PROBLEM 11.1

Part a

We add up all the volume force densities on the elastic material, and with the help of equation 11.1.4, we write Newton's law as

$$\rho \frac{\partial^2 \delta_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} - \rho g \tag{a}$$

where we have taken $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0$. Since this is a static problem, we let $\frac{\partial}{\partial t} = 0$. Thus,

$$\frac{\partial T_{11}}{\partial x_1} = \rho g. \tag{b}$$

From 11.2.32, we obtain

$$T_{11} = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} \tag{c}$$

Therefore

$$(2G + \lambda) \frac{\partial^2 \delta_1}{\partial x_1^2} = \rho g \tag{d}$$

Solving for δ_1, we obtain

$$\delta_1 = \frac{\rho g}{2(2G+\lambda)} x_1^2 + C_1 x + C_2 \tag{e}$$

where C_1 and C_2 are arbitrary constants of integration, which can be evaluated by the boundary conditions

$$\delta_1(0) = 0 \tag{f}$$

and

$$T_{11}(L) = (2G + \lambda) \frac{\partial \delta_1}{\partial x_1} (L) = 0 \tag{g}$$

since $x_1 = L$ is a free surface. Therefore, the solution is

$$\delta_1 = \frac{\rho g\, x_1}{2(2G+\lambda)} [x_1 - 2L]. \tag{f}$$

Part b

Again applying 11.2.32

PROBLEM 11.1 (Continued)

$$T_{11} = (2G+\lambda)\,\frac{\partial \delta_1}{\partial x_1} = \rho g[x_1 - L)$$

$$T_{12} = T_{21} = 0$$

$$T_{13} = T_{31} = 0$$

$$T_{22} = \lambda\,\frac{\partial \delta_1}{\partial x_1} = \frac{\lambda \rho g}{(2G+\lambda)}\,[x_1 - L] \tag{g}$$

$$T_{33} = \lambda\,\frac{\partial \delta_1}{\partial x_1} = \frac{\lambda \rho g}{(2G+\lambda)}\,[x_1 - L]$$

$$T_{32} = T_{23} = 0$$

$$\bar{\bar{T}} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \tag{h}$$

PROBLEM 11.2

Since the electric force only acts on the surface at $x_1 = -L$, the equation of motion for the elastic material $(-L \leq x_1 \leq 0)$ is from Eqs. (11.1.4) and (11.2.32),

$$\rho\,\frac{\partial^2 \delta_1}{\partial t^2} = (2G+\lambda)\,\frac{\partial^2 \delta_1}{\partial x_1^2} \tag{a}$$

The boundary conditions are

$$\delta_1(0,t) = 0$$

and

$$M\,\frac{\partial^2 \delta_1(-L,t)}{\partial t^2} = aD(2G+\lambda)\,\frac{\partial \delta_1}{\partial x_1}(-L,t) + f^e \tag{b}$$

f^e is the electric force in the x_1 direction at $x_1 = -L$, and may be found by using the Maxwell Stress Tensor $T_{ij} = \varepsilon E_i E_j - \frac{1}{2}\delta_{ij}\,\varepsilon E_k E_k$ to be (see Appendix G for discussion of stress tensor),

$$f^e = -\frac{\varepsilon}{2}\,E^2\,aD$$

with

$$E = \frac{V_o + V_1 \cos \omega t}{d + \delta_1(-L,t)} \tag{c}$$

PROBLEM 11.2 (continued)

Expanding f^e to linear terms only, we obtain

$$f^e = -\frac{\varepsilon aD}{2}\left[\frac{V_o^2}{d^2} + \frac{2V_oV_1\cos\omega t}{d^2} - \frac{2V_o^2}{d^3}\delta_1(-L,t)\right] \tag{d}$$

We have neglected all second order products of small quantities.

Because of the constant bias V_o, and the sinusoidal nature of the perturbations, we assume solutions of the form

$$\delta_1(x_1,t) = \delta_1(x_1) + \mathrm{Re}\,\hat{\delta}e^{j(\omega t - kx_1)} \tag{e}$$

where

$$\hat{\delta} << \delta_1(x_1) << L$$

The relationship between ω and k is readily found by substituting (e) into (a), from which we obtain

$$k = \pm\frac{\omega}{v_p} \text{ with } v_p = \sqrt{\frac{2G+\lambda}{\rho}} \tag{f}$$

We first solve for the equilibrium configuration which is time independent. Thus

$$\frac{\partial^2\delta_1(x_1)}{\partial x_1^2} = 0 \tag{g}$$

This implies

$$\delta_1(x_1) = C_1x_1 + C_2$$

Because $\delta_1(0) = 0$, $C_2 = 0$.

