PROBLEM 9.1

The equation of motion for a static rod is

$$0 = E \frac{d^2\delta}{dx^2} + F_x \text{ where } F_x = \rho g \quad \text{(a)}$$

We can integrate this equation directly and get

$$\delta(x) = -\frac{\rho g}{E}\left(\frac{x^2}{2}\right) + Cx + D, \quad \text{(b)}$$

where C and D are arbitrary constants.

Part a

The stress function is $T(x) = E\frac{d\delta}{dx}$, and therefore

$$T(x) = -\rho g x + CE. \quad \text{(c)}$$

We have a free end at $x = \ell$ and this implies $T(x=\ell)=0$. Now we can write the stress as

$$T(x) = -\rho g x + \rho g \ell. \quad \text{(d)}$$

The maximum stress occurs at $x = 0$ and is $T_{max} = \rho g \ell$. Equating this to the maximum allowable stress, we have

$$2 \times 10^9 = (7.8 \times 10^3)(9.8)\ell$$

hence

$$\ell = 2.6 \times 10^4 \text{ meters.}$$

Part b

From part (a)

$$T(x) = -\rho g x + \rho g \ell \quad \text{(e)}$$

The fixed end at $x = 0$ implies that $D = 0$, so now we can write the displacement

$$\delta(x) = -\frac{\rho g}{E}\left(\frac{x^2}{2}\right) + \frac{\rho g \ell}{E}(x) \quad \text{(f)}$$

Part c

$$\delta(\ell) = -\frac{\rho g}{E}\frac{\ell^2}{2} + \frac{\rho g \ell}{E}(\ell) = \frac{\rho g \ell^2}{2E} \quad \text{(g)}$$

For $\ell = 2.6 \times 10^4$ meters, $\delta(\ell) = 129$ meters. This appears to be a large displacement, but note that the total unstressed length is 26,000 meters.

PROBLEM 9.2

Part a

The equation of motion for a static rod is

$$0 = E\frac{d^2\delta}{dx^2} + \rho g \quad \text{(a)}$$

If we define $x' = x-L_1$, we can write the solutions for δ in rod 1 and in rod 2 as

$$\delta_1(x) = -\frac{\rho_1 g}{E_1}\left(\frac{x^2}{2}\right) + C_1 x + D_1 \quad \text{(b)}$$

and

$$\delta_2(x') = -\frac{\rho_2 g}{E_2}\left(\frac{x'^2}{2}\right) + C_2 x' + D_2 \quad \text{(c)}$$

where C_1, C_2, D_1, and D_2 are arbitrary constants. Since $T = E\frac{d\delta}{dx}$ we can also write the tensions,

$$T_1(x) = -\rho_1 g x + E_1 C_1 \quad \text{(d)}$$

and

$$T_2(x') = -\rho_2 g x' + E_2 C_2 \quad \text{(e)}$$

We must have four boundary conditions to evaluate the constants and they are:

$$\delta_1(x{=}0) = 0, \quad \text{(f)}$$

$$\delta_2(x'{=}0) = \delta_1(x{=}L_1) \quad \text{(g)}$$

$$0 = -A_1 T_1(x{=}L_1) + A_2 T_2(x'{=}0) + mg, \quad \text{(h)}$$

and

$$0 = -A_2 T_2(x'{=}L_2) + Mg + f_x^e \quad \text{(i)}$$

where f_x^e is found using the Maxwell stress tensor

$$f_x^e = \frac{\varepsilon_o A_M V_o^2}{2d^2} \quad \text{(j)}$$

where we assume $d \gg \delta^{(2)}(x'{=}L_2)$.

Equations (f), (g), (h) and (i) serve to define the constants of integration. Substitution of (b)-(e) shows that

$$D_1 = 0 \quad \text{(k)}$$

$$-\frac{\rho_1 g}{E_1}\left(\frac{L_1^2}{2}\right) + C_1 L_1 + D_1 - D_2 = 0 \quad \text{(ℓ)}$$

PROBLEM 9.2 (Continued)

$$-A_1[-\rho_1 g L_1 + E_1 C_1] + A_2[E_2 C_2] + mg = 0 \tag{m}$$

$$-A_2[-\rho_2 g L_2 + E_2 C_2] + Mg + \frac{\varepsilon_o A_M V_o^2}{2d^2} = 0 \tag{n}$$

Solution of these expressions, beginning with (n), gives

$$C_2 = \left[Mg + \frac{\varepsilon_o A_M V_o^2}{2d^2} + \rho_2 g L_2 A_2\right]\frac{1}{A_2 E_2} \tag{o}$$

and hence

$$C_1 = \left[mg + \rho_1 g L_1 A_1 + A_2 E_2 C_2\right]\frac{1}{A_1 E_1}$$

$$= \left\{[(M+m) + \rho_1 L_1 A_1 + \rho_2 L_2 A_2]g + \frac{\varepsilon_o A_M V_o^2}{2d^2}\right\}\frac{1}{A_1 E_1} \tag{p}$$

$$D_2 = \frac{L_1}{A_1 E_1}\left\{[(M+m) + \frac{\rho_1 L_1 A_1}{2} + \rho_2 L_2 A_2]g + \frac{\varepsilon_o A_M V_o^2}{2d^2}\right\} \tag{q}$$

$$D_1 = 0 \tag{r}$$

Thus, (b) and (c) are determined.

PROBLEM 9.3

Part a

Longitudinal displacements on the rod satisfy the wave equation

$$\rho\frac{\partial^2\delta}{\partial t^2} = E\frac{\partial^2\delta}{\partial x^2} \text{ and the stress } T = E\frac{\partial\delta}{\partial x}\,. \tag{a}$$

We can write $\delta(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$ for sinusoidal excitations. $\hat{\delta}(x)$ can be written as $\hat{\delta}(x) = C_1\sin\beta x + C_2\cos\beta x$ where $\beta = \omega\sqrt{\rho/E}$. The two constants are found from the boundary conditions

$$M\frac{\partial^2\delta}{\partial t^2}(\ell,t) = -AT(\ell,t) + f(t) \tag{b}$$

$$\delta(0,t) = 0. \tag{c}$$

These conditions become

$$-M\omega^2\,\hat{\delta}(\ell) = -AE\frac{d\hat{\delta}}{dx}(\ell) + f_o \tag{d}$$

and

$$\hat{\delta}(0) = 0 \tag{e}$$

PROBLEM 9.3 (Continued)

for sinusoidal excitations.

Now we find $C_2 = 0$ and

$$C_1 = \frac{f_o}{AE\beta\cos\beta\ell - M\omega^2\sin\beta\ell} \quad \text{(f)}$$

Hence,

$$\delta(x,t) = \frac{\sin\beta x}{AE\beta\cos\beta\ell - M\omega^2\sin\beta\ell}\,Re[f_o e^{j\omega t}] \quad \text{(g)}$$

and

$$T(x,t) = E\frac{\partial\delta}{\partial x} = \frac{E\beta\cos\beta x}{AE\beta\cos\beta\ell - M\omega^2\sin\beta\ell}\,Re[f_o e^{j\omega t}] \quad \text{(h)}$$

Part b

At $x = \ell$,

$$\delta(\ell,t) = \frac{1}{AE\beta\cot\beta\ell - M\omega^2}\,Re[f_o e^{j\omega t}] \quad \text{(i)}$$

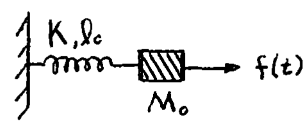
where $\beta\cot\beta\ell = \omega\sqrt{\rho/E}\,\cot\,(\omega\ell\sqrt{\rho/E})$.

For small ω, $\cot(\omega\ell\sqrt{\rho/E}) \to \frac{1}{\omega\ell\sqrt{\frac{\rho}{E}}}$ and

$$\delta(\ell,t) \to \frac{1}{\frac{AE}{\ell} - M\omega^2}\,f(t) \quad \text{(j)}$$

This equation is as used to describe a mass on the end of a massless spring:

$$M_o\frac{d^2x}{dt^2} = -Kx + f(t) \quad \text{(k)}$$



and

$$x = Re[\hat{x}e^{j\omega t}],$$

$$-M_o\omega^2\hat{x} = -K\hat{x} + f_o, \quad (\ell)$$

or

$$x = \frac{1}{K - M_o\omega^2}\,f(t) \quad \text{(m)}$$

Comparing (j) and (ℓ) we note that

$$K = \frac{AE}{\ell} \text{ and } \ell_o = \ell. \quad \text{(n)}$$

Our comparison is complete and since $M >> \rho A\ell$ we can use the massless spring model with a mass $M_o = M$ on the end.

PROBLEM 9.4

A response that can be represented purely as a wave traveling in the negative x direction implies that there be no wave reflection at the left-hand boundary. We must have

$$v(0,t) + \frac{1}{\sqrt{\rho E}} T(0,t) = 0 \tag{a}$$

as seen in Sec. 9.1.1b.

This condition can be satisfied by a viscous damper alone:

$$AT(0,t) + Bv(0,t) = 0 \tag{b}$$

Hence, we can write

$$B = A\sqrt{\rho E}$$
$$M = 0 \tag{c}$$
$$K = 0.$$

PROBLEM 9.5

Part a

At $x = \ell$ the boundary condition is

$$0 = - AT(\ell,t) - B \frac{\partial\delta}{\partial t} (\ell,t) + f(t) \tag{a}$$

Part b

We can write the solution as

$$\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x, \tag{b}$$

where $\beta = \omega\sqrt{\frac{\rho}{E}}$. At $x = 0$ there is a fixed end, hence $\hat{\delta}(x=0) = 0$ and $C_2 = 0$. At $x = \ell$ our boundary condition becomes

$$F_o = j\omega B\hat{\delta}(x=\ell) + AE \frac{d\hat{\delta}}{dx} (x=\ell), \tag{c}$$

or in terms of C_1;

$$F_o = j\omega BC_1 \sin \beta\ell + AE\beta C_1 \cos \beta\ell \tag{d}$$

After solving for C_1, we can write our solution as

$$\hat{\delta}(x) = \frac{F_o \sin\beta x}{AE\beta\cos\beta\ell + j\omega B\sin\beta\ell} \tag{e}$$

Part c

For ω real and $B>0$, δ cannot be infinite with a finite-applied force, because the denominator of $\hat{\delta}(x)$ can never be zero.

Physically, $B>0$ implies that the system is damped and energy would be

PROBLEM 9.5 (Continued)

dissipated for each cycle of operation, hence a perfect resonance cannot occur. However, there will be frequencies which will maximize the amplitude.

