2}\right)^2 + j \frac{\omega}{\mu\sigma}} \right] \quad (c)$$

or in terms of  $R_m = \mu\sigma VL$  and  $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$

$$\left(k \frac{\pm}{\pm}\right)L = j \frac{R_m}{2} \pm j \sqrt{\left(\frac{R_m}{2}\right)^2 + j2 \left(\frac{L}{\delta}\right)^2} \quad (d)$$

From Fig. 7.1.16 of the text we see that

$$k^+ = k_r^+ + jk_i^+, \quad k^- = k_r^- + jk_i^-$$

where

$$-k_r^- = k_r^+ > 0 \quad \text{and} \quad k_i^- > -k_i^+ > 0 \quad (e)$$

To meet the boundary condition of part (a) we must have

$$B_y(z,t) = C \left[ e^{-jk^+z} - e^{-jk^-z} \right] e^{j\omega t} \quad (f)$$

Using the boundary condition at  $z = -L$

$$B_y(z,t) = \frac{B_0}{(e^{jk^+L} - e^{jk^-L})} (e^{-jk^+z} - e^{-jk^-z}) e^{j\omega t} \quad (g)$$

Part c

$$\nabla_x \bar{B} = -\bar{i}_x \frac{\partial B_y}{\partial z} = J_x \bar{i}_x / \mu_0 \quad (h)$$

$$J_x = \frac{jB_0 / \mu_0}{(e^{jk^+L} - e^{jk^-L})} (k^+ e^{-jk^+z} - k^- e^{-jk^-z}) e^{j\omega t} \quad (i)$$

Part d

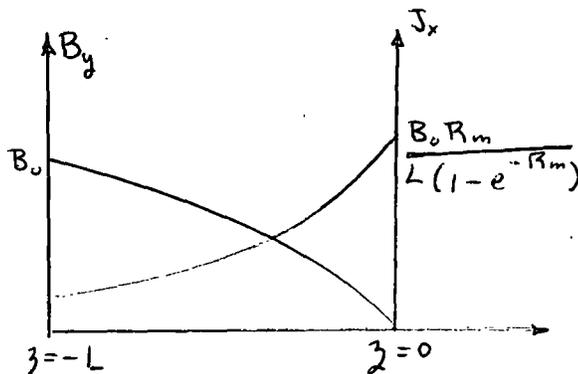
As  $\omega \rightarrow 0$   $k^+ \rightarrow 0$ ,  $k^- \rightarrow j \frac{R_m}{L}$

$$B_y = \frac{B_0}{(1 - e^{-R_m})} \left( 1 - e^{\frac{(R_m/L)z}{L}} \right) \quad (j)$$

$$J_x = \frac{B_0/L}{(1 - e^{-R_m})} R_m e^{\frac{(R_m/L)z}{L}} \quad (k)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

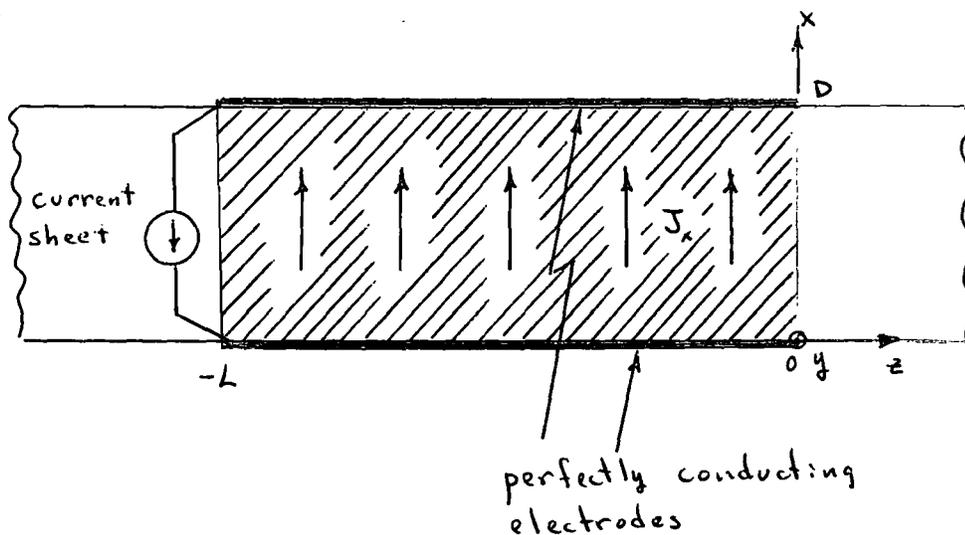
PROBLEM 7.13 (Continued)



As the sketch Fig. 7.1.9 of the text suggests, we could realize this problem by placing a current sheet source

$$\bar{K} = -\frac{B_0}{\mu_0} \bar{i}_x e^{j\omega t}$$

across the end  $z = -L$  and providing perfect conductors to slide against the slab at  $x = 0, D$ . The top view of the slab then appears as shown in the figure.



Note from (j) and (k) that as  $R_m \rightarrow 0$ , the current density  $J_x$  is uniform and  $B_y$  is a linear function of  $z$ . This limiting case is as would be obtained with the given driving arrangement.