From the boundary condition at $x_1 = -L$ ((b) & (d))

$$aD(2G+\lambda)C_1 - \frac{\varepsilon}{2}\frac{V_o^2}{d^2}aD = 0 \tag{h}$$

Therefore

$$\delta_1(x_1) = +\frac{\varepsilon}{2}\frac{V_o^2}{d^2(2G+\lambda)}x_1 \tag{i}$$

Note that $\delta_1(x_1 = -L)$ is negative, as it should be.

For the time varying part of the solution, using (f) and the boundary condition

$$\delta(0,t) = 0$$

PROBLEM 11.2 (continued)

we can let the perturbation δ_1 be of the form

$$\delta_1(x_1,t) = \mathrm{Re}\ \hat{\delta} \sin kx_1\ e^{j\omega t} \tag{j}$$

Substituting this assumed solution into (b) and using (d), we obtain

$$+ Mw^2\ \hat{\delta} \sin kL = aD(2G+\lambda)k\ \hat{\delta} \cos kL - \frac{\varepsilon aDV_oV_1}{d^2} - \frac{\varepsilon aDV_o^2}{d^3}\hat{\delta} \sin kL \tag{k}$$

Solving for $\hat{\delta}$, we have

$$\hat{\delta} = -\frac{\varepsilon aDV_oV_1}{d^2\left[Mw^2 \sin kL - aD(2G+\lambda)k \cos kL + \frac{\varepsilon aDV_o^2}{d^3} \sin kL\right]}$$

Thus, because $\hat{\delta}$ has been shown to be real,

$$\delta_1(-L,t) = -\frac{\varepsilon}{2}\frac{V_o^2\ L}{d^2(2G+\lambda)} - \hat{\delta} \sin kL \cos \omega t \tag{m}$$

Part b

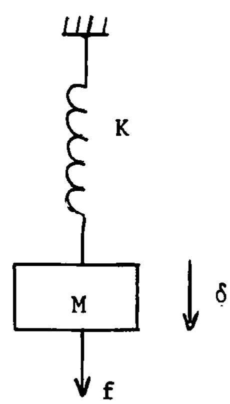
If $k\ell << 1$, we can approximate the sinusoidal part of (m) as

$$\delta_1(-L,t) = \frac{\varepsilon aDV_oV_1 \cos \omega t}{d^2\left[Mw^2 - \frac{aD(2G+\lambda)}{L} + \frac{\varepsilon aDV_o^2}{d^3}\right]} \tag{n}$$

We recognize this as a force-displacement relation for a mass on the end of a spring.

Part c

We thus can model (n) as



PROBLEM 11.2 (Continued)

where

$$f = -\frac{\varepsilon a D V_o V_1 \cos \omega t}{d^2}$$

and

$$K = \frac{aD(2G+\lambda)}{L} - \frac{\varepsilon a D V_o^2}{d^3}$$

We see that the electrical force acts like a negative spring constant.

PROBLEM 11.3

Part a

From (11.1.4), we have the equation of motion in the x_2 direction as

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = \frac{\partial T_{21}}{\partial x_1} \qquad \text{(a)}$$

From(11.2.32),

$$T_{21} = G\left[\frac{\partial \delta_2}{\partial x_1}\right] \qquad \text{(b)}$$

Therefore, substituting (b) into (a), we obtain an equation for δ_2

$$\rho \frac{\partial^2 \delta_2}{\partial t^2} = G\frac{\partial^2 \delta_2}{\partial x_1^2} \qquad \text{(c)}$$