PROBLEM 9.6

First, we can calculate the force of magnetic origin, f_x, on the rod. If we define $\delta(\ell,t)$ to be the a.c. deflection of the rod at $x = \ell$, then using Ampere's law and the Maxwell stress tensor (Eq. 8.5.41 with magnetostriction ignored) we find

$$f_x = \frac{\mu_o AN^2I^2}{2[d-\delta(\ell,t)]^2} \tag{a}$$

This result can also be obtained using the energy methods of Chap. 3 (See Appendix E, Table 3.1). Since $d >> \delta(\ell,t)$, we may linearize f_x:

$$f_x \simeq \frac{\mu_o AN^2I^2}{2d^2} + \frac{\mu_o AN^2I^2}{d^3}\ \delta(\ell,t) \tag{b}$$

The first term represents a constant force which is balanced by a static deflection on the rod. If we assume that this static deflection is included in the equilibrium length ℓ, then we need only use the last term of f_x to compute the dynamic deflection $\delta(\ell,t)$. In the bulk of the rod we have the wave equation; for sinusoidal variations

$$\delta(x,t) = \text{Re}[\hat{\delta}(x)e^{j\omega t}] \tag{c}$$

we can write the complex amplitude $\hat{\delta}(x)$ as

$$\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x \tag{d}$$

where $\beta = \omega\sqrt{\frac{\rho}{E}}$. At $x = 0$ we have a fixed end, so $\hat{\delta}(o) = 0$ and $C_2 = 0$. At $x = \ell$ the boundary condition is

$$0 = f_x - AE\frac{\partial\delta}{\partial x}(\ell,t), \tag{e}$$

or

$$0 = \frac{\mu_o AN^2I^2}{d^3}\ \hat{\delta}(x{=}\ell) - AE\frac{d\hat{\delta}}{dx}(x{=}\ell) \tag{f}$$

Substituting we obtain

$$\frac{\mu_o AN^2I^2}{d^3} C_1 \sin \beta\ell = C_1\ AE\beta \cos \beta\ell \tag{g}$$

Our solution is $\hat{\delta}(x) = C_1 \sin \beta x$ and for a non-trivial solution we must have $C_1 \neq 0$. So, divide (g) by C_1 and obtain the resonance condition:

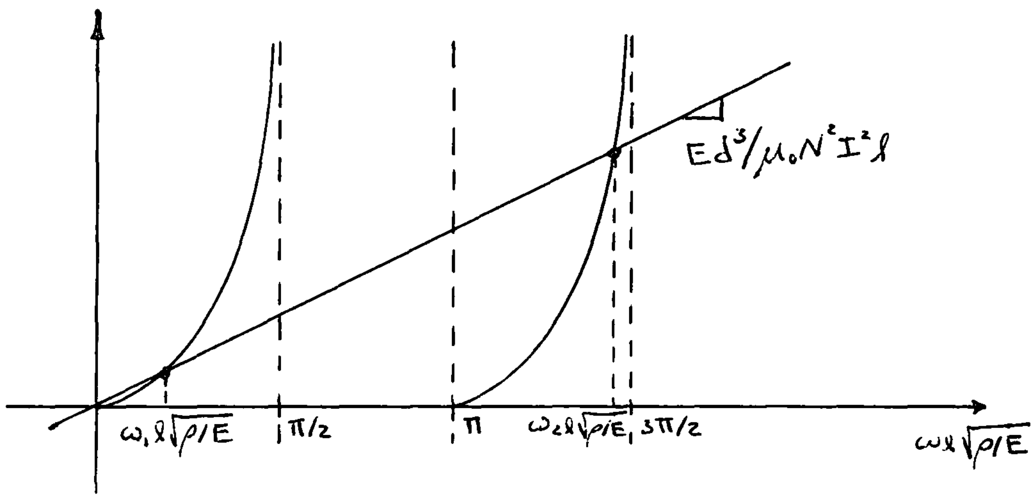
PROBLEM 9.6 (Continued)

$$\left(\frac{\mu_o A N^2 I^2}{d^3}\right) \sin \beta\ell = AE\beta \cos \beta\ell \qquad \text{(h)}$$

Substituting $\beta = \omega\sqrt{\frac{\rho}{E}}$ and rearranging, we have

$$\frac{Ed^3}{\mu_o N^2 I^2 \ell}\left(\omega\ell\sqrt{\frac{\rho}{E}}\right) = \tan\left(\omega\ell\sqrt{\frac{\rho}{E}}\right) \qquad \text{(i)}$$

which, when solved for ω, yields the eigenfrequencies. Graphically, the first two eigenfrequencies are found from the sketch.



Notice that as the current I is increased, the slope of the straight line decreases and the first eigenfrequency (denoted by ω_1) goes to zero and then seemingly disappears for still higher currents. Actually ω_1 now becomes <u>imaginary</u> and can be found from the equation

$$\frac{Ed^3}{\mu_o N^2 I^2 \ell}\left(|\omega_1|\ell\sqrt{\frac{\rho}{E}}\right) = \tanh\left(|\omega_1|\ell\sqrt{\frac{\rho}{E}}\right) \qquad \text{(j)}$$

Just as there are negative solutions to (i), $-\omega_1$, $-\omega_2$ .. etc., so there are now solutions $\pm\, j|\omega_1|$. Thus, because ω_1 is imaginary, the system is <u>unstable</u>,

PROBLEM 9.6 (Continued)

(amplitude of one solution growing in time).

Hence when the slope of the straight line becomes less than unity, the system is unstable. This condition can be stated as:

$$\text{STABLE} \longrightarrow \frac{Ed^3}{\mu_o N^2 I^2 \ell} > 1 \tag{k}$$

or

$$\text{UNSTABLE} \longrightarrow \frac{Ed^3}{\mu_o N^2 I^2 \ell} < 1 \tag{ℓ}$$

PROBLEM 9.7

Part a

$\delta(x,t)$ satisfies the wave equation

$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x^2} \tag{a}$$

and the stress is $T = E \frac{\partial \delta}{\partial x}$. We can write

$$\delta(x,t) = \text{Re}[\hat{\delta}(x) e^{j\omega t}] \tag{b}$$

and substitution into the wave equation gives

$$\frac{d^2\hat{\delta}}{dx^2} + \beta^2 \hat{\delta} = 0. \tag{c}$$

$$\beta = \omega\sqrt{\frac{\rho}{E}}$$

For $x > 0$ we have,

$$\hat{\delta}_a(x) = \hat{C}_1 \sin \beta x + \hat{C}_2 \cos \beta x \tag{d}$$

and

$$\hat{T}_a(x) = \hat{C}_1 E\beta \cos \beta x - \hat{C}_2 E\beta \sin \beta x \tag{e}$$

and for $x < 0$ we have,

$$\hat{\delta}_b(x) = \hat{C}_3 \sin \beta x + \hat{C}_4 \cos \beta x \tag{f}$$

and

$$\hat{T}_b(x) = \hat{C}_3 E\beta \cos \beta x - \hat{C}_4 E\beta \sin \beta x \tag{g}$$

Part b

There are four constants to be determined; thus we need four boundary conditions. At the right end (x=L), we have

$$\hat{\delta}_a(x=L) = 0 \tag{h}$$

PROBLEM 9.7 (Continued)

and the left end,

$$\hat{\delta}_b(x=-L) = \delta_o e^{-j(\frac{\pi}{2})} \qquad (i)$$

There are two conditions at the middle (x=0),

$$\hat{\delta}_a(x=0^+) = \hat{\delta}_o(x=0^-) \qquad (j)$$

and

$$-M\omega^2\,\hat{\delta}_a(x=0) = A\hat{T}_a(x=0^+) - A\hat{T}_b(x=0^-) - 4K\hat{\delta}_a(x=0) \qquad (k)$$

Part c

Solving for $\hat{C}_1, \hat{C}_2, \hat{C}_3$, and $\hat{C}_4$ we obtain

$$\hat{C}_1 = \frac{-\delta_o AE\beta e^{-j\frac{\pi}{2}}\cot\beta L}{\sin\beta L(4K+2AE\beta\cot\beta L-M\omega^2)} \qquad (\ell)$$

$$\hat{C}_2 = \frac{\delta_o AE\beta e^{-j\frac{\pi}{2}}}{\sin\beta L(4K+2AE\beta\cot\beta L-M\omega^2)} \qquad (m)$$

$$\hat{C}_3 = \frac{\delta_o AE\beta e^{-j\frac{\pi}{2}}\cot\beta L}{\sin\beta L(4K+2AE\beta\cot\beta L-M\omega^2)} - \frac{\delta_o e^{-j\frac{\pi}{2}}}{\sin\beta L} \qquad (n)$$

$$\hat{C}_4 = \hat{C}_2 \qquad (o)$$

Thus, (b), (e), and (g) with these constants give the desired stress distribution.

PROBLEM 9.8

In terms of the complex amplitudes, (ℓ) and (r) become

$$\hat{T}'(0) = \frac{L_o I}{aA}\,\hat{i}_i \qquad (\ell) \text{ - text} \qquad (a)$$

and

$$\hat{T}'(\ell) = \frac{L_o I}{aA}\,\hat{i}' \qquad (r) \text{ - text} \qquad (b)$$

where $\hat{i}' = -G\hat{v}_o$.

Equation (t) without the approximation becomes

$$\hat{v}_o = -j\omega\,\frac{GL_o(\mu+\mu_o)}{\mu-\mu_o}\,\hat{v}_o + j\omega\,\frac{L_o I}{a}\,\hat{\delta}_o \qquad (c)$$

Using the steady-state solutions for the rod, we can solve for T(x) in terms of the boundary values $\hat{T}(0)$ and $\hat{T}(\ell)$:

PROBLEM 9.8 (Continued)

$$\hat{T}(x) = \hat{T}(0)\,\frac{\sin[k(\ell-x)]}{\sin[k\ell]} + \hat{T}(\ell)\,\frac{\sin[kx]}{\sin[k\ell]} \tag{d}$$

then

$$\hat{\delta} = \frac{1}{\omega\sqrt{\rho E}}\left[\hat{T}(0)\,\frac{\cos[k(\ell-x)]}{\sin[k\ell]} - \hat{T}(\ell)\,\frac{\cos[kx]}{\sin[k\ell]}\right] \tag{e}$$

From (a) and (b), this becomes

$$\hat{\delta}(\ell) = \hat{\delta}_o = \frac{1}{\omega\sqrt{\rho E}}\left[\frac{L_o I}{aA}\,\hat{i}_i\,\frac{1}{\sin[k\ell]} + \frac{GL_o I}{aA}\,\hat{v}_o\,\frac{\cos[k\ell]}{\sin[k\ell]}\right] \tag{f}$$

Thus, in view of (c) solved for $\hat{\delta}_o$, we obtain the system function

$$H(\omega) = \frac{\hat{v}_o}{\hat{i}_i} = \frac{-\frac{1}{G}}{\cos[k\ell]+j\sqrt{\rho E}\,(\frac{A}{G})(\frac{a}{L_o I})^2\sin[k\ell] - \left[\frac{\omega G L_o(\mu+\mu_o)}{(\mu-\mu_o)}\right]\sqrt{\rho E}\,(\frac{A}{G})(\frac{a}{L_o I})^2\sin[k\ell]} \tag{g}$$

PROBLEM 9.9

Part a

First of all, $y(t) = \delta(-L,t)$ where $\delta(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$. We can write the solution for $\hat{\delta}$ as $\hat{\delta}(x) = C_1\sin\beta x + C_2\cos\beta x$, where $\beta = \omega\sqrt{\rho/E}$. The C_2 is zero because of the fixed end at $x = 0(\hat{\delta}(0) = 0)$. At the other end we have

$$M\,\frac{\partial^2\delta}{\partial t^2}(-L,t) = A_2E\,\frac{\partial\delta}{\partial x}(-L,t) + f^e(t) \tag{a}$$

Using the Maxwell stress tensor, (or the energy method of Chap. 3) we find

$$f^e(t) = \frac{A\mu_o N^2}{2}\left\{\frac{[I_o-I(t)]^2}{[d-D+\delta(-L,t)]^2} - \frac{[I_o+I(t)]^2}{[d-D-\delta(-L,t)]^2}\right\} \tag{b}$$

which when linearized becomes,

$$f^e(t) \simeq -C_I I(t) - C_y\,\delta(-L,t), \tag{c}$$

where

$$C_I = \frac{2N^2\mu_o A I_o}{(d-D)^2};\; C_y = \frac{2N^2\mu_o A I_o^2}{(d-D)^3}$$

Our boundary condition (a) becomes

$$-M\omega^2\,\hat{\delta}(-L) = A_2E\,\frac{d\hat{\delta}}{dx}(-L) - C_I\hat{I} - C_y\,\hat{\delta}(-L) \tag{d}$$

Solving for C_1 we obtain

PROBLEM 9.9 (Continued)

$$C_1 = \frac{C_I \hat{I}}{A_2 E\beta\cos\beta L - (M\omega^2 - C_y)\sin\beta L}, \tag{e}$$

and we can write our solution as

$$y(t) = \text{Re}[-C_1 \sin\beta L \, e^{j\omega t}]. \tag{f}$$

Part b

The transducer is itself made from solid materials having characteristics that do not differ greatly from those of the rod. Thus, there is the question of whether the elastic response of the transducer materials is of importance. Under the assumption that the rod and transducer are constructed from materials having essentially the same elastic properties, the assumption that the yoke and plunger are rigid, but that the rod supports acoustic waves, is justified provided the rod is long compared to the largest dimension of the transducer, and that an acoustic wavelength is long compared to the largest transducer dimension. (See Sec. 9.1.3).