PROBLEM 7.14

Part a

Since  $\bar{J}' = \bar{J}$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.14 (Continued)

$$\begin{aligned}\bar{K} &= \bar{i}_z K_o \cos(kUt-kx) & (a) \\ &= \bar{i}_z K_o \cos(\omega t-kx); \omega = kU\end{aligned}$$

Part b

The track can be taken as large in the y direction when it is many skin depths thick

$$L = \text{track thickness} \gg \delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} = \sqrt{\frac{2}{kU\mu_o\sigma}} \quad (b)$$

In the track we have the diffusion equation

$$\frac{1}{\mu_o\sigma} \nabla^2 \bar{B} = \frac{\partial \bar{B}}{\partial t} \quad (c)$$

or, with  $\bar{B} = \text{Re } \hat{B} \exp j(\omega t-kx)$ ,

$$\frac{1}{\mu_o\sigma} \left( \frac{\partial^2 \hat{B}_x}{\partial y^2} - k^2 \hat{B}_x \right) = j\omega \hat{B}_x \quad (d)$$

Let  $\hat{B}_x(y) = C e^{\alpha y}$ , then

$$\frac{1}{\mu_o\sigma} \alpha^2 = j\omega + \frac{k^2}{\mu_o\sigma} \quad (e)$$

$$\alpha = k \sqrt{1+jS}; S = \frac{\omega\mu_o\sigma}{k^2} = \frac{U\mu_o\sigma}{k} \quad (f)$$

Since the track is modeled as infinitely thick

$$B_x = C e^{\alpha y} e^{j(\omega t-kx)} \quad (g)$$

The gap between track and train is very thin; thus,

$$-\bar{i}_y \times \frac{\bar{B}}{\mu_o} = \bar{K} = K_o e^{j(\omega t-kx)} \bar{i}_z \quad (h)$$

which yields

$$B_x(x,y,t) = \mu_o K_o e^{\alpha y} e^{j(\omega t-kx)} \quad (i)$$

We must also have  $\nabla \cdot \bar{B} = \partial B_x / \partial x + \partial B_y / \partial y = 0$  or

$$B_y = \frac{j k}{\alpha} B_x(x,y,t) \quad (j)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.14 (Continued)

To compute the current in the track we note that

$$\nabla_{\mathbf{x}} \bar{\mathbf{B}} = \bar{\mathbf{i}}_z \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \mu_0 \bar{\mathbf{J}} \quad (\text{k})$$

$$\bar{\mathbf{J}} = - \left( j \frac{S}{\alpha} k^2 \right) \frac{B_x}{\mu_0} (x, y, t) \bar{\mathbf{i}}_z \quad (\text{l})$$

Part c

The time average force density in the track is (see footnote, page 368)

$$\langle F_y \rangle = \frac{1}{2} \text{Re}(J_z B_x^*) \quad (\text{m})$$

Hence the time average lifting force per unit x-z area on the train is

$$\langle T_y \rangle = - \int_{-\infty}^0 \langle F_y \rangle dy = - \text{Re} \int_{-\infty}^0 \frac{1}{2} J_z B_x^* dy \quad (\text{n})$$

$$= \frac{1}{4} \mu_0 K_0^2 \left( \frac{\sqrt{1+S^2} - 1}{\sqrt{1+S^2}} \right) > 0 \quad (\text{o})$$

See Fig. 7.1.21 of the text for a plot of this lifting force.

Part d

The time average force density in the track in the x direction is

$$\langle F_x \rangle = - \frac{1}{2} \text{Re}(J_z B_y^*) \quad (\text{p})$$

The force on the train in the x direction is then

$$\begin{aligned} \langle T_x \rangle &= - \int_{-\infty}^0 \langle F_x \rangle dy = \frac{1}{2} \text{Re} \int_{-\infty}^0 J_z B_y^* dy \quad (\text{q}) \\ &= - \frac{\mu_0 K_0^2}{4} \frac{S}{\sqrt{1+S^2} \text{Re} \sqrt{1+jS}} < 0 \end{aligned}$$

The problem is that this force drags the train instead of propelling it in the x direction. (See Fig. 7.1.20 of the text for a plot of the magnitude of this drag force). To make matters worse, if the train stops, the magnetic levitation force becomes zero.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.15

Part a :

Let the current sheet lie in the plane  $y = -s$ . In the region  $-s < y < 0$  we have the "diffusion equation"

$$\nabla^2 B_z = 0 \quad (a)$$

If  $B_z(x, y, t) = B_z(y) e^{j(\omega t - kx)}$  this equation yields

$$\frac{\partial^2 B_z}{\partial y^2} = k^2 B_z \quad (b)$$

Hence we can conclude that

$$B_z = [A \cosh k(y+s) + B \sinh k(y+s)] e^{j(\omega t - kz)} \quad (c)$$

At  $y = -s$  we have the boundary condition

$$\bar{i}_y \times \bar{B}_z = \mu_0 K_o \cos(\omega t - kz) \bar{i}_x \quad (d)$$

Thus

$$B_z = [\mu_0 K_o \cosh k(y+s) + B \sinh k(y+s)] e^{j(\omega t - kz)} \quad (e)$$

Since  $\nabla \cdot \bar{B} = \partial B_y / \partial y + \partial B_z / \partial z = 0$  we must have

$$B_y = [j(\mu_0 K_o \sinh k(y+s) + B \cosh k(y+s))] e^{j(\omega t - kz)} \quad (f)$$

In the conductor the diffusion equation is

$$\frac{1}{\mu_0 \sigma} \nabla^2 \bar{B} = \frac{\partial \bar{B}}{\partial t} + v \frac{\partial \bar{B}}{\partial z} \quad (g)$$

Then

$$\frac{\partial^2 B_z}{\partial y^2} = (j\mu_0 \sigma (\omega - kv) + k^2) B_z \quad (h)$$

which suggests a solution

$$B_z(y) = C e^{-\alpha y}, \quad \alpha = k\sqrt{1+jS}, \quad S = \frac{\mu_0 \sigma (\omega - kv)}{k^2} \quad (i)$$

Since  $\nabla \cdot \bar{B} = 0$  in the conductor too, we must have

$$B_y = -j \frac{k}{\alpha} B_z \quad (j)$$

As the boundary  $y = 0$  we must have

$$B_{y1} = B_{y2}, \quad H_{z1} = H_{z2} \quad (k)$$

Note that.