We assume solutions of the form

$$\delta_2 = \text{Re}\ \hat{\delta}_2\ e^{j(\omega t - kx_1)} \qquad \text{(d)}$$

where from (c) we obtain

$$k = \pm \frac{w}{v_p} \qquad v_p{}^2 = \frac{G}{\rho}$$

Thus we let

$$\delta_2 = \text{Re}\left[\delta_a\ e^{j(\omega t - kx_1)} + \delta_b\ e^{j(\omega t + kx_1)}\right] \qquad \text{(e)}$$

$$\text{with } k = \frac{\omega}{v_p}$$

The boundary conditions are

$$\delta_2(\ell,t) = \delta_o e^{j\omega t} \qquad \text{(f)}$$

PROBLEM 11.3 (continued)

and

$$\left.\frac{\partial \delta_2}{\partial x_1}\right|_{x_1=0} = 0 \tag{g}$$

since the surface at $x_1 = 0$ is free.
Therefore

$$\delta_a\, e^{-jk\ell} + \delta_b\, e^{jk\ell} = \delta_o \tag{h}$$

and

$$-jk\,\delta_a + jk\,\delta_b = 0 \tag{i}$$

Solving, we obtain

$$\delta_a = \delta_b = \frac{\delta_o}{2\cos k\ell} \tag{j}$$

Therefore

$$\delta_2(x_1,t) = \text{Re}\left[\frac{\delta_o}{\cos k\ell}\cos kx_1\, e^{j\omega t}\right] = \frac{\delta_o}{\cos k\ell}\cos kx_1 \cos \omega t \tag{k}$$

and

$$T_{21}(x_1,t) = -\text{Re}\left[\frac{G\delta_o k}{\cos k\ell}\sin kx_1\, e^{j\omega t}\right] \tag{ℓ}$$

$$= -\frac{G\delta_o k}{\cos k\ell}\sin kx_1 \cos \omega t$$

Part b

In the limit as ω gets small

$$\delta_2(x_1,t) \to \text{Re}[\delta_o\, e^{j\omega t}] \tag{m}$$

In this limit, δ_2 varies everywhere in phase with the source. The slab of elastic material moves as a rigid body. Note from (ℓ) that the force per unit area at $x_1 = \ell$ required to set the slab into motion is $T_{21}(\ell,t) = \rho\ell \frac{d^2}{dt^2}(\delta_o \cos \omega t)$ or the mass $/(x_2-x_3)$ area times the rigid body acceleration.

Part c

The slab can resonate if we can have a finite displacement, even as $\delta_o \to 0$. This can happen if the denominator of (k) vanishes

$$\cos k\ell = 0 \tag{n}$$

or

$$\omega = \frac{(2n+1)\pi v_p}{2\ell} \qquad n = 0,1,2,\ldots \tag{o}$$

PROBLEM 11.3 (continued)

The lowest frequency is for n = 0

$$\text{or } \omega_{low} = \frac{\pi v_p}{2\ell} \qquad \text{(p)}$$

PROBLEM 11.4

Part a

We have that

$$\tau_i = T_{ij}n_j = \alpha\delta_{ij}n_j$$

It is given that the T_{ij} are known, thus the above equation may be written as three scalar equations $(T_{ij} - \alpha\delta_{ij})n_j = 0$, or:

$$(T_{11} - \alpha)n_1 + T_{12}n_2 + T_{13}n_3 = 0$$

$$T_{21}n_1 + (T_{22} - \alpha)n_2 + T_{23}n_3 = 0 \qquad \text{(a)}$$

$$T_{31}n_1 + T_{32}n_2 + (T_{33} - \alpha)n_3 = 0$$

Part b

The solution for these homogeneous equations requires that the determinant of the coefficients of the n_i's equal zero.

Thus

$$(T_{11} - \alpha)[(T_{22} - \alpha)(T_{33} - \alpha) - (T_{23})^2]$$

$$- T_{12}[T_{12}(T_{33} - \alpha) - T_{13}T_{23}] \qquad \text{(b)}$$

$$+ T_{13}[T_{12}T_{23} - T_{13}(T_{22} - \alpha)] = 0$$

where we have used the fact that

$$T_{ij} = T_{ji}. \qquad \text{(c)}$$

Since the T_{ij} are known, this equation can be solved for α.

Part c

Consider $T_{12} = T_{21} = T_o$, with all other components equal to zero. The determinant of coefficients then reduces to

$$-\alpha^3 + T_o^2\alpha = 0 \qquad \text{(d)}$$

for which

$$\alpha = 0 \qquad \text{(e)}$$

or

$$\alpha = \pm T_o \qquad \text{(f)}$$

The $\alpha = 0$ solution indicates that with the normal in the x_3 direction, there is no normal stress. The $\alpha = \pm T_o$ solution implies that there are two surfaces where the net traction is purely normal with stresses $\pm T_o$, respectively, as

PROBLEM 11.4 (continued)

found in example 11.2.1. Note that the normal to the surface for which the shear stress is zero can be found from (a), since α is known, and it is known that $|\bar{n}| = 1$.