PROBLEM 9.10

Part a

At the outset, we can write the equation of motion for the massless plate:

$$-aT(\ell,t) + f^e(t) = M \frac{\partial^2 \delta}{\partial t^2}(\ell,t) \cong 0 \tag{a}$$

Using the Maxwell stress tensor we find the force of electrical origin $f^e(t)$ to be

$$f^e(t) = \frac{\varepsilon_o A}{2}\left[\frac{(V_o + v(t))^2}{(d-\delta(\ell,t))^2} - \frac{(V_o - v(t))^2}{(d+\delta(\ell,t))^2}\right] \tag{b}$$

Since $v(t) \ll V_o$ and $\delta(\ell,t) \ll d$, we can linearize $f^e(t)$:

$$f^e(t) \simeq \left[\frac{2\varepsilon_o A V_o^2}{d^3}\right]\delta(\ell,t) + \left[\frac{2\varepsilon_o A V_o}{d^2}\right]v(t) \tag{c}$$

Recognizing that $T(\ell,t) = E\frac{\partial\delta}{\partial x}(\ell,t)$ we can write our boundary condition at $x = \ell$ in the desired form:

$$aE\frac{\partial\delta}{\partial x}(\ell,t) = \frac{2\varepsilon_o A V_o^2}{d^3}\delta(\ell,t) + \frac{2\varepsilon_o A V_o}{d^2}v(t) \tag{d}$$

PROBLEM 9.10 (Continued)

Longitudinal displacements in the rod obey the wave equation and for an assumed form of $\delta(x,t) = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$ we can write $\hat{\delta}(x) = C_1\sin\beta x + C_2\cos\beta x$, where $\beta = \omega\sqrt{\rho/E}$. At x = 0 we have a fixed end, thus $\hat{\delta}(x=0) = 0$ and $C_2 = 0$. From part (a) and assuming sinusoidal time dependence, we can write our boundary condition at $x = \ell$ as

$$aE\frac{d\hat{\delta}}{dx}(\ell) = \frac{2\varepsilon_o AV_o^2}{d^3}\hat{\delta}(\ell) + \frac{2\varepsilon_o AV_o}{d^2}\hat{v} \tag{e}$$

Solving

$$C_1 = \frac{2\varepsilon_o AV_o\hat{v}}{aEd^2\beta\cos\beta\ell - \frac{2\varepsilon_o AV_o^2}{d}\sin\beta\ell} \tag{f}$$

Finally, we can write our solution as

$$\delta(x,t) = \left[\frac{2\varepsilon_o AV_o\sin\beta x}{aEd^2\beta\cos\beta\ell - \frac{2\varepsilon_o AV_o^2}{d}\sin\beta\ell}\right]\mathrm{Re}[\hat{V}e^{j\omega t}] \tag{g}$$

*

PROBLEM 9.11

Part a

For no elastic wave reflection at the right-hand boundary we must have a boundary condition of the form

$$v(0,t) + \frac{1}{\sqrt{\rho E}}T(0,t) = 0 \tag{a}$$

(from Sec. 9.1.1b). Since $v(0,t) = \frac{\partial\delta}{\partial t}(0,t)$, we can write

$$-\sqrt{\rho E}\frac{\partial\delta}{\partial t}(0,t) = T(0,t) \tag{b}$$

If we write the boundary condition at x = 0 for our example we obtain

$$0 = -ST(0,t) + f_x^e(t), \tag{c}$$

or for perturbations

$$0 = -ST(0,t) + f_{a.c.}^e(t) \tag{d}$$

Combining (b) and (d)

$$f_{a.c.}^e(t) = -S\sqrt{\rho E}\frac{\partial\delta}{\partial t}(0,t) \tag{e}$$

PROBLEM 9.11 (Continued)

and since $\partial\delta/\partial t\ (0,t) = dy_s/dt$,

$$f^e_{a.c.}(t) = -\ S\sqrt{\rho E}\ \frac{dy_s}{dt} \tag{f}$$

The perturbation electric force can be found using the Maxwell stress tensor (using a surface of integration similar to that illustrated by Prob. 8.10):

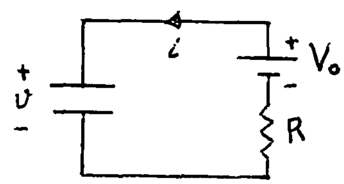
$$f^e_x(t) = \frac{\varepsilon_o v^2 D}{a} \simeq \frac{\varepsilon_o V_o^2 D}{a} + \frac{2\varepsilon_o V_o D v_s}{a} \tag{g}$$

where we associate $f^e_{a.c.}(t) = \frac{2\varepsilon_o V_o D v_s}{a}$.

Equation (f) now becomes

$$\frac{2\varepsilon_o V_o D v_s}{a} = -\ S\sqrt{\rho E}\ \frac{dy_s}{dt} \tag{h}$$

Now that we have dealt with the force balance we can write the circuit equations.



The capacitance of the device is found to be

$$C = \frac{2\varepsilon_o y D}{a}$$

Note that $q = Cv$ and $i = \frac{dq}{dt}$. The basic circuit equation is

$$v + iR = V_o = v + R\frac{dq}{dt} = v + R\left[C\frac{dv}{dt} + v\frac{dC}{dt}\right] \tag{i}$$

Substituting, we obtain

$$V_o = v + RC\frac{dv}{dt} + \frac{2\varepsilon_o DR}{a}\frac{dy}{dt} \tag{j}$$

and for perturbation quantities,

$$0 = v_s + RC_o\frac{dv_s}{dt} + \frac{2\varepsilon_o DV_o R}{a}\ \frac{dy_s}{dt}\ . \tag{k}$$

Since $\omega \ll \frac{1}{RC_o}$, $v_s \gg RC_o\ dv_s/dt$ and now we have

$$0 = v_s + \frac{2\varepsilon_o DV_o R}{a}\frac{dy_s}{dt} \tag{ℓ}$$

PROBLEM 9.11 (Continued)

Equations (h) and (ℓ) must be satisfied simultaneously and this can occur only if

$$\frac{2\varepsilon_o D V_o R}{a} = \frac{aS\sqrt{\rho E}}{2\varepsilon_o V_o D} \tag{m}$$

Finally from (m) we have the condition on the d.c. voltage,

$$V_o = \frac{a}{2\varepsilon_o D}\left[\frac{S\sqrt{\rho E}}{R}\right]^{1/2} \tag{n}$$

PROBLEM 9.12

Part a

Note that there is no mutual capacitance between the two pairs. We can find the capacitance of the left-hand pair of plates to be

$$C_2 = \frac{\varepsilon d(\frac{w}{2} - y_2)}{h} + \frac{\varepsilon_o d(\frac{w}{2} + y_2)}{h} \tag{a}$$

The current i_2 can be found from $i_2 = dq_2/dt = d(V_oC_2)/dt = V_o\, dC_2/dt$, and upon substitution of C_2 we obtain

$$i_2 = \left[\frac{-(\varepsilon-\varepsilon_o)V_o d}{h}\right]\frac{dy_2}{dt} \tag{b}$$

If we solve for y_2 in terms of v_s our job will be done.

Define the y-axis from left to right with $y = 0$ at $y_1 = 0$. Assume all constant forces (with $v_s = 0$) to be balanced and consider only the perturbations. If we assume for the rod $\delta(y,t) = \text{Re}[\hat{\delta}(y)e^{j\omega t}]$ then we can write

$$\hat{\delta}(y) = C_1 \sin\beta y + C_2 \cos\beta y \tag{c}$$

where $\beta = \omega\sqrt{\rho/E}$. (We have assumed that the electrical forces act only on the surfaces of the rod. This is evident from the form of the force density, Eq. 8.5.45, if the effect of electrostriction can be ignored.) At $y = 0$ there is no perturbation force and for a.c. deflections we have a free end condition:

$$0 = T(0,t) \Rightarrow E\frac{d\hat{\delta}}{dy}(y = 0) = 0 \tag{d}$$

This forces C_1 to be zero. At $y = \ell$ we can write the boundary condition as

$$0 = -hdT(\ell,t) + f^e_{a.c.}(t)$$

PROBLEM 9.12 (Continued)

Using the Maxwell stress tensor (or energy methods, as in Sec. 8.54)

$$f_1^e(t) = \frac{(\varepsilon-\varepsilon_o)d}{2h}(V_o + v_s)^2 \tag{e}$$

Linearizing and ignoring the d.c. term we have

$$f_{1a.c.}^e(t) = \frac{(\varepsilon-\varepsilon_o)V_o d}{h} v_s.$$

From the boundary condition for complex amplitudes we obtain

$$0 = -\, hdE\,\frac{d\hat{\delta}}{dy}(\ell) + \frac{(\varepsilon-\varepsilon_o)V_o d}{h}\hat{v}_s \tag{f}$$

Substituting and solving for C_2;

$$C_2 = \frac{-(\varepsilon-\varepsilon_o)V_o}{h^2 E\beta\sin\beta\ell}\hat{v}_s. \tag{g}$$

Recognizing that $y_2(t) = \delta(0,t)$, we can now write

$$y_2(t) = \text{Re}\left[\frac{-(\varepsilon-\varepsilon_o)V_o\hat{v}_s}{h^2\, E\beta\sin\beta\ell}e^{j\omega t}\right] \tag{h}$$

Since $i_2 = \frac{-(\varepsilon-\varepsilon_o)V_o d}{h}\frac{dy_2}{dt}$, we have

$$i_2 = \text{Re}\left[\frac{j\omega(\varepsilon-\varepsilon_o)^2 V_o^2 d}{h^3\, E\beta\sin\beta\ell}\hat{v}_s\, e^{j\omega t}\right] \tag{i}$$

Finally, we can write

$$Y(j\omega) = \frac{\hat{i}_2}{\hat{v}_s} = \frac{j\omega(\varepsilon-\varepsilon_o)^2 V_o^2 d}{h^3\, E\beta\sin\beta\ell} \tag{j}$$

Part b

The poles can be found from

$$h^3\, E\beta\sin\beta\ell = 0 \tag{k}$$

where $\beta = \omega\sqrt{\rho/E}$. The lowest nonzero frequency can be found from $\sin(\omega\ell\sqrt{\rho/E} = 0$ to be

$$\omega = \frac{\pi}{\ell\sqrt{\rho/E}} \tag{ℓ}$$

Note that the $\omega = 0$ is a pole because the rod is free to translate slowly between the plates.