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.15 (Continued)

$$\begin{aligned} & \cosh ks B_{z2} + j \sinh ks B_{y2} \\ & = \mu_o K_o (\cosh^2 ks - \sinh^2 ks) = \mu_o K_o \end{aligned} \quad (l)$$

Then we must also have

$$\begin{aligned} \mu_o K_o & = \cosh ks B_{z1} + j \sinh ks B_{y1} \\ & = C (\cosh ks + \frac{k}{\alpha} \sinh ks) \end{aligned} \quad (m)$$

It follows that the  $\bar{B}$  field for  $y>0$  is

$$\bar{B} = \frac{\mu_o K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \left( -j \frac{k}{\alpha} \bar{i}_y + \bar{i}_z \right) e^{-\alpha y} e^{j(\omega t - kz)} \quad (n)$$

Comparing with Eq. 7.1.91 of Sec. 7.1.4 of the text we see that it is only necessary to replace

$$K_o \text{ by } \frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks}$$

starting with Eq. 7.1.90. The average forces depend on the magnitude, not the phase, of  $K_o$ , which is reduced by this substitution.

Part b

We note that if  $ks \ll 1$

$$\frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \approx K_o \quad (o)$$

which shows that the results of Sec. 7.1.4 are valid when  $ks \ll 1$ .

Part c

When  $ks \rightarrow \infty$

$$\frac{K_o}{\cosh ks + \frac{k}{\alpha} \sinh ks} \rightarrow 0$$

No fields will then be present in the conductor.

PROBLEM 7.16

Part a

Because the charge needs time to move through the conductor, at  $t=0^+$  there is only free charge on the plates. The electric fields are directed in the negative vertical direction and satisfy

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.16 (Continued)

$$E_{\ell} b + E_g a = V_o \quad (a)$$

at the interface at  $t=0^+$

$$\epsilon E_{\ell} = \epsilon_o E_g \quad (b)$$

Hence at  $t=0^+$

$$E_{\ell} = \frac{V_o}{b + \frac{\epsilon}{\epsilon_o} a}, \quad E_g = \frac{V_o}{\frac{\epsilon_o}{\epsilon} b + a} \quad (c)$$

Part b

As  $t \rightarrow \infty$  the charge on the interface excludes the fields from the conducting liquid, hence

$$E_{\ell} = 0 \quad E_g = \frac{V_o}{a} \quad (f)$$

Part c

The charge on the interface at any time is

$$\sigma_f = \epsilon E_{\ell} - \epsilon_o E_g \quad (g)$$

Conservation of charge requires

$$\frac{d\sigma_f}{dt} = -\sigma E_{\ell} \quad (h)$$

The voltage across the plates is  $V_o$  for  $t > 0$

$$V_o = E_{\ell} b + E_g a \quad (i)$$

Solving g, h, i we find that the charge obeys

$$\frac{(\epsilon + \epsilon_o b/a)}{\sigma} \frac{d\sigma_f}{dt} + \sigma_f = -\frac{\epsilon_o}{a} V_o \quad (j)$$

Let  $\tau = \frac{\epsilon + \epsilon_o b/a}{\sigma}$ , then

$$\sigma_f = -\frac{\epsilon_o V_o}{a} (1 - e^{-t/\tau}), \quad t \geq 0 \quad (k)$$

$$q_f = -\frac{\epsilon_o A V_o}{a} (1 - e^{-t/\tau}), \quad t \geq 0 \quad (l)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.17

Part a

In the inner sphere

$$\frac{\sigma_1}{\epsilon_0} \rho_f + \frac{\partial \rho_f}{\partial t} = 0 \quad (a)$$

So we find that

$$\rho_f(r, t) = \rho_0(r) e^{-\sigma_1 / \epsilon_0 t}, \quad t \geq 0 \quad r < R_1 \quad (b)$$

A similar equation holds for the charge in the outer sphere, but it has no initial charge distribution at  $t = 0$ , so

$$\rho_f(r, t) = 0, \quad t \geq 0 \quad R_1 < r < R_0 \quad (c)$$

Part b

Let

$$Q_0 = \int_0^{R_1} 4\pi r^2 \rho_0(r) dr \quad (d)$$

Also define

$\sigma_A$  = the surface charge density at  $r = R_1$

$\sigma_B$  = the surface charge density at  $r = R_0$

The field at  $R_0^+$  is, by Gauss' law

$$E(R_0^+) = \frac{Q_0}{4\pi\epsilon_0 R_0^2} \quad (e)$$

Then, conservation of charge requires that the electric field at  $r = R_0^-$  obey

$$\sigma_0 E(R_0^-) + \epsilon_0 \frac{\partial E}{\partial t}(R_0^-) = 0 \quad (f)$$

$$E(R_0^-) = \frac{Q_0}{4\pi\epsilon_0 R_0^2} e^{-\sigma_0 / \epsilon_0 t}, \quad t \geq 0 \quad (g)$$

We can thus conclude that

$$\sigma_B = \frac{Q_0}{4\pi R_0^2} (1 - e^{-\sigma_0 / \epsilon_0 t}), \quad t \geq 0 \quad (h)$$