PROBLEM 11.5

From Eqs. 11.2.25 - 11.2.28, we have

$$e_{11} = \frac{1}{E}[T_{11} - \nu(T_{22} + T_{33})] \tag{a}$$

$$e_{22} = \frac{1}{E}[T_{22} - \nu(T_{33} + T_{11})] \tag{b}$$

$$e_{33} = \frac{1}{E}[T_{33} - \nu(T_{11} + T_{22})] \tag{c}$$

and

$$e_{ij} = \frac{T_{ij}}{2G} \qquad i \neq j \tag{d}$$

These relations must still hold in a primed coordinate system, where we can use the transformations

$$T'_{ij} = a_{ik}a_{j\ell}T_{k\ell} \tag{e}$$

and

$$e'_{ij} = a_{ik}a_{j\ell}e_{k\ell} \tag{f}$$

For an example, we look at e'_{11}

$$e'_{11} = a_{1k}a_{1\ell}e_{k\ell} = \frac{1}{E}[T'_{11} - \nu(T'_{22} + T'_{33})] \tag{g}$$

This may be rewritten as

$$a_{1k}a_{1\ell}e_{k\ell} = \frac{1}{E}[(1+\nu)a_{1k}a_{1\ell}T_{k\ell} - \nu\,\delta_{k\ell}T_{k\ell}] \tag{h}$$

where we have used the relation from Eq.(8.2.23), page G10 or 439.

$$a_{pr}a_{ps} = \delta_{ps} \tag{i}$$

Consider some values of k and ℓ where $k \neq \ell$.

Then, from the stress-strain relation in the unprimed frame,

$$a_{1k}a_{1\ell}e_{k\ell} = a_{1k}a_{1\ell}\frac{T_{k\ell}}{2G} = \frac{a_{1k}a_{1\ell}}{E}(1+\nu)T_{k\ell} \tag{j}$$

Thus

$$\frac{1}{2G} = \frac{1+\nu}{E} \tag{k}$$

or $E = 2G(1+\nu)$ which agrees with Eq. (g) of example 11.2.1.

PROBLEM 11.6

Part a

Following the analysis in Eqs. 11.4.16 - 11.4.26, the equation of motion for the bar is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho} \frac{\partial^4 \xi}{\partial x_1^4} = 0 \qquad \text{(a)}$$

where ξ measures the bar displacement in the x_2 direction, T_2 in Eq. 11.4.26 = 0 as the surfaces at $x_2 = \pm b$ are free. The boundary conditions for this problem are that at $x_1 = 0$ and at $x_1 = L$

$$T_{21} = 0 \text{ and } T_{11} = 0 \qquad \text{(b)}$$

as the ends are free.

We assume solutions of the form

$$\xi = \text{Re } \hat{\xi}(x) e^{j\omega t} \qquad \text{(c)}$$

As in example 11.4.4, the solutions for $\hat{\xi}(x)$ are

$$\hat{\xi}(x) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1 \qquad \text{(d)}$$

with $$\alpha = \left[\omega^2 \left(\frac{3\rho}{Eb^2}\right)\right]^{1/4}$$

Now, from Eqs. 11.4.18 and 11.4.21,

$$T_{21} = \frac{(x_2^2 - b^2)E}{2} \frac{\partial^3 \xi}{\partial x_1^3} \qquad \text{(e)}$$

which implies

$$\frac{\partial^3 \xi}{\partial x_1^3} = 0 \qquad \text{(f)}$$

at $x_1 = 0$, $x_1 = L$

and

$$T_{11} = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2} \qquad \text{(g)}$$

which implies $$\frac{\partial^2 \xi}{\partial x_1^2} = 0 \qquad \text{(h)}$$

at $x_1 = 0$ and $x_1 = L$

PROBLEM 11.6 (continued)

With these relations, the boundary conditions require that

$$\begin{aligned} -A \qquad\qquad\qquad\qquad + C \qquad\qquad\qquad\qquad &= 0 \\ -A\cos\alpha L + B\sin\alpha L + C\cosh\alpha L + D\sinh\alpha L &= 0 \\ -B \qquad\qquad\qquad\qquad + D &= 0 \\ -A\sin\alpha L - B\cos\alpha L + C\sinh\alpha L + D\cosh\alpha L &= 0 \end{aligned} \qquad \text{(i)}$$