PROBLEM 9.13

Part a

The flux for the left-hand transducer is

$$\lambda_\ell = \frac{\mu_o N^2}{g} 2\pi R(a-\delta(0,t))i_\ell \quad \text{(a)}$$

and for the right-hand one,

$$\lambda_r = \frac{\mu_o N^2}{g} 2\pi R(a+\delta(L_1,t))i_r \quad \text{(b)}$$

For this electrically linear situation we have $W_m' = \frac{1}{2} Li^2 = \frac{1}{2} \lambda i$ and $f = \frac{\partial W_m'}{\partial \delta}$. Hence we find, to linear terms

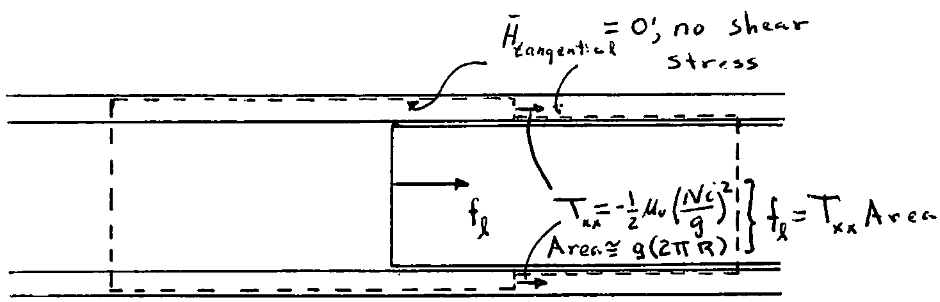
$$f_\ell = -\frac{\mu_o N^2}{g} \pi R(I_o^2 + 2I_o i) \quad \text{(c)}$$

and, because $i_r = I_o - Gv_{out}$

$$f_r = \frac{\mu_o N^2}{g} \pi R(I_o^2 - 2I_o Gv_{out}) \quad \text{(d)}$$

Part b

For the left-hand transducer, an acceptable stress-tensor surface is shown below,



and the mirror-image is acceptable for the right-hand transducer. Application of $f_x = \oint_S T_{xj} n_j da$ to the two surfaces yields the same result as in part (a).

PROBLEM 9.13 (Continued)

Part c

The wave equation holds in the rod for $\delta(x,t)$. Assuming $\delta = \mathrm{Re}[\hat{\delta}(x)e^{j\omega t}]$, we have $\hat{\delta}(x) = C_1 \sin \beta x + C_2 \cos \beta x$, where $\beta = \omega\sqrt{\rho/E}$. At $x = 0$, $f_\ell = -T(0,t)(\pi R^2)$ which yields

$$\hat{T}(0) = \frac{2\mu_o N^2 I_o \hat{I}}{Rg} = C_I \hat{I},$$

which in turn implies $C_1 = \frac{C_I \hat{I}}{E\beta}$. At $x = \ell$, $f_r = T(L,t)(\pi R^2)$, which will yield C_2.

The only other relation we need is the electrical circuit equation, which we can find from

$$\hat{v}_{out} = \frac{d\lambda_r}{dt}$$

$$C_2 = \frac{C_I}{E\beta \sin\beta L}\left(\hat{I}\cos\beta L + G\hat{v}_{out}\right)$$

to be

$$\hat{v}_{out} = \frac{I_o L_1 \, j\omega \, \hat{\delta}(L)}{a(1+j\, GL_1\omega)} \tag{e}$$

where $L_1 = \mu_o N^2 (2\pi Ra)/g$.

Finally we can write $G(\omega)$ as

$$G(\omega) = \frac{\hat{v}_{out}}{\hat{I}} = \frac{j\omega I_o L_1 C_I}{aE\beta \sin\beta L(1+jGL_1\omega) - j\omega G C_I I_o L_1 \cos\beta\ell} \tag{f}$$

Part d

If $G << \frac{1}{L_1\omega}$ so that the self inductance of the output transducer is negligible and the system is matched so that $a\sqrt{E\rho} = G\, C_I\, I_o L_1$ we have

$$\frac{\hat{v}_{out}}{\hat{I}} = \frac{j I_o L_1 C_I}{a\sqrt{E\rho}\,[\sin\beta L - j\cos\beta L]} \tag{g}$$

and

$$\left|\frac{\hat{v}_{out}}{\hat{I}}\right| = \frac{I_o\, L_1\, C_I}{a\sqrt{E\rho}} \tag{h}$$

PROBLEM 9.14

Part a

With no perturbations and no volume force in the rod we know that the stress, $T(x_1)$, will be constant. At $x_1 = 0$,

$$0 = -\,AT(x_1 = 0) + f^e \tag{a}$$

where, using the Maxwell stress tensor, $f^e = \frac{\varepsilon_o V_o^2 A_1}{2d^2}$. Hence,

PROBLEM 9.14 (Continued)

$$T(x_1) = \frac{\varepsilon_o V_o^2 A_1}{2Ad^2} . \qquad (b)$$

Part b

The velocity of the wave will be $v_p = \sqrt{E/\rho}$ and the transit time will be $t_d = L/v_p$.

Using Table 9.1 we have

$$t_d = \frac{1}{5100} = 1.96 \times 10^{-4} \text{ sec.}$$

Part c

This part is similar to Prob. 9.11, where our condition for no reflection is

$$f^e_{a.c.}(t) = - A\sqrt{\rho E}\, \frac{\partial\delta}{\partial t}\,(0,t) \qquad (c)$$

Using the Maxwell stress tensor

$$f^e = \frac{\varepsilon_o A_1 v^2}{2d^2} \cong \frac{\varepsilon_o A_1 V_o^2}{2d^2} + \frac{\varepsilon_o A_1 V_o}{d^2}\, v'$$

where $v = v' + V_o$. Here, we ignore the effect on f^e of the change in d resulting from the motion of the plate.

Writing the circuit equation we have

$$iR + v = V_o = R\,\frac{dq}{dt} + v = R\left(C\,\frac{dv}{dt} + v\,\frac{dC}{dt}\right) \qquad (d)$$

The capacitance C is

$$\frac{\varepsilon_o A_1}{d-\delta(0,t)} \simeq \frac{\varepsilon_o A_1}{d} + \frac{\varepsilon_o A_1}{d^2}\,\delta(0,t) \qquad (e)$$

Our equation becomes

$$0 = v' + R\,\frac{\varepsilon_o A_1}{d}\,\frac{dv'}{dt} + RV_o\,\frac{\varepsilon_o A_1}{d^2}\,\frac{\partial\delta}{\partial t}\,(0,t) \qquad (f)$$

and since

$$v' \gg \frac{\varepsilon_o A_1 R}{d}\;\frac{dv'}{dt},$$

we have

PROBLEM 9.14 (continued)

$$v' = -\frac{RV_o\varepsilon_o A_1}{d^2}\frac{\partial\delta}{\partial t}(0,t) \tag{g}$$

Now we can use this result to write $f^e_{a.c.} = \varepsilon_o A_1 V_o v'/d^2$, and the condition that this force take the form of (c) requires

$$A\sqrt{\rho E}\, d^4 = RV_o^2\,\varepsilon_o^2 A_1^2\ , \tag{h}$$

or equivalently

$$R = \frac{A\sqrt{\rho E}\ d^4}{\varepsilon_o^2 A_1^2 V_o^2} \tag{i}$$

PROBLEM 9.15

Part a

We have from the problem statement

$$\psi(z+\Delta z) - \psi(z) = \beta\tau\Delta z\ .$$

If we take the limit $\Delta z \to 0$, then we obtain

$$\tau = \frac{1}{\beta}\frac{\partial\psi}{\partial z}\ .$$

Part b

We can write the equation of motion directly as

$$(J\Delta z)\frac{\partial^2\psi}{\partial t^2} = \tau(z+\Delta z,t) - \tau(z,t).$$

Dividing by Δz we have

$$J\frac{\partial^2\psi}{\partial t^2} = \frac{\tau(z+\Delta z,t)-\tau(z,t)}{\Delta z}$$

Taking the limit $\Delta z \to 0$ we obtain

$$J\frac{\partial^2\psi}{\partial t^2} = \frac{\partial\tau}{\partial z}$$

Part c

Substituting the result of part (a) into the result of part (b) we get

$$J\beta\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial z^2}\ .$$

PROBLEM 9.16

Part a

We seek to write Newton's law for motions in the z direction of a slice of the material having x thickness dx. In our situation the mass is $\rho adzdx$, where the acceleration is $\partial^2\delta_z/\partial t^2$. The net force due to the stress is

$$F_z = [T_{zx}(x+dx) - T_{zx}(x)]a\ dz \tag{a}$$

and

$$\rho \frac{\partial^2\delta_z}{\partial t^2} dx\ adz = [T_{zx}\ (x+dx) - T_{zx}(x)]a\ dz \tag{b}$$

Finally, in the limit $dx \to 0$ we have

$$\rho \frac{\partial^2\delta_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} \tag{c}$$

Part b

The shear strain, e_{zx}, is defined so that it is proportional to $\delta_z(x+dx) - \delta_z(x)$ normalized to the distance between points dx. If $T_{zx}=2G\ e_{zx}$ then in the limit $dx \to 0$ $T_{zx} = G\ \partial\delta_z/\partial x$ if we define

$$e_{zx} = \frac{1}{2}\frac{\partial\delta_z}{\partial x}\ . \tag{d}$$

The 1/2 is included to subtract out rigid body rotation, a point that is important in dealing with three-dimensional motions (see Chap. 11, Sec. 11.2.1a).

Part c

From part (a),

$$\rho \frac{\partial^2\delta_z}{\partial t^2} = \frac{\partial T_{zx}}{\partial x} \tag{e}$$

Using the result of part (b) we have

$$\rho \frac{\partial^2\delta_z}{\partial t^2} = G \frac{\partial^2\delta_z}{\partial x^2}\ ; \tag{f}$$

the wave equation for shear waves with the propagational velocity

$$v_p = \sqrt{\frac{G}{\rho}}\ .$$

PROBLEM 9.17

Part a

Conservation of mass implies: net mass out per unit time = time rate of decrease of stored mass

$$[\rho v + \frac{\partial(\rho v)}{\partial x} \Delta x]A - (\rho v)A = -\frac{\partial}{\partial t}[\rho(\Delta x)A] \qquad (a)$$

As $\Delta x \to 0$, we have

$$\frac{\partial}{\partial x}(\rho v) + \frac{\partial \rho}{\partial t} = 0 \qquad (b)$$

If we write $\rho = \rho_o + \rho'(x,t)$ and $v = v(x,t)$ then we obtain by substitution

$$\rho_o \frac{\partial v}{\partial x} + \frac{\partial(\rho' v)}{\partial x} = -\frac{\partial \rho'}{\partial t} \qquad (c)$$

Retaining only first-order terms we have

$$\rho_o \frac{\partial v}{\partial x} = -\frac{\partial \rho'}{\partial t} \qquad (d)$$

as desired.