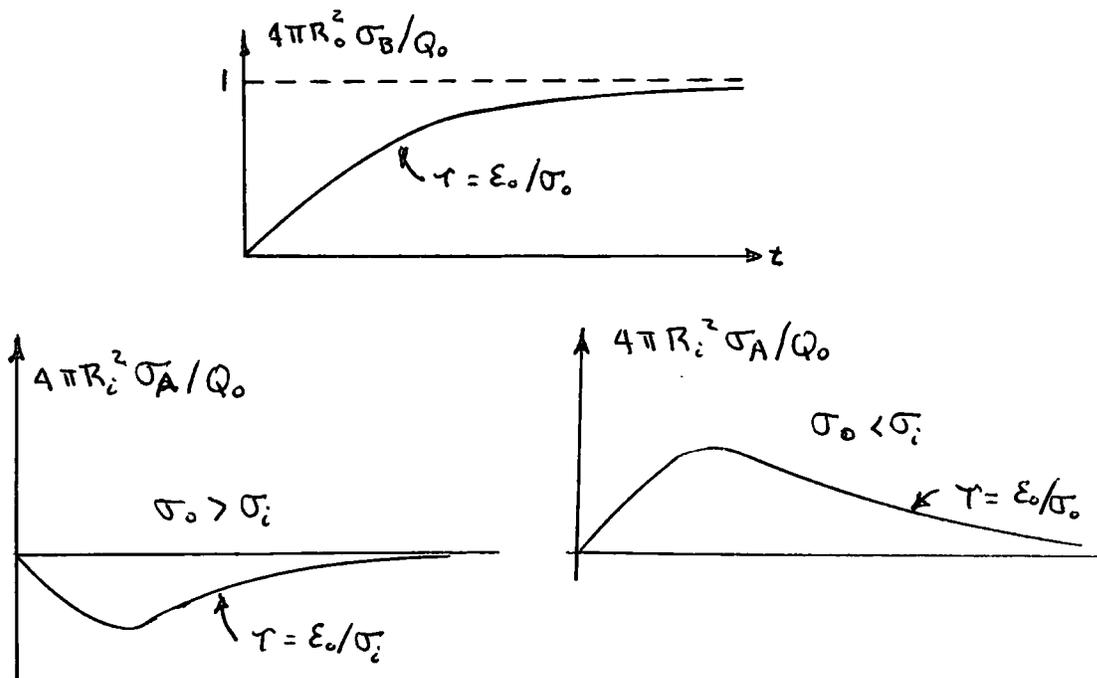
Since charge is conserved we now know that

$$\sigma_A = \frac{Q_0}{4\pi R_1^2} (e^{-\sigma_0 / \epsilon_0 t} - e^{-\sigma_1 / \epsilon_0 t}), \quad t \geq 0 \quad (i)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.17 (Continued)

Part c



PROBLEM 7.18

Part a

At the radius b

$$\epsilon[E(b^+) - E(b^-)] = \sigma_f \quad (a)$$

$$\sigma[E(b^+) - E(b^-)] = -\frac{\partial \sigma_f}{\partial t} = -\epsilon \frac{\partial}{\partial t} [E(b^+) - E(b^-)] \quad (b)$$

For  $t < 0$  when the system has come to rest

$$\nabla \cdot \vec{J} = (\sigma/\epsilon)\nabla \cdot \vec{E} = -\frac{\partial \rho_f}{\partial t} = 0 \quad (c)$$

For cylindrical geometry this has the solution

$$\vec{E} = +\frac{A}{r} \vec{i}_r; \quad V_o = + \int_a^b E_r dr = A \ln(b/a) \quad (d)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.18 (Continued)

then

$$\left. \begin{aligned} E(r=b^-) &= + \frac{V_o}{\ln(b/a)} \frac{1}{b} \\ E(r=b^+) &= 0 \end{aligned} \right\} t = 0^- \quad (e)$$

Since  $E(b^+) - E(b^-) = \sigma_f/\epsilon$  it cannot change instantaneously, so

$$E(b^+) - E(b^-) = - \frac{V_o}{b \ln(b/a)} e^{-(\sigma/\epsilon)t}, \quad t \geq 0 \quad (f)$$

Because there is no initial charge between the shells, there will be no charge between the shells for  $t \geq 0$ , thus

$$E_r = \begin{cases} + \frac{C_1(t)}{r} & a < r < b \\ + \frac{C_2(t)}{r} & b < r < c \end{cases} \quad t \geq 0 \quad (g)$$

The battery adds the constraint

$$V_o = C_1 \ln(b/a) + C_2 \ln(c/b) \quad (h)$$

while (f) becomes

$$C_1 - C_2 = \frac{V_o}{\ln(b/a)} e^{-(\sigma/\epsilon)t} \quad (i)$$

Solving (h) and (i) for  $C_1, C_2$

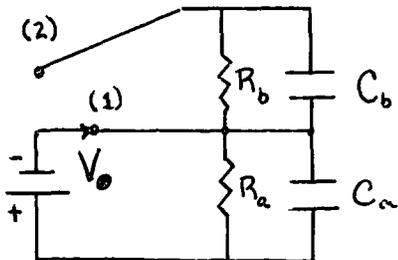
$$C_2 = \frac{V_o}{\ln c/a} (1 - e^{-(\sigma/\epsilon)t}) \quad (j)$$

$$C_1 = \frac{V_o}{\ln c/a} \left(1 + \frac{\ln(c/b)}{\ln(b/a)} e^{-(\sigma/\epsilon)t}\right) \quad (k)$$

Part b

$$\sigma_f = \epsilon(E(b^+) - E(b^-)) = - \frac{\epsilon}{b \ln(b/a)} V_o e^{-(\sigma/\epsilon)t} \quad (l)$$

Part c



$$R_b = \frac{\ln c/b}{2\pi\sigma}, \quad C_b = \frac{2\pi\epsilon}{\ln c/b}$$

$$R_a = \frac{\ln(b/a)}{2\pi\sigma}, \quad C_a = \frac{2\pi\epsilon}{\ln b/a}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.19

While the potential  $v$  is applied the system reaches an equilibrium. During this time

$$\nabla \cdot \bar{J} = \frac{\sigma}{\epsilon} \rho_f = - \frac{\partial \rho_f}{\partial t} \quad (a)$$

in the bulk of the liquid. If the potential  $V$  is applied for many time constants ( $\tau = \epsilon/\sigma$ ) any charge in the fluid decays away. For  $t > 0$  if the fluid is incompressible ( $\nabla \cdot \bar{v} = 0$ ) and  $\bar{J} = \sigma \bar{E} + \rho_f \bar{v}$  we know that

$$\nabla \cdot \bar{J} = (\sigma/\epsilon) \rho_f + \bar{v} \cdot \nabla \rho_f = - \frac{\partial \rho_f}{\partial t} \quad (b)$$

But in a frame moving with the particles of fluid

$$\frac{d}{dt} \rho_f = \frac{\partial \rho_f}{\partial t} + \bar{v} \cdot \nabla \rho_f = - (\sigma/\epsilon) \rho_f \quad (c)$$

$$\rho_f(t) = \rho_f(t=0) e^{-(\sigma/\epsilon)t} \quad t \geq 0 \quad (d)$$

where  $\rho_f(t)$  is the local charge seen by a moving particle. But for all fluid particles

$$\rho_f(t = 0) = 0 \quad (e)$$

Hence the charge remains zero everywhere for  $t \geq 0$ .