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, and D equal zero. Performing this operation, we obtain

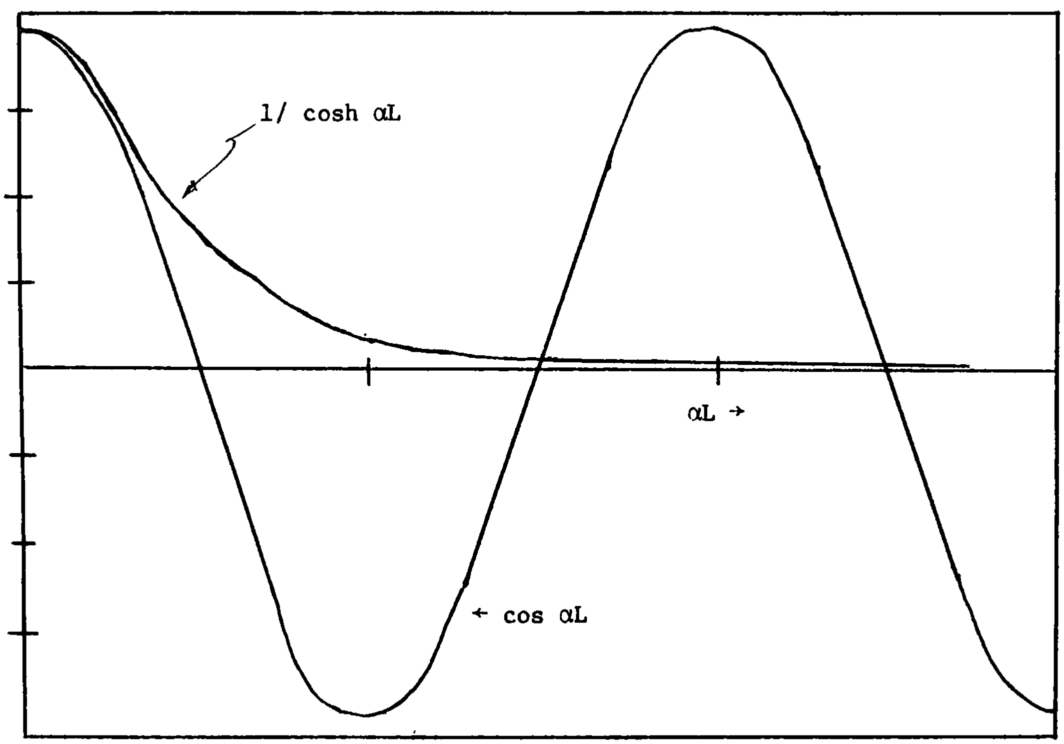
$$\cos\alpha L \cosh\alpha L = 1 \qquad \text{(j)}$$

Thus,

$$\beta = \alpha L = \left[\omega^2 \left(\frac{3\rho}{Eb^2}\right)\right]^{1/4} L \qquad \text{(k)}$$

Part b

The roots of $\cos\beta = \dfrac{1}{\cosh\beta}$ follow from the figure.



Note from the figure that the roots αL are essentially the roots $3\pi/2$, $5\pi/2$, ... of $\cos\alpha L = 0$.