Part b

Conservation of momentum implies:

time rate of increase of stored momentum = net momentum in per unit time + externally applied force

$$\frac{\partial}{\partial t}(\rho v \Delta x A) = -[\rho v^2 + \frac{\partial(\rho v^2)}{\partial x}\Delta x]A + (\rho v^2)A + pA - (p + \frac{\partial p}{\partial x}\Delta x)A \qquad (e)$$

as $\Delta x \to 0$, we have

$$\frac{\partial(\rho v)}{\partial t} = -\frac{\partial(\rho v^2)}{\partial x} - \frac{\partial p}{\partial x} \qquad (f)$$

Expanding we have

$$\rho(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}) + \underbrace{v\left(\frac{\partial(\rho v)}{\partial x} + \frac{\partial \rho}{\partial t}\right)} = -\frac{\partial p}{\partial x} \qquad (g)$$

this term is zero by conservation of mass

PROBLEM 9.17 (continued)

Finally we have

$$\rho\left(\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial x} \tag{h}$$

Substituting the perturbation quantities and retaining only the first order terms we obtain

$$\rho_o\,\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} \tag{i}$$

Part c

In terms of perturbation quantities we can write

$$p' = a^2\,\rho' \tag{j}$$

where

$$a^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\rho_o} .$$

Substitution for p' yields the two equations

$$\rho_o\,\frac{\partial v}{\partial x} = -\frac{\partial \rho'}{\partial t} \tag{k}$$

and

$$-a^2\,\frac{\partial \rho'}{\partial x} = \rho_o\,\frac{\partial v}{\partial t}. \tag{ℓ}$$

Combining we obtain

$$\frac{\partial^2 v}{\partial t^2} = a^2\,\frac{\partial^2 v}{\partial x^2} \quad \text{(scalar wave equation)} \tag{m}$$

Part d

If we substitute $v = \text{Re}[\hat{v}(x)e^{j\omega t}]$ in the above equation we obtain

$$\frac{d^2\hat{v}(x)}{dx^2} + \frac{\omega^2}{a^2}\,\hat{v}(x) = 0 \tag{n}$$

which has solutions of the form

$$\hat{v}(x) = C_1\,\sin\left(\frac{\omega}{a}\,x\right) + C_2\,\cos\left(\frac{\omega}{a}\,x\right). \tag{o}$$

PROBLEM 9.17 (continued)

A rigid wall at x = 0 imples that $\hat{v}(x=0) = 0$. The drive at $x = \ell$ and the equations of part (c) imply that

$$\frac{d\hat{v}}{dx} = -\frac{j\omega\hat{p}_o}{a^2\rho_o} \tag{p}$$

at $x = \ell$.

The solution for $\hat{v}$ is

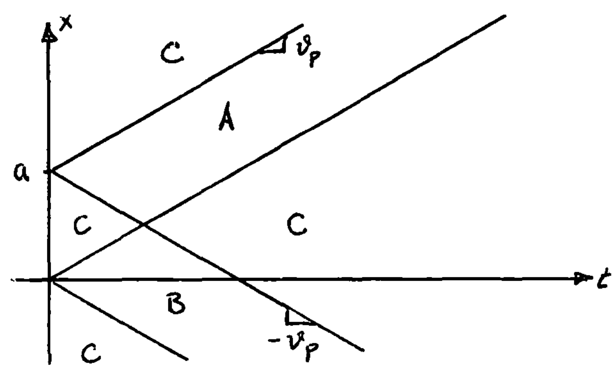
$$\hat{v}(x) = -\frac{j\hat{p}_o}{a\rho_o}\frac{\sin(\frac{\omega}{a}x)}{\cos(\frac{\omega}{a}\ell)} \tag{q}$$

and we can now obtain v(x,t): for $\hat{p}_o$ real

$$v(x,t) = \frac{\hat{p}_o}{a\rho_o}\frac{\sin(\frac{\omega}{a}x)}{\cos(\frac{\omega}{a}\ell)}\sin\omega t. \tag{r}$$

PROBLEM 9.18

We can calculate the values of $d\delta+/d\alpha$ and $d\delta-/d\beta$ for three regions of the x-t plane as defined below.



Referring to equations from text, 9.1.23 and 9.1.24, 9.1.27 and 9.1.28:

Region A:

$$\frac{d\delta+}{d\alpha} = -\frac{1}{2}\frac{v_m}{v_p},\ \frac{d\delta-}{d\beta} = 0 \tag{a}$$

and

$$T = -\frac{E}{2}\frac{v_m}{v_p} \tag{b}$$

PROBLEM 9.18 (continued)

Region B:

$$\frac{d\delta+}{d\alpha} = 0, \ \frac{d\delta-}{d\beta} = \frac{1}{2}\frac{v_m}{v_p} \qquad (c)$$

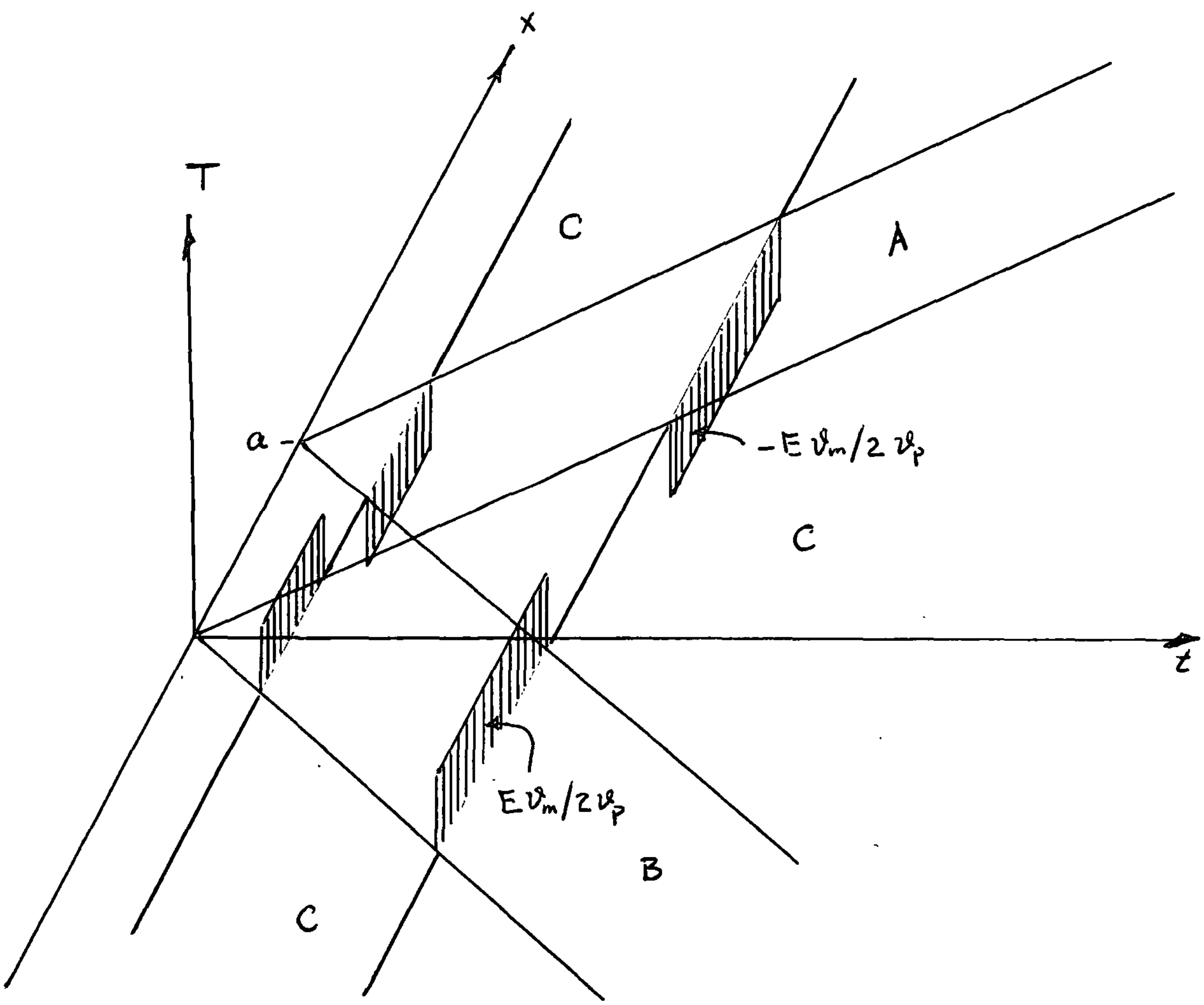
and

$$T = \frac{E}{2}\frac{v_m}{v_p} \qquad (d)$$

Region C:

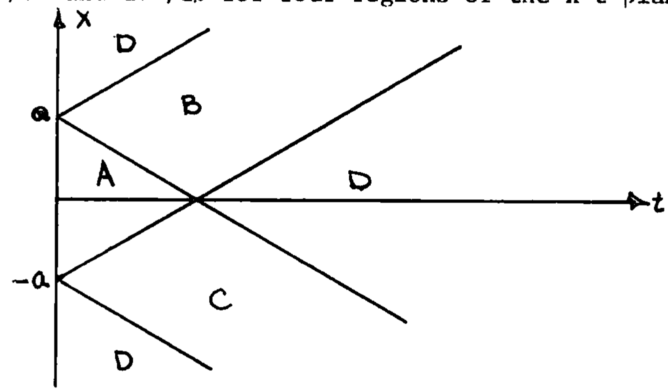
$$\frac{d\delta+}{d\alpha} = \frac{d\delta-}{d\beta} = 0 \text{ and } T = 0. \qquad (e)$$

Plotting T(x,t) in the x-t plane we have



PROBLEM 9.19

We can find $d\delta+/d\alpha$ and $d\delta-/d\beta$ for four regions of the x-t plane:



Referring to equations from the text 9.1.23, 9.1.24 and 9.1.27, 9.1.28 we have,

Region A:

$$\frac{d\delta+}{d\alpha} = \frac{1}{2}\,\frac{T(\alpha)}{E},\ \frac{d\delta-}{d\beta} = \frac{1}{2}\,\frac{T(\beta)}{E} \tag{a}$$

Region B:

$$\frac{d\delta+}{d\alpha} = \frac{1}{2}\,\frac{T(\alpha)}{E},\ \frac{d\delta-}{d\beta} = 0 \tag{b}$$

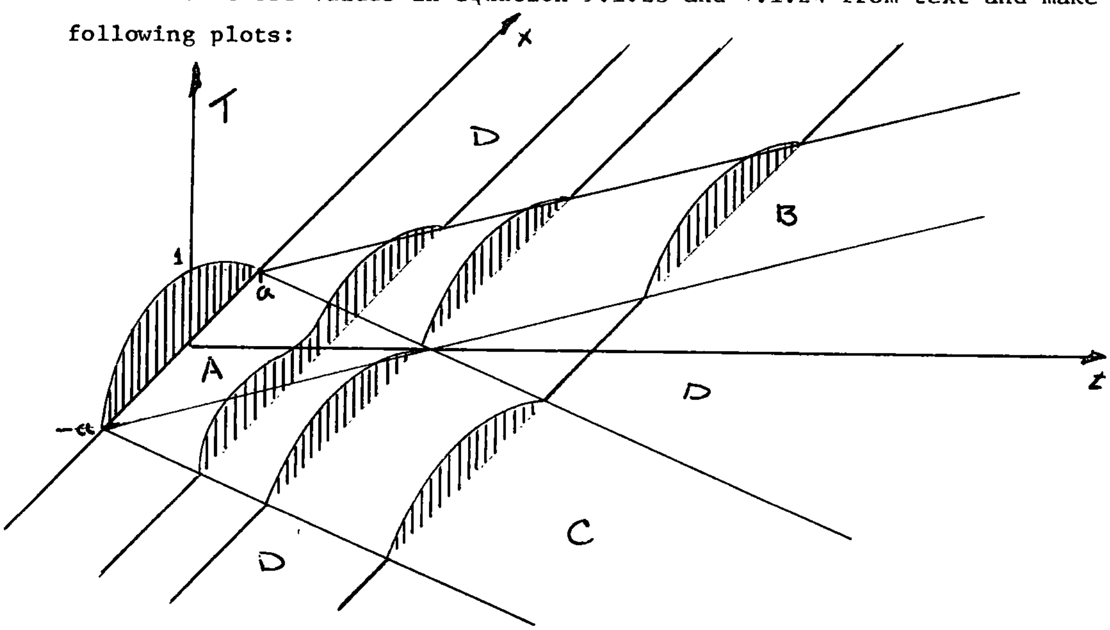
Region C:

$$\frac{d\delta+}{d\alpha} = 0,\ \frac{d\delta-}{d\beta} = \frac{1}{2}\,\frac{T(\alpha)}{E} \tag{c}$$