Now draw a volume around the upper sphere big enough to enclose it for a few seconds even though it is moving.

$$\oint_S \bar{J} \cdot d\bar{a} = - \frac{d}{dt} \int_V \rho_f dV \quad (f)$$

Now because  $\rho_f = 0$  in the fluid

$$\bar{J} = \sigma \bar{E}, \quad \oint_S \bar{J} \cdot d\bar{a} = (\sigma/\epsilon) \oint_S \epsilon \bar{E} \cdot d\bar{a} = (\sigma/\epsilon) Q(t) \quad (g)$$

Then

$$(\sigma/\epsilon) Q(t) = - \frac{d}{dt} \int_V \rho_f dV = - \frac{d}{dt} Q(t) \quad (h)$$

which has solution

$$Q(t) = Q e^{-t/\tau}; \tau = \epsilon/\sigma$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.20

Part a

We can use Gauss' law

$$\oint_S \epsilon_0 \vec{E} \cdot d\vec{a} = \int_V \rho_f dV \quad (a)$$

to determine the electric field if we note that there is no net charge in the system, which means that

$$\vec{E} = E_x \vec{i}_x = 0 \quad x < 0 \text{ and } x > 3d \quad (b)$$

$$\epsilon_0 E_x(x) = \int_0^x \frac{Q}{D^2 d} dx = \frac{Q}{D^2} \frac{x}{d} \quad \left\{ \begin{array}{l} 0 < x < d \\ t = 0 \end{array} \right. \quad (c)$$

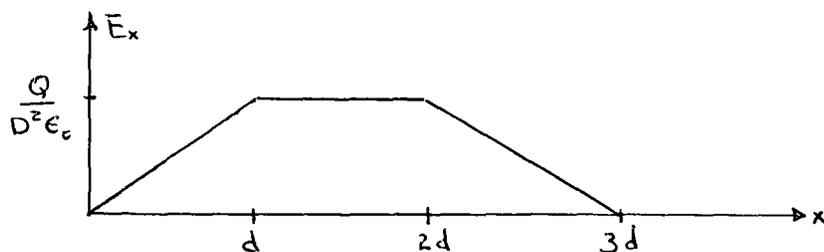
There is no charge in the middle region so

$$E_x = \frac{Q}{D^2 \epsilon_0} \quad d < x < 2d; \quad t = 0 \quad (d)$$

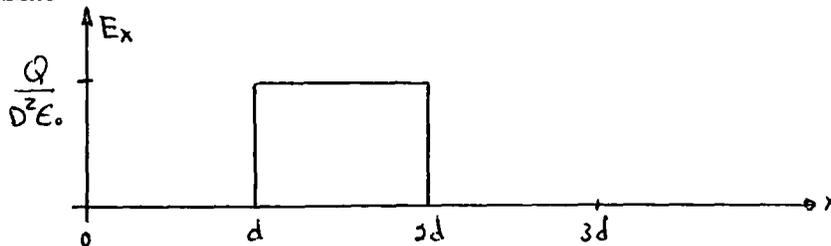
In the region  $2d < x < 3d$

$$\epsilon_0 (E_x(x) - E_x(2d)) = \int_{2d}^x \frac{-Q}{D^2 d} dx = -\frac{Q}{D^2} \frac{(x-2d)}{d} \quad (e)$$

$$E_x(x) = \frac{Q}{D^2 \epsilon_0} \frac{(3d-x)}{d} \quad \left\{ \begin{array}{l} 2d < d < 3d \\ t = 0 \end{array} \right. \quad (f)$$



As  $t \rightarrow \infty$  all the charge on the lower plate relaxes to the surface  $x = d$ , while the charge on the upper plate relaxes to the surface  $x = 2d$ . The electric field then looks like

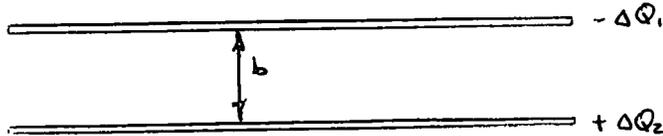


MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.20 (Continued)

Part b

Each charge distribution can be thought of as made up of many thin charges sheets; any two such sheets,



one located somewhere in the top conductor, one located somewhere in the bottom conductor, attract each other with a force

$$\Delta F = \frac{\Delta Q_1 \Delta Q_2}{2\epsilon_0 D^2} \quad (g)$$

which is independent of their separation, hence the net attractive force between plates does not change with time. At  $t \rightarrow \infty$  there is a surface charge

$$\sigma_T = -\frac{Q}{D^2} \quad x = 2d \quad (h)$$

$$\sigma_B = +\frac{Q}{D^2} \quad x = d$$

and the force per unit area  $T_x$  is simply that found for a pair of capacitor plates having separation  $d$  and supporting surface charge densities  $\pm Q$ . (See Sec. 3.1.2b).

$$T_x = \frac{Q^2}{2\epsilon_0 D^2} \quad t \geq 0 \quad (i)$$

This force can be easily seen to be constant from the viewpoint taken in Chapter 8, where the force on the lower plate can be found from the Maxwell Stress Tensor. The only contribution comes from  $T_{xx} = \frac{1}{2} \epsilon_0 E_x^2$  evaluated at  $x = d$ , and thus  $T_{xx}(x = d) = T_x$  as given by (i) regardless of  $t$ . Problem 8.23 is worked out following the stress-tensor approach.