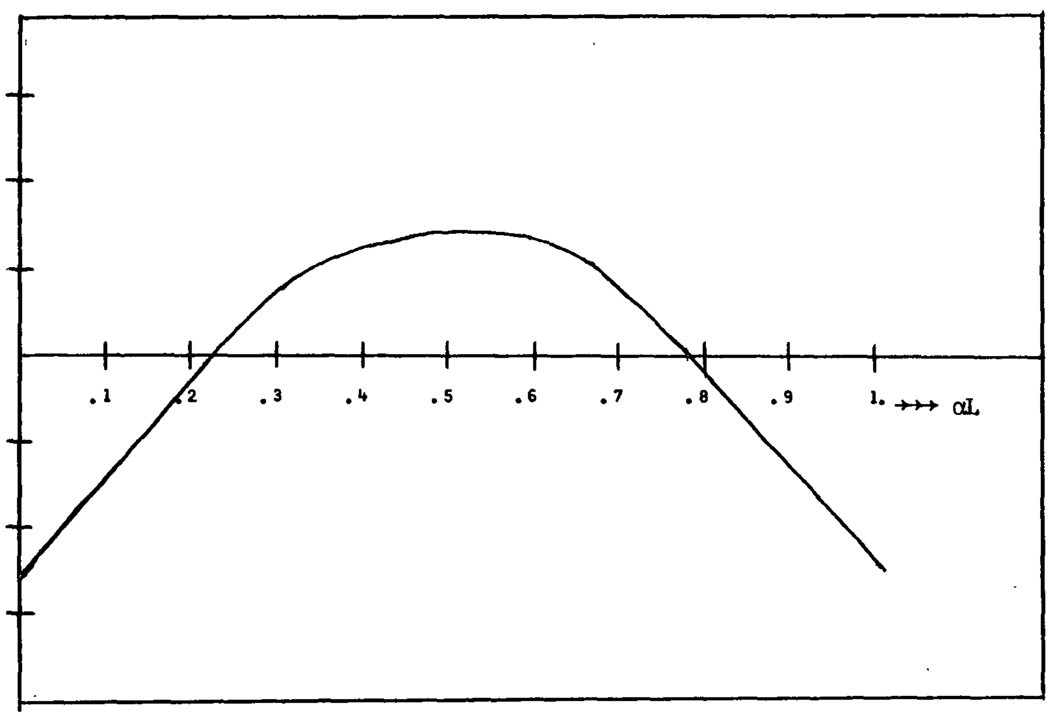
PROBLEM 11.6 (continued)

Part c

It follows from (i) that the eigenfunction is

$$\hat{\xi} = A'[(\sin \alpha x_1 + \sinh \alpha x_1)(\sin \alpha L + \sinh \alpha L) + (\cos \alpha L - \cosh \alpha L)(\cos \alpha x_1 + \cosh \alpha x_1) \qquad (\ell)$$

where A' is an arbitrary amplitude. This expression is found by taking one of the constants A ... D as known, and solving for the others. Then, (d) gives the required dependence on x_1 to within an arbitrary constant. A sketch of this function is shown in the figure.



PROBLEM 11.7

As in problem 11.6, the equation of motion for the elastic beam is

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4 \xi}{\partial x_1^4} = 0 \qquad \text{(a)}$$

The four boundary conditions for this problem are:

$$\xi(x_1 = 0) = 0 \qquad \xi(x_1 = L) = 0$$

$$\delta_1(0) = -x_2 \left.\frac{\partial \xi}{\partial x_1}\right|_{x_1=0} = 0 \quad \delta_1(L) = -x_2 \left.\frac{\partial \xi}{\partial x_1}\right|_{x_1=L} = 0 \qquad \text{(b)}$$

We assume solutions of the form

$\xi(x_1,t) = \text{Re}\, \hat{\xi}(x_1)e^{j\omega t}$, and as in problem 11.6, the solutions for (c) $\hat{\xi}(x_1)$ are

$$\hat{\xi}(x_1) = A \sin \alpha x_1 + B \cos \alpha x_1 + C \sinh \alpha x_1 + D \cosh \alpha x_1$$

$$\text{with } \alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4} \qquad \text{(d)}$$