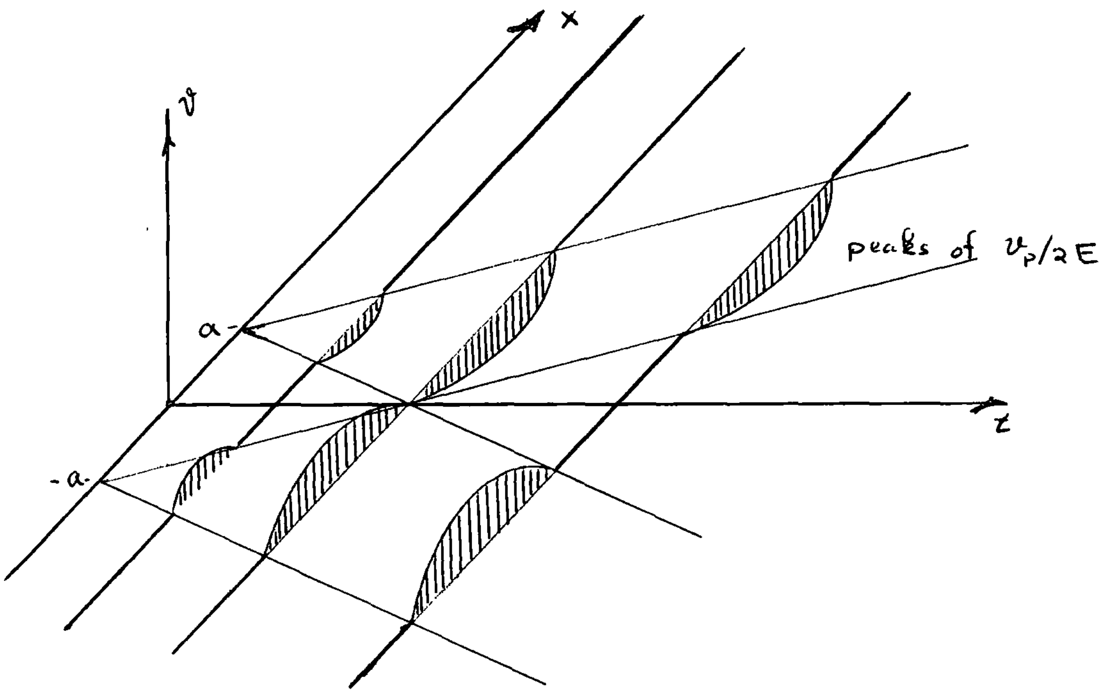
Region D:

$$\frac{d\delta+}{d\alpha} = \frac{d\delta-}{d\beta} = 0 \tag{d}$$

We can use these values in equation 9.1.23 and 9.1.24 from text and make the following plots:



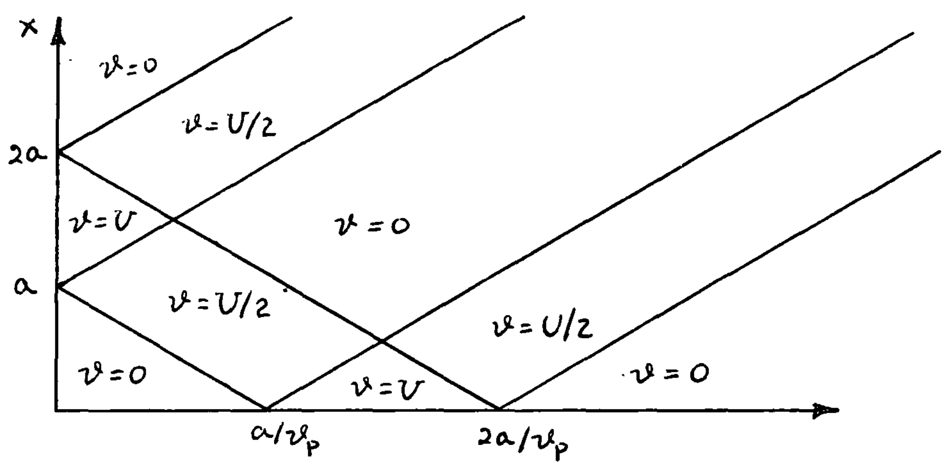
PROBLEM 9.19 (continued)



PROBLEM 9.20

Part a

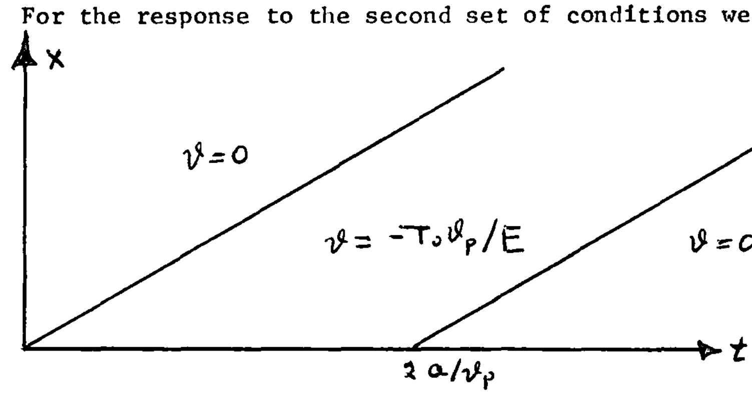
The free end at x = 0 implies that T(0,t) = 0 and using equations 9.1.23 through 9.1.26 we can easily find that velocity pulses "bounce off" x = 0 boundary with the same sign and magnitude. For the x-t plane we can indicate the values for v(x,t):



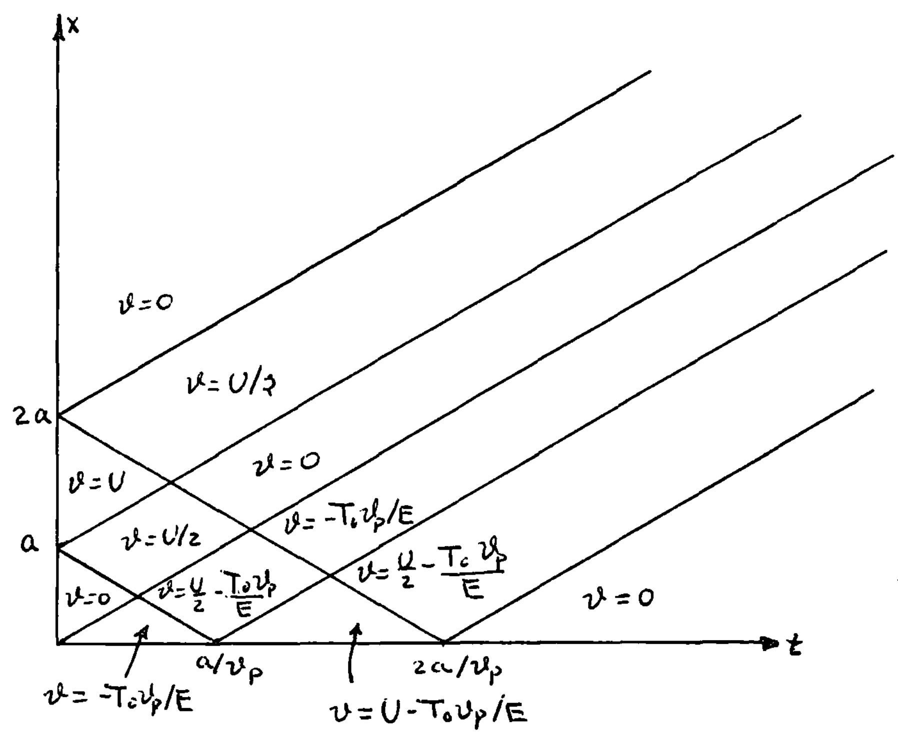
PROBLEM 9.20 (continued)

Part b

We can make use of part (a) if we use superposition. Consider the superposition of boundary and initial conditions; a free end, T(0,t) = 0 with the initial conditions in part (a) and the T(0,t) as shown in Fig. 9.P20b with initial conditions on T and v zero. Since the system is linear, we can add the velocities that result from the two situations and thus have the net velocity. For the response to the second set of conditions we have



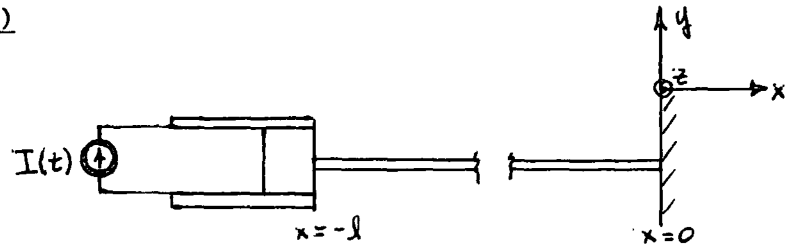
Add this velocity set to the set in part (a) and we obtain:



PROBLEM 9.21

Part a

With the current returned on the inside surface the field in the air gap is $H_z = -\frac{I(t)}{D}$



and the force per unit area acting on the inside surface is

$$T_x = \frac{-1}{2}\mu_o \frac{I^2}{D^2} \tag{a}$$

The force is $f_x = -T_x aD = \frac{1}{2}\frac{\mu_o a}{D} I^2(t)$ and the boundary condition at $x = -\ell$ is

$$M \frac{\partial^2 \delta}{\partial t^2}(-\ell,t) = \frac{1}{2}\frac{\mu_o a}{D} I^2(t) + AT(-\ell,t) \tag{b}$$

Part b

The current will flow on the surface when the time τ is much shorter than the characteristic diffusion time τ_d over the length b:

$$\tau_d >> \tau \quad \text{or } \mu_o \sigma b^2 >> \tau \tag{c}$$

Part c

In order to ignore the mass M, the inertial term must be small compared to $AT(-\ell,t)$. For $t < \tau$, $\delta_- = 0$ on the rod, and from Eqs. 9.1.23 and 9.1.24,

$$T(-\ell,t) = -\frac{E}{v_p}\frac{\partial \delta}{\partial t}(-\ell,t) \tag{d}$$

Thus

$$\left| M \frac{\partial^2 \delta}{\partial t^2}(-\ell,t) \right| << \left| \frac{AE}{v_p}\frac{\partial \delta}{\partial t} \right| \tag{e}$$

or

$$M << AE\, \tau/v_p \tag{f}$$

Our boundary condition in part (a) now becomes:

$$0 = \frac{1}{2}\frac{\mu_o a}{D} I^2(t) + AT(-\ell,t) \tag{g}$$

Since there is a fixed end at x = 0 we know that a stress wave traveling in the +x direction will reflect at x = 0 with the same wave returning in the -x direction. To satisfy the condition $v(0,t) = 0$, Eq. 9.1.23 shows that $d\delta_+/d\alpha = d\delta_-/d\beta$ at x = 0. Thus, from Eq. 9.1.24, the stress is twice that