PROBLEM 7.21

Part a

If the electric field beyond the plates is zero the conservation of charge equation

$$\oint_S \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int_V \rho_f dV = -\frac{\partial}{\partial t} \oint_S \epsilon \vec{E} \cdot d\vec{a} \quad (a)$$

MAGNETIC DIFFUSION OF CHARGE RELAXATION

PROBLEM 7.21 (Continued)

becomes

$$\sigma \hat{E}_x(x) - \frac{\hat{I}}{A} = -j\omega \epsilon \hat{E}_x(x) \quad (b)$$

That is, the equation for  $\hat{E}_x$  is as given by (f) of Example 7.2.3, with  $\epsilon$  now a function of  $x$ .

$$\hat{E}_x(x) = \frac{\hat{I}}{A(\sigma + j\omega\epsilon)} = \frac{\hat{I}/A}{[\sigma_1 + \frac{\sigma_2}{\ell}x + j\omega(\epsilon_1 + \frac{\epsilon_2}{\ell}x)]} \quad (c)$$

From Coulomb's law

$$\hat{\rho}_f = \frac{d}{dx}(\epsilon \hat{E}_x) = -\frac{\epsilon \hat{I}}{A} \frac{(j\omega \frac{d\epsilon}{dx} + \frac{d\sigma}{dx})}{(j\omega\epsilon + \sigma)^2} + \frac{\hat{I}}{A} \frac{d\epsilon}{dx} \quad (d)$$

$$\frac{\hat{\rho}_f A}{\hat{I}} = -\frac{(\epsilon_1 + \frac{\epsilon_2}{\ell}x)(j\omega \frac{\epsilon_2}{\ell} + \frac{\sigma_2}{\ell})}{[(\sigma_1 + \frac{\sigma_2}{\ell}x) + j\omega(\epsilon_1 + \frac{\epsilon_2}{\ell}x)]^2} + \frac{\frac{\epsilon_2}{\ell}}{[(\sigma_1 + \frac{\sigma_2}{\ell}x) + j\omega(\epsilon_1 + \frac{\epsilon_2}{\ell}x)]} \quad (e)$$

Part b

Consider the effect of a small change in  $\epsilon$  alone

$$\sigma_2 = 0; \epsilon_2/\epsilon_1 \ll 1$$

then

$$\hat{\rho}_f \approx \frac{\sigma_1 \epsilon_2 \hat{I}}{A \ell (j\omega \epsilon_1 + \sigma_1)^2} \quad (f)$$

It is seen from (f) that in the presence of conduction the gradient of  $\epsilon$  causes free charge to be stored in the bulk of the fluid. This effect is highly dependent on frequency, being greatest at zero frequency and disappearing when the cycle time is short compared to the relaxation time of the material.

PROBLEM 7.22

Part a

In the fluid the constitutive law for conduction is

$$\bar{J} = \sigma \bar{E} + \rho_f \bar{v} \quad (a)$$

Since the given velocity distribution has the property

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.22 (Continued)

$$\nabla \cdot \bar{v} = 0 \quad (b)$$

$$\nabla \cdot \bar{J} = \frac{\sigma}{\epsilon} \nabla \cdot (\epsilon E) + \bar{v} \cdot \nabla \rho_f = \frac{\sigma}{\epsilon} \rho_f + U \frac{\partial}{\partial x} \rho_f = - \frac{\partial \rho_f}{\partial t} \quad (c)$$

or

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \rho_f = - \frac{\sigma}{\epsilon} \rho_f \quad (d)$$

The charge is relaxing in the frame of the moving fluid. The solution has the form

$$\rho_f = \text{Re } \hat{\rho}_0 e^{-\frac{\sigma}{\epsilon} \frac{x}{U}} e^{j\omega(t - \frac{x}{U})} \quad -\frac{\Delta}{2} < y < \frac{\Delta}{2} \quad (e)$$

= 0 elsewhere in the channel

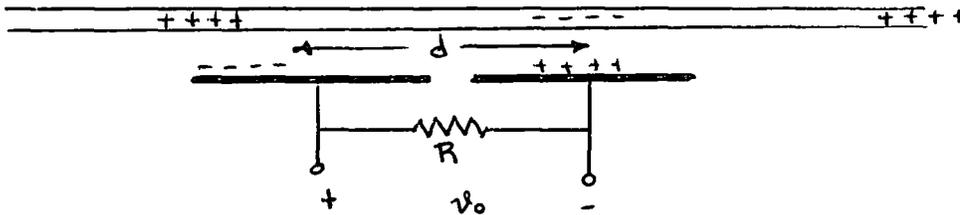
where  $y = 0$  is the channel center. Note that (e) satisfies the boundary condition at  $x = 0$  and states that a charge at  $x$  at time  $t$  has been decaying  $\frac{x}{U}$  seconds (since it left the source) and was dumped in the channel at time

$$t' = t - \frac{x}{U}$$

Substitution of (e) into (d) verifies that it is a solution.

Part b

From (e) it is clear that the wave length of the sinusoidally (and decaying) charge stream is  $2\pi U/\omega$ . Thus, the wave length can be altered simply by changing  $\omega$ . One technique for measuring the flow velocity would consist in measuring the voltage induced across the resistance  $R$  (as shown in the figure) as a function of the frequency. With the distance between electrode centers  $d$  equal to 1/2 wave length, a peak in the output signal would be expected. If we call the frequency at which this peak occurs  $\omega_p$ , then



MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.22 (Continued)

$$\frac{2\pi U}{\omega_p} = 2d$$

or

$$U = \frac{d\omega_p}{\pi}$$

Thus, a determination of  $\omega_p$  gives U. There are, of course, problems with this approach. For example, there would be lesser peaks in the output at harmonic frequencies that could be mistaken for the desired peak. Alternatives are to use the decay rate, but such techniques are vulnerable to conductivity variations which are likely to be large.