Applying the boundary conditions, we obtain

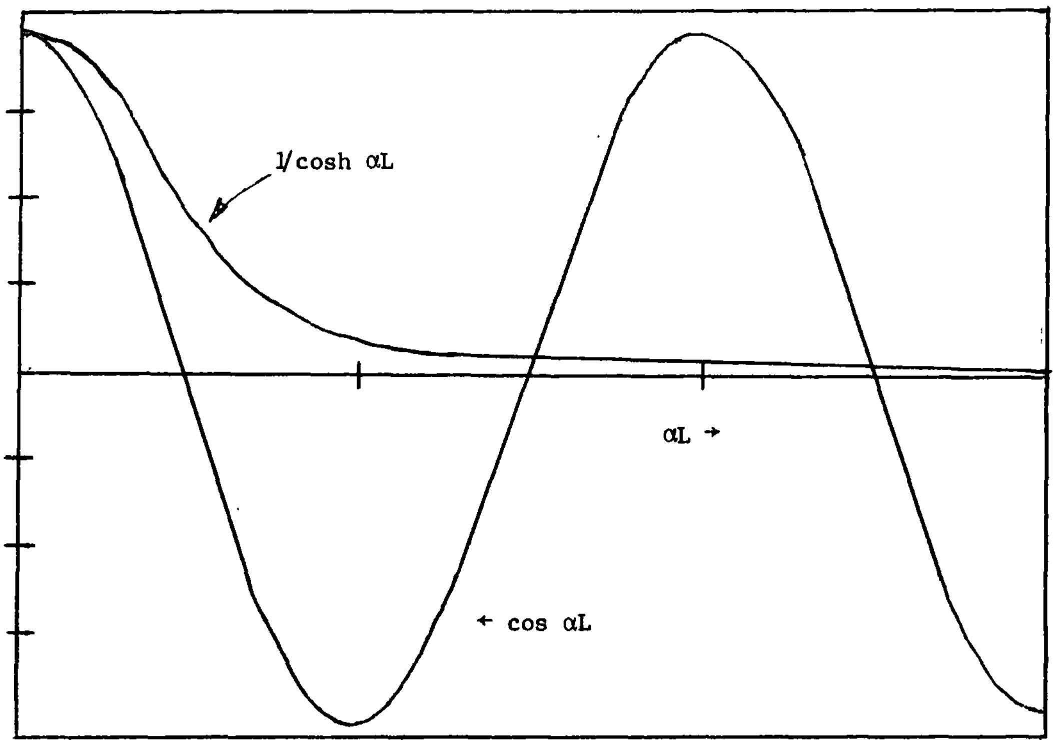
$$\begin{aligned}
B \qquad\qquad\qquad + D &= 0\\
A \sin \alpha L + B \cos \alpha L + C \sinh \alpha L + D \cosh \alpha L &= 0 \qquad \text{(e)}\\
A \qquad\qquad + C &= 0\\
A \cos \alpha L - B \sin \alpha L + C \cosh \alpha L + D \sinh \alpha L &= 0
\end{aligned}$$

The solution to this set of homogeneous equations requires that the determinant of the coefficients of A, B, C, D, equal zero. Performing this operation, we obtain

$$\cos \alpha L \cosh \alpha L = +1 \qquad \text{(f)}$$

To solve for the natural frequencies, we must use a graphical procedure.

PROBLEM 11.7 (continued)



The first natural frequency is at about

$$\alpha L = \frac{3\pi}{2}$$

Thus

$$\omega^2 \left(\frac{3\rho}{Eb^2}\right) L^4 = \left(\frac{3\pi}{2}\right)^4$$

or

$$\omega = \frac{\left(\frac{3\pi}{2}\right)^2}{L^2} \left(\frac{Eb^2}{3\rho}\right)^{1/2} \qquad \text{(g)}$$

Part b

We are given that $L = .5$ m and $b = 5 \times 10^{-4}$ m

From Table 9.1, Appendix G, the parameters for steel are:

$$E \approx 2 \times 10^{11}\ \text{N/m}^2$$

$$\rho \approx 7.75 \times 10^3\ \text{kg/m}^3$$

PROBLEM 11.7 (continued)

$$\omega \approx 120 \text{ rad/sec.}$$

Then, $f_1 = \frac{\omega}{2\pi} \approx 19$ Hz.

Part c

For the next higher resonance, $\alpha L \approx \frac{5}{2}\pi$

Therefore, $f_2 = \left(\frac{5}{2}\right)^2 f_1 \approx 53$ Hz.

PROBLEM 11.8

Part a

As in Prob. 11.7, the equation of motion for the beam is

$$\frac{\partial^2\xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4\xi}{\partial x_1^4} = 0 \tag{a}$$

At $x_1 = L$, there is a free end, so the boundary conditions are:

$$T_{11}(x_1{=}L) = 0$$

and

$$T_{21}(x_1{=}L) = 0 \tag{b}$$

The boundary conditions at $x_1 = 0$ are

$$M\frac{\partial^2\xi(0,t)}{\partial t^2} = +\int (T_{21})_{x_1=0}\, D\, dx_2 + \overline{f}_e + \overline{F}_o \tag{c}$$

and

$$\delta_1(x_1 = 0) = 0 \tag{d}$$

The $\overline{H}$ field in the air gap and in the plunger is

$$\overline{H} = \frac{Ni}{D}\overline{i}_1 \tag{e}$$

Using the Maxwell stress tensor

$$\overline{f}^{\,e} = -\frac{(\mu-\mu_o)}{2}\left(\frac{N^2 i^2}{D^2}\right) D^2\,\overline{i}_2 = -\frac{N^2 i^2}{2}(\mu-\mu_o)\overline{i}_2 \tag{f}$$

with $i = I_o + i_1\cos\omega t = I_o + \text{Re}\, i_1 e^{j\omega t}$

PROBLEM 11.8 (continued)