PROBLEM 9.21 (continued)

initiated at the left end

$$T_r = \frac{\mu_o a}{DA} I_o^2 \tag{h}$$

PROBLEM 9.22

Part a

We have W = W' and U = C + U' where W' and U' are perturbations from equilibrium. Rewriting the equations we have

$$\frac{\partial W'}{\partial t} + (W')\frac{\partial W'}{\partial x} + \frac{\partial U'}{\partial x} + \frac{K}{(C+U')^3}\frac{\partial U'}{\partial x} = 0 \tag{a}$$

and

$$\frac{\partial U'}{\partial t} + (C+U')\frac{\partial W'}{\partial x} + (W')\frac{\partial U'}{\partial x} = 0 \tag{b}$$

Neglecting all second-order perturbation terms we have

$$\frac{\partial W'}{\partial t} + (1 + \frac{K}{C^3})\frac{\partial U'}{\partial x'} = 0 \tag{c}$$

$$\frac{\partial U'}{\partial t} + (C)\frac{\partial W'}{\partial x} = 0 \tag{d}$$

Part b

Multiplying the above two equations by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively, we have

$$\frac{\partial^2 W'}{\partial t^2} + (1 + \frac{K}{C^3})\frac{\partial^2 U'}{\partial x \partial t} = 0 \tag{e}$$

and

$$\frac{\partial^2 U'}{\partial x \partial t} + (C)\frac{\partial^2 W'}{\partial x^2} = 0 \tag{f}$$

Eliminating U' we obtain

$$\frac{\partial^2 W'}{\partial t^2} = C(1 + \frac{K}{C^3})\frac{\partial^2 W'}{\partial x^2} \tag{g}$$

which is the familiar wave equation with wave velocity $v_p = \sqrt{C(1 + \frac{K}{C^3})}$.
We can write the solution as $W' = \text{Re}[\hat{W}(x)e^{j\omega t}]$ where

$$\hat{W}(x) = C_1 \sin \beta x + C_2 \cos \beta x \tag{h}$$

with $\beta = \omega/v_p$.

At x = 0, W = W' = 0 and hence $C_2 = 0$. At x = − L, $W = W' = W_o \cos \omega t$, or equivalently $\hat{W}(-L) = W_o$, hence $C_1 = -W_o/\sin\beta L$. Upon substitution we find that

PROBLEM 9.22 (continued)

the solution is

$$W = W' = -\frac{W_o \sin \beta x}{\sin \beta L} \cos \omega t \, . \tag{i}$$

PROBLEM 9.23

Part a

This part is similar to Prob. 9.24 with two simplifications:

$V_o = 0$ and

the mass is M/unit width (M_w) instead of 2M. The two separate relations yielding the natural frequencies are

$$\sin\left(\omega L \sqrt{\frac{\sigma_m}{S}}\right) = 0 \tag{a}$$

and

$$\frac{2\sigma_m}{M\omega\sqrt{\frac{\sigma_m}{S}}} = \tan\left(\omega L \sqrt{\frac{\sigma_m}{S}}\right) \tag{b}$$

(a) yields $\omega L \sqrt{\sigma_m/S} = n\pi$ where $n = 1, 2, \ldots$ and corresponds to solutions which are "odd", or $\xi(x) = -\xi(-x)$. (b) can be solved graphically and corresponds to solutions which are "even", or $\xi(x) = \xi(-x)$.

Part b

The effect of raising M is to reduce the eigenfrequencies of the "even" modes. The "odd" solutions predicted by (a) are independent of the mass M. This is physically reasonable since there is a node at the mass, and since the mass doesn't move there is no inertial force. For the "even" solutions predicted by (b), we notice that if M = 0 we have essentially the natural frequencies of a membrane of length 2L. As $M \to \infty$, the system responds like two different membranes of length L. The infinite mass acts like a rigid boundary.

PROBLEM 9.24

Part a

We can use the Maxwell Stress Tensor to find the forces of electric origin. If f_u^e corresponds to the force due to the upper electrode and f_ℓ^e corresponds to the force due to the lower electrode, then we have:

$$\vec{f}_u^e(t) = \frac{\varepsilon_o V_o^2 A}{2[d-\xi(0,t)]^2} \, \vec{i}_y \tag{a}$$

PROBLEM 9.24 (Continued)

$$\vec{f}_\ell^e\ (t) = -\frac{\varepsilon_o V_o^2 A}{2[d+\xi(0,t)]^2}\vec{i}_y \tag{b}$$

Our equation for the membranes is $\sigma_m\ \partial^2\xi/\partial t^2 = S\ \frac{\partial^2\xi}{\partial x^2}$ and if we assume $\xi=\mathrm{Re}[\hat{\xi}(x)e^{j\omega t}]$, then we can write

$$\hat{\xi}(x) = C_1 \sin\beta x + C_2 \cos\beta x \tag{c}$$

for $x > 0$ and

$$\hat{\xi}(x) = C_3 \sin\beta x + C_4 \cos\beta x \tag{d}$$

for $x < 0$ where $\beta = \omega\sqrt{\sigma_m/S}$.

Our boundary condition will yield the four constants. We have

$$\begin{aligned} &\hat{\xi}(x = -L) = 0 \\ &\hat{\xi}(x = L) = 0 \\ &\hat{\xi}(x = 0^+) = \hat{\xi}(x = 0^-) \end{aligned} \tag{e}$$

and

$$2M\frac{\partial^2\xi}{\partial t^2}(0,t) = Sw\left[\frac{\partial\xi}{\partial x}(0^+) - \frac{\partial\xi}{\partial x}(0^-)\right] + f_u^e(t) + f_\ell^e(t) \tag{f}$$

which reduces to

$$-2M\omega^2\ \hat{\xi}(0) = Sw\left[\frac{d\hat{\xi}}{dx}(0^+) - \frac{d\hat{\xi}}{dx}(0^-)\right] + \frac{2\varepsilon_o V_o^2 A}{d^3}\hat{\xi}(0) \tag{g}$$

after we linearize $[f_u^e(t) + f_\ell^e(t)]$. Substituting, we immediately find $C_2 = C_4$. Writing the remaining equations we have

$$0 = -C_3 \sin\beta L + C_2 \cos\beta L \tag{h}$$

$$0 = C_1 \sin\beta L + C_2 \cos\beta L \tag{i}$$

$$0 = Sw\beta\ C_1 + \left[\frac{2\varepsilon_o V_o^2 A}{d^3} + 2M\omega^2\right] C_2 - Sw\beta\ C_3 \tag{j}$$

If we eliminate the constants by setting the determinant of the coefficients C_1, C_2, and C_3 equal to zero, we obtain two separate relations:

$$\sin\beta L = 0 \text{ and } \frac{Sw\beta}{\frac{\varepsilon_o V_o^2 A}{d^3} + M\omega^2} = \tan\beta L. \tag{k}$$

PROBLEM 9.24 (continued)

Substituting for β we have

$$\sin\left(\omega L\sqrt{\frac{\sigma_m}{S}}\right) = 0 \text{ and } \frac{Sw\omega\sqrt{\frac{\sigma_m}{S}}}{\frac{\varepsilon_o V_o^2 A}{d^3} + M\omega^2} = \tan \omega L \sqrt{\frac{\sigma_m}{S}} \qquad (\ell)$$

The first relation implies that $\omega L\sqrt{\sigma_m/S} = n\pi$ where $n = 1, 2, \ldots$. The second relation can be solved graphically.

Part b

As V_o is increased from $V_o = 0$, the lowest natural frequency decreases. When V_o approaches the value

$$\sqrt{\frac{Swd^3}{\varepsilon_o AL}},$$

the lowest natural frequency approaches zero; as V_o is further increased, there will be an imaginary solution for ω and the system will be unstable.

PROBLEM 9.25

Part a

The force of the lower plunger is $f_\ell^m = \frac{\partial W'}{\partial \xi} = \frac{L_o}{2a} i_\ell^2$. By symmetry the upper plunger has a force $f_u^m = -(L_o/2a)\, i_u^2$. From the circuit $i_\ell^2 = (I_o + i_1)^2 \simeq I_o^2 + 2I_o i_1$ and $i_u^2 = (I_o - i_1)^2 \simeq I_o^2 - 2I_o i_1$. Hence the total magnetic force is

$$f^m = \frac{2L_o I_o i_1}{a} = \frac{2L_o I_o G}{a} \frac{\partial \xi}{\partial x}(0,t) \qquad (a)$$

Writing the force balance on the tip of the wire at $x = -\ell$ we have

$$f \frac{\partial \xi}{\partial x}(-\ell,t) + \frac{2L_o I_o G}{a}\frac{\partial \xi}{\partial x}(0,t) = 0 \qquad (b)$$

Part b

Away from the ends

$$M \frac{\partial^2 \xi}{\partial t^2} = f \frac{\partial^2 \xi}{\partial x^2} \qquad (c)$$

and if $\xi = \text{Re}[\hat{\xi}(x)e^{j\omega t}]$ then

$$\hat{\xi}(x) = C_1 \sin \beta x + C_2 \cos \beta x \qquad (d)$$

where $\beta \simeq \omega\sqrt{m/f}$. $\xi(0,t) = 0$ implies that $C_2 = 0$. From part (a) we have

PROBLEM 9.25 (continued)

$$f\,\frac{d\hat{\xi}}{dx}(-\ell) + \frac{2L_oI_oG}{a}\,\frac{d\hat{\xi}}{dx}(0) = 0. \qquad \text{(e)}$$

Upon substitution we obtain

$$f\beta\,C_1\cos\beta\ell + \frac{2L_oI_o}{a}\,G\beta C_1 = 0 \qquad \text{(f)}$$

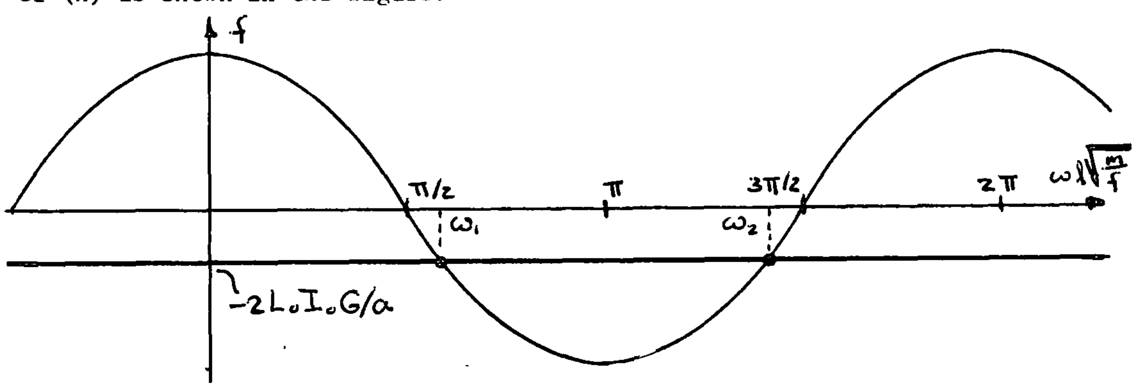
Since C_1 must be finite for a finite response, we have

$$f\beta\cos\beta\ell + \frac{2L_oI_o}{a}\,G\beta = 0, \qquad \text{(g)}$$

or

$$f\cos\omega\ell\sqrt{\frac{m}{f}} + \frac{2L_oI_oG}{a} = 0 \qquad \text{(h)}$$

(We have ruled out one solution, because it is trivial.) A graphical solution of (h) is shown in the figure.