PROBLEM 7.23

Part a

Current is carried by the conductor because of normal conduction and also because of convection of a net charge.

$$\bar{J} = \sigma \bar{E} + \rho_f \bar{v}$$

Also

$$\nabla \cdot \bar{E} = \rho_f / \epsilon = \nabla \cdot \frac{(\bar{J} - \rho_f \bar{v})}{\sigma}$$

But

$$\nabla \cdot \bar{J} = - \frac{\partial \rho_f}{\partial t} = 0 \text{ in steady state}$$

$$\nabla \cdot \bar{v} = \nabla \cdot (U \bar{i}_x) = 0 \text{ also, so that}$$

$$\rho_f / \epsilon = - \frac{\bar{v} \cdot \nabla \rho_f}{\sigma} = - \frac{U}{\sigma} \frac{\partial \rho_f}{\partial x}$$

The solution to this last equation is

$$\rho = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right) \left(\frac{x}{U}\right)}$$

i.e., the charge relaxes in the conductor; the time  $\tau = \frac{x}{U}$  is a measure of how long since the charge left the source at the first screen.

Part b

Let

$$E_x(x=0) = E_0$$

$$\frac{\partial E_x}{\partial x} = \frac{\rho(x)}{\epsilon} = \frac{\rho_0}{\epsilon} e^{-\left(\frac{\sigma}{\epsilon}\right) \left(\frac{x}{U}\right)}$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.23 (Continued)

$$E_x(x) = E_o + \int_0^x \frac{\rho_f(x)}{\epsilon} dx = E_o + \frac{\rho_o U}{\sigma} \left(1 - e^{-\frac{\sigma}{\epsilon} \frac{x}{U}}\right)$$

Note that since  $J_x(x=0) = \sigma E_o + \rho_o U = \frac{V}{RA}$

$$E_x(x) = \frac{V}{RA\sigma} - \frac{\rho_o U}{\sigma} e^{-\left(\frac{\sigma}{\epsilon}\right)\left(\frac{x}{U}\right)}$$

We must finish the problem to know V

Part c

$$V = - \int_0^l E_x(x) dx = - \frac{Vl}{RA\sigma} + \rho_o \epsilon_o \left(\frac{U}{\sigma}\right)^2 \left(1 - e^{-\left(\frac{\sigma}{\epsilon}\right) \frac{l}{U}}\right)$$

$$V = \left(\frac{1}{1 + \frac{l}{RA\sigma}}\right) \rho_o \epsilon \left(\frac{U}{\sigma}\right)^2 \left(1 - e^{-\left(\frac{\sigma}{\epsilon}\right) \left(\frac{l}{U}\right)}\right)$$

PROBLEM 7.24

Part a

The model for this problem is similar to that used in Example 7.2.6 of the text. Each ring induces a charge on the stream having opposite polarity to its potential. Thus, conservation of charge for the can at potential  $v_3$  (under the ring at potential  $v_1$ ) is

$$-C_1 n v_1 = C \frac{dv_3}{dt} + \frac{v_3}{R} \tag{a}$$

Similarly, for the other two cans,

$$-C_1 n v_2 = C \frac{dv_1}{dt} + \frac{v_1}{R} \tag{b}$$

$$-C_1 n v_3 = C \frac{dv_2}{dt} + \frac{v_2}{R} \tag{c}$$

To solve these three equations, we assume solutions of the form

$$v_i = \hat{v}_i e^{st} \tag{d}$$

and the complex amplitudes  $\hat{v}_i$  are governed by the conditions that follow from substitution of (d) into (a)-(c)

$$\begin{bmatrix} C_1 n & 0 & (Cs + \frac{1}{R}) \\ (Cs + \frac{1}{R}) & C_1 n & 0 \\ 0 & (Cs + \frac{1}{R}) & C_1 n \end{bmatrix} = 0 \tag{e}$$

PROBLEM 7.24 (Continued)

The solution for  $s$  is

$$s = -\frac{1}{RC} + \frac{C_1 n}{C} \left[ \frac{1}{2} \pm \frac{j\sqrt{3}}{2}, -1 \right] \quad (f)$$

Part b

Thus, the system is unstable if

$$\frac{1}{RC} < \frac{C_1 n}{2C} \quad (g)$$

Part c

In particular, from (g), the system is self-excited as

$$\frac{1}{R} = \frac{C_1 n}{2} \quad (h)$$

Part d

The frequency of oscillation under condition (h) follows from (f) and (h) as

$$\omega = \frac{C_1 n \sqrt{3}}{2C} = \frac{\sqrt{3}}{RC} \quad (i)$$

PROBLEM 7.25

The crucial quantities in the respective systems are the magnetic diffusion time (Eq. 7.1.28) and the charge relaxation time (Eq. 7.2.11) relative to the period of excitation  $T = 1/f$ . The conductivities required to make these respective times equal to the excitation period  $T$  are

$$\sigma = \pi^2 T / \mu_0 d^2 \quad (a)$$

$$\sigma = \epsilon / T \quad (b)$$

In terms of the given numbers,

$$\begin{aligned} \sigma &= (3.14)^2 (10^{-5}) / (4) (3.14 \times 10^{-7}) (10^{-4}) \\ &= 7.85 \times 10^7 \text{ mhos/m} \end{aligned} \quad (c)$$

and

$$\sigma = (81) (8.85 \times 10^{-12}) / 10^{-5} = 7.16 \times 10^{-5} \text{ mhos/m} \quad (d)$$

For the change in depth to have a large effect on the inductance, the conductivity must be greater than that given by (c). Thus, the magnetic device would not be satisfactory. By contrast, (d) indicates that the conductivity of the electric apparatus is more than sufficient to make a change in capacitance with liquid depth apparent even if  $\epsilon = \epsilon_0$ . Both devices would be attractive for this application only if the conductivity exceeded that given by (c).

## MAGNETIC DIFFUSION AND CHARGE RELAXATION

### PROBLEM 7.26

This problem depends on the same physical reasoning as used in connection with Prob. 7.25. There are two modes in which either device can operate. Consider configuration (a): the inductance can change either because of the magnetization of the water, or because of currents induced in the water. However, water is only weakly magnetic and so the first mode of operation is not attractive. Moreover, the frequency is too low to induce appreciable currents, as can be seen by comparing the magnetic diffusion time to the period of excitation. Hence, configuration (a) does not represent an attractive approach to the engineering problem.