Part c

If G = 0, then

$$\omega\ell\sqrt{\frac{m}{f}} = \left(\frac{2n+1}{2}\right)\pi \qquad \text{(i)}$$

with n = 0, 1, 2,...

Part d

From the figure, ω_1 increases toward $\omega_1\ell\sqrt{m/f} = \pi$ and ω_2 decreases toward the same value. They come together at $G = af/2L_oI_o$ and seemingly disappear if $G > \frac{af}{2L_oI_o}$.

PROBLEM 9.25 (continued)

Part e

If $|G| > \frac{af}{2L_oI_o}$, then (h) has imaginary solutions for ω, hence the system will be unstable.

PROBLEM 9.26

Part a

First of all we notice that $y(t) = \xi(-L,t)$. For the membrane $\sigma_m \frac{\partial^2\xi}{\partial t^2} = S\frac{\partial^2\xi}{\partial x^2}$ and if $\xi = \text{Re}[\hat{\xi}(x)e^{j\omega t}]$ then $\hat{\xi}(x) = C_1\sin\beta x + C_2\cos\beta x$ where $\beta = \omega\sqrt{\sigma_m/S}$. At $x = 0$, $\hat{\xi}(x=0) = 0$ and therefore $C_2 = 0$. At $x = -L$, we can write the boundary condition

$$M\frac{\partial^2\xi}{\partial t^2}(-L,t) = SD\frac{\partial\xi}{\partial x}(-L,t) + f_y^m(t) \qquad \text{(a)}$$

We can find $f_y^e(t)$ using Ampere's Law and the Maxwell stress tensor

$$f_y^e(t) = \frac{\mu_o AN^2}{2}\left[\frac{(I_o + I(t))^2}{(d-D-\xi(-L,t))^2} - \frac{(I_o - I(t))^2}{(d-D+\xi(-L,t))^2}\right] \qquad \text{(b)}$$

Since $I_o \gg I(t)$ and $(d-D) \gg \xi(-L,t)$ then we can linearize:

$$f_y^e(t) \simeq 2N^2A\mu_o\left[\frac{I_o}{(d-D)^2}I(t) + \frac{I_o^2}{(d-D)^3}\xi(-L,t)\right] \qquad \text{(c)}$$

Substitution of (c) into (a) and definition of $C_I \equiv \frac{2N^2A\mu_oI_o}{(d-D)^2}$ and $C_y \equiv \frac{2N^2A\mu_oI_o^2}{(d-D)^3}$ gives

$$M\frac{\partial^2\xi}{\partial t^2}(-L,t) = SD\frac{\partial\xi}{\partial x}(-L,t) + C_I I(t) + C_y\,\xi(-L,t) \qquad \text{(d)}$$

or in complex form,

$$-M\omega^2\,\hat{\xi}(-L) = SD\frac{\partial\hat{\xi}}{\partial x}(-L) + C_I\hat{I} + C_y\,\hat{\xi}(-L) \qquad \text{(e)}$$

After solving for C_1, we can write

$$\hat{\xi}(x) = \frac{C_I\sin\beta x\,\hat{I}}{(M\omega^2+C_y)\sin\beta L - SD\beta\cos\beta L} \qquad \text{(f)}$$

or finally

PROBLEM 9.26 (continued)

$$\hat{y} = \frac{C_I \sin \beta L \, \hat{I}}{SD\beta \cos \beta L - (M\omega^2 + C_y)\sin \beta L} \qquad (g)$$

where $y(t) = Re[\hat{y}\, e^{j\omega t}]$.

Part b

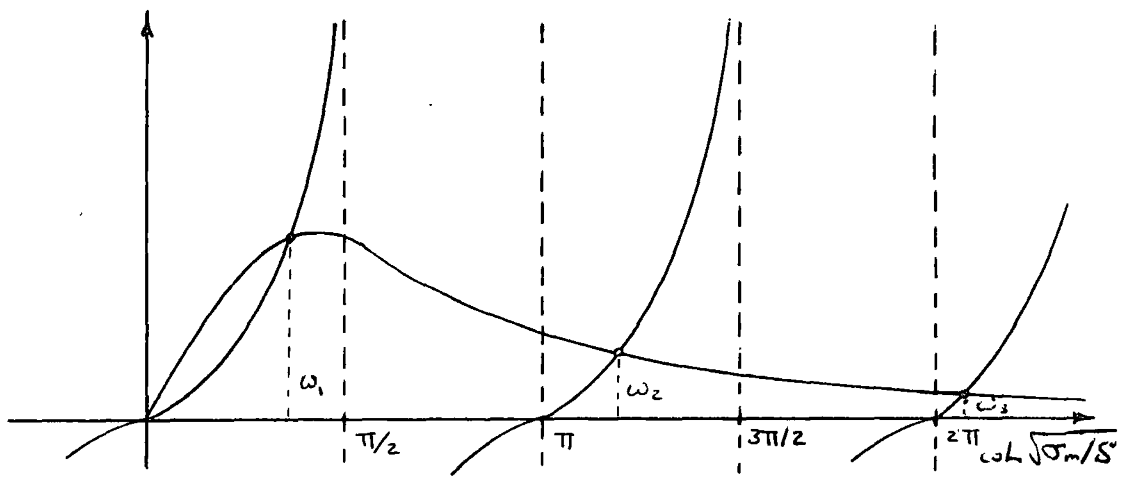
To find the resonance frequencies we look at the poles of $\hat{y}/\hat{I}$. This amounts to finding the zeros of the denominator of $\hat{y}/\hat{I}$. We have

$$SD\omega \sqrt{\frac{\sigma_m}{S}} \cos \omega L \sqrt{\frac{\sigma_m}{S}} = [M\omega^2 + C_y] \sin \omega L \sqrt{\frac{\sigma_m}{S}} \qquad (h)$$

or

$$\frac{SD\omega \sqrt{\frac{\sigma_m}{S}}}{M\omega^2 + C_y} = \tan(\omega L \sqrt{\frac{\sigma_m}{S}}) \qquad (i)$$

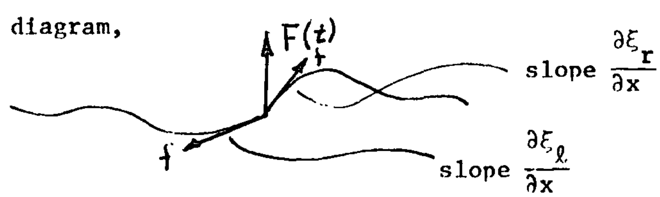
We can represent the solution graphically:



PROBLEM 9.27

Part a

The boundary condition may be obtained by applying force equilibrium using the following diagram,



PROBLEM 9.27 (continued)

thus

$$F(t) = f\left[\frac{\partial \xi_\ell}{\partial x}(0^-) - \frac{\partial \xi_r}{\partial x}(0^+)\right] \qquad \text{(a)}$$

Part b

For the odd solution, $\xi_\ell(x,t) = -\xi_r(-x,t)$ and it follows that $\frac{\partial \xi_\ell}{\partial x} - \frac{\partial \xi_r}{\partial x} = 0$. This implies that the odd solution is not excited by the force F(t).

Part c

For the even solution, $\xi_\ell(x,t) = \xi_r(-x,t)$, we have $\frac{\partial \xi_\ell}{\partial x} = -\frac{\partial \xi_r}{\partial x}$ and the boundary condition (a) from part (a) becomes

$$F(t) = -2f\frac{\partial \xi_r}{\partial x} \text{ at } x = 0 \qquad \text{(b)}$$

For $0<x<\ell$ we have

$$m\frac{\partial^2 \xi}{\partial t^2} = f\frac{\partial^2 \xi}{\partial x^2} \qquad \text{(c)}$$

with $\xi(x,t) = 0$ at $x = \ell$.

For $t < 0$, this reducss to

$$\frac{\partial^2 \xi}{\partial x^2} = 0 \qquad \text{(d)}$$

and we obtain

$$\xi(x) = \frac{F_o \ell}{2f}\left(1 - \frac{x}{\ell}\right) \text{ for } 0<x<\ell \qquad \text{(e)}$$

Part d

We now have a combined transient and driven response, as discussed in Sec. 9.2.1. By contrast with the developments of that section, we now have a boundary condition at $x = 0$ on the slope $\partial \xi_r/\partial x$ (see (b) of part (c)). Our program is: ($\xi \equiv \xi_r$ in the following)

i. Find the driven sinusoidal steady-state response. This satisfies the boundary conditions:

$$F_o \cos \omega t = -2f\frac{\partial \xi}{\partial x}(0,t) \qquad \text{(f)}$$

$$\xi(\ell,t) = 0 \qquad \text{(g)}$$

ii. Find normal modes, which satisfy homogeneous boundary conditions;

$$\frac{\partial \xi}{\partial x}(0,t) = 0 \qquad \text{(h)}$$

$$\xi(\ell,t) = 0 \qquad \text{(i)}$$

The sum of these modes takes the form of a Fourier series.

PROBLEM 9.27 (continued)

iii. Superimpose (i) and (ii) and use the initial conditions found in parts (a)-(c) to evaluate the arbitrary coefficients.

The driven response is of the form

$$\xi = \mathrm{Re}(C_1 \sin \beta x + C_2 \cos \beta x)e^{j\omega t}; \quad \beta = \omega\sqrt{\frac{m}{f}} \tag{j}$$

a linear combination which satisfies (g)

$$\xi = \mathrm{Re} C_3 \sin \beta(x-\ell)e^{j\omega t} \tag{k}$$

while (f) evaluates C_3 and the driven response is

$$\xi = -\mathrm{Re}\frac{F_o \sin \beta(x-\ell)e^{j\omega t}}{2f\beta \cos \beta\ell} \tag{ℓ}$$

The normal modes are in this lossless case the resonances of the driven response and occur as $\cos \beta\ell = 0$. Thus

$$\omega_n \ell\sqrt{\frac{m}{f}} = \left(\frac{2n+1}{2}\right)\pi, \quad n = 0, 1, 2, 3... \tag{m}$$

and the total solution for $0<x<\ell$ is

$$\xi = -\frac{F_o \sin\beta(x-\ell)}{2f\beta \cos \beta\ell}\cos \omega t + \sum_{n=0}^{\infty}[A_n^+ e^{j\omega_n t} + A_n^- e^{-j\omega_n t}]\sin[(\frac{2n+1}{2})\frac{\pi}{\ell}(x-\ell)] \tag{n}$$

The coefficients A_n^+ and A_n^- are evaluated by requiring that

$$\xi(x,0) = \frac{F_o \ell}{2f}(1-\frac{x}{\ell}) = -\frac{F_o \sin\beta(x-\ell)}{2f\beta \cos \beta\ell} + \sum_{n=0}^{\infty}(A_n^+ + A_n^-)\sin[(\frac{2n+1}{2})\frac{\pi}{\ell}(x-\ell)] \tag{o}$$

and

$$\frac{\partial\xi}{\partial t}(x,0) = 0 = \sum_{n=0}^{\infty}[j\omega_n A_n^+ - j\omega_n A_n^-]\sin[(\frac{2n+1}{2})\frac{\pi}{\ell}(x-\ell)] \tag{p}$$

This last condition is satisfied if $A_n^+ = A_n^-$. The A_n^+'s follow from (o) by using the orthogonality of the functions $\sin[(2n+1/2)\frac{\pi}{\ell}(x-\ell)]$ and $\sin[(2m+1\ /2)\frac{\pi}{\ell}(x-\ell)$, $m \neq n$, over the interval ℓ.