On the other hand, configuration (b) can operate either because of a change in capacitance between the electrodes due to the change in position of the polarized liquid (at high frequencies) or due to a change in position of a perfectly conducting liquid (low frequencies). As the calculations of Prob. 7.26 show, it is this last mode of operation that is appropriate in this case.

### PROBLEM 7.27

#### Part a

Because we have changed only a boundary condition, the potentials in regions (a) and (b) are still of the general form

$$\begin{aligned}\hat{\phi}_a &= A \sinh kx + B \cosh kx \\ \hat{\phi}_b &= C \sinh kx + D \cosh kx\end{aligned}\tag{a}$$

There are now four boundary conditions:

$$\hat{\phi}_a(d) = \hat{V}\tag{b}$$

$$\hat{\phi}_a(0) = \hat{\phi}_b(0)\tag{c}$$

$$-\sigma \frac{\partial \phi_b(0)}{\partial x} = \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right) \left( -\epsilon_0 \frac{\partial \phi_a(0)}{\partial x} + \epsilon \frac{\partial \phi_b(0)}{\partial x} \right)\tag{d}$$

$$\phi_b(-f) = 0\tag{e}$$

Only boundary condition (e) is new; it has replaced the assumption that  $\phi_b$  must go to zero as  $x \rightarrow -\infty$ .

Solving for A, B, C and D we find that

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.27 (Continued)

$$\phi_a = \text{Re} \frac{\hat{V}}{\Delta} [(1+jS \epsilon/\epsilon_o) \sinh kx + jS \tanh kf \cosh kx] e^{j(\omega t - kz)} \quad (f)$$

$$\phi_b = \text{Re} \frac{\hat{V}}{\Delta} [jS \sinh kx + jS \tanh kf \cosh kx] e^{j(\omega t - kz)} \quad (g)$$

where  $\Delta = (1+jS \epsilon/\epsilon_o) \sinh kd + jS \tanh kf \cosh kd$ .

Part b

If

$$|fk| \gg 1$$

$$\tanh kf \rightarrow 1 \quad (h)$$

A comparison shows that in this limit the results agree with Sec. 7.2.4 if we note that

$$e^{kx} = \cosh kx + \sinh kx \quad (i)$$

PROBLEM 7.28

Part a

The regions between the traveling wave electrodes and the moving sheet are free space, and therefore the fields are governed by

$$\nabla^2 \phi = 0 \quad (a)$$

where

$$\bar{E} = -\nabla\phi \quad (b)$$

Moreover, solutions that have the same (z-t) dependence as the imposed traveling wave potentials, and that satisfy (a) are

$$\phi_a = \text{Re}[A_1 \cosh kx + A_2 \sinh kx] e^{j(\omega t - kz)} \quad (c)$$

$$\phi_b = \text{Re}[B_1 \cosh kx + B_2 \sinh kx] e^{j(\omega t - kz)} \quad (d)$$

The constants  $A_1, A_2, B_1, B_2$  must be adjusted to make these solutions satisfy the boundary conditions

$$\hat{\phi}_a = \hat{V}_o \quad \text{at} \quad x = c \quad (e)$$

$$\hat{\phi}_b = \hat{V}_o \quad \text{at} \quad x = -c \quad (f)$$

$$\hat{\phi}_a = \hat{\phi}_b \quad \text{at} \quad x = 0 \quad (g)$$

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) (\epsilon_a E_x^a - \epsilon_b E_x^b) + \sigma_s \frac{\partial E_z^a}{\partial z} = 0 \quad (h)$$

Part b

The symmetry requires that

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.28 (Continued)

$$\phi_a(x, z, t) = \phi_b(-x, z, t) \quad (1)$$

and this implies that  $A_1 = B_1$ ,  $A_2 = -B_2$ . The boundary conditions become

$$A_1 \cosh kc + A_2 \sinh kc = \hat{V}_0 \quad (j)$$

$$jS (2A_2) = A_1 \quad (k)$$

where

$$S = (\omega - kU)\epsilon_0 / k\sigma_s \quad (l)$$

Thus,

$$A_1 = B_1 = 2j \hat{S} \hat{V}_0 / (\sinh kc + 2j S \cosh kc) \quad (m)$$

and

$$A_2 = -B_2 = \hat{V}_0 / (\sinh kc + 2j S \cosh kc) \quad (n)$$

Part b

A section of the sheet can be enclosed by a thin volume of small area in the y-z plane to give the force per unit area as

$$T_z = 2T_{zx}^a \quad (x = 0) \quad (o)$$

where the symmetry has been used to set

$$T_{zx}^a = -T_{zx}^b \quad (p)$$

Thus, the time average force per unit area is

$$\langle T_z \rangle = \text{Re}[\epsilon_0 \hat{E}_x^a(0) \hat{E}_z^{a*}(0)] \quad (q)$$

and from (m) and (n),

$$\langle T_z \rangle = \text{Re}[\epsilon_0 (-jk) A_1^* (-k) A_2] \quad (r)$$

$$= \text{Re} \left[ \frac{2\epsilon_0 k^2 |\hat{V}_0|^2 S}{\sinh^2 kc + 4S^2 \cosh^2 kc} \right] \quad (s)$$

$$= \frac{2\epsilon_0 k^2 |\hat{V}_0|^2 S}{(\sinh^2 kc + 4S^2 \cosh^2 kc)} \quad (t)$$

It follows from (t) that the maximum occurs as

$$S' = \frac{1}{2} \tanh kc \quad (u)$$

or

$$\omega = kU + \frac{\sigma_s k}{2\epsilon_0} \tanh kc \quad (v)$$

MAGNETIC DIFFUSION AND CHARGE RELAXATION

PROBLEM 7.28 (Continued)

Part c

Note that if  $S$  is held fixed at the value given by (u), the force per unit area remains fixed. Thus, as  $\sigma_s \rightarrow 0$ , the velocities of the potential wave and the sheet must become equal to retain the force at a constant value

$$\omega \rightarrow kU$$

(w)