We linearize $\bar{f}^e$ to obtain

$$\bar{f}^e = -\frac{N^2}{2}(\mu-\mu_o)[I_o^2 + 2I_o i_1 \cos \omega t]\bar{i}_2 \qquad (g)$$

For equilibrium

$$\bar{F}_o - \frac{N^2}{2}(\mu-\mu_o)I_o^2\,\bar{i}_2 = 0$$

Thus

$$\bar{F}_o = \frac{N^2}{2}(\mu-\mu_o)I_o^2\,\bar{i}_2 \qquad (h)$$

Part b

We write the solution to Eq. (a) in the form

$$\xi(x_1,t) = \text{Re}\ \hat{\xi}(x_1)\, e^{j\omega t}$$

where, from example 11.4.4

$$\hat{\xi}(x_1) = A_1 \sin \alpha x_1 + A_2 \cos \alpha x_1 + A_3 \sinh \alpha x_1 + A_4 \cosh \alpha x_1 \qquad (i)$$

with

$$\alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4}$$

Now, from Eqs. 11.4.6 and 11.4.16

$$T_{11}(x=L) = E\frac{\partial \delta_1}{\partial x_1} = -Ex_2\frac{\partial^2 \xi}{\partial x_1^2} = 0 \qquad (j)$$

Thus

$$\frac{\partial^2 \xi}{\partial x_1^2}(x_1 = L) = 0$$

From Eq. 11.4.21

$$T_{21} = \frac{(x_2^2 - b^2)}{2}\, E\frac{\partial^3 \xi}{\partial x_1^3} \qquad (k)$$

and from Eq. 11.4.16

$$\delta_1(x_1 = 0) = -x_2\left(\frac{\partial \xi}{\partial x_1}\right)_{x_1=0} = 0 \qquad (\ell)$$

Thus

$$\left(\frac{\partial \xi}{\partial x_1}\right)_{x_1=0} = 0$$

PROBLEM 11.8(continued)

Applying the boundary conditions from Eqs. (b), (c), (d) to our solution of Eq. (i), we obtain the four equations

$$\begin{aligned}
A_1 \quad\quad + A_3 \quad\quad &= 0\\
- A_1 \sin \alpha L - A_2 \cos \alpha \ell + A_3 \sinh \alpha L + A_4 \cosh \alpha L &= 0\\
- A_1 \cos \alpha L + A_2 \sin \alpha L + A_3 \cosh \alpha L + A_4 \sinh \alpha L &= 0\\
- \frac{2}{3} \alpha^3 b^3 EDA_1 + M\omega^2 A_2 + \frac{2}{3}\alpha^3 b^3 EDA_3 + M\omega^2 A_4 &= + N^2 I_o i_1 (\mu - \mu_o)
\end{aligned} \quad \text{(m)}$$

Now

$$v = \frac{d\lambda}{dt} = \frac{d}{dt}\left\{\frac{N^2 i}{D} D\left[\mu_o \xi(0) + \mu\left(D - \xi(0)\right)\right]\right\} \quad \text{(n)}$$

or

$$\hat{v} = - N^2 I_o (\mu - \mu_o) j\omega (A_2 + A_4) + N^2 i_1 \mu D j\omega \quad \text{(o)}$$

We solve Eqs. (m) for A_2 and A_4 using Cramer's rule to obtain

$$A_2 = \frac{N^2 I_o i_1 (\mu - \mu_o)(-1 + \sin \alpha L \sinh \alpha L - \cos \alpha L \cosh \alpha L)}{- 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3}(\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \quad \text{(p)}$$

and

$$A_4 = \frac{N^2 I_o i_1 (\mu - \mu_o)(- 1 - \cos \alpha L \cosh \alpha L - \sin \alpha L \sinh \alpha L)}{- 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3}(\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)} \quad \text{(q)}$$

Thus

$$Z(j\omega) = \frac{\hat{v}(j\omega)}{i_1} = \frac{+\left[N^2 I_o (\mu - \mu_o)\right]^2 j\omega (+ 2 + 2 \cos \alpha L \cosh \alpha L)}{- 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) + \frac{4}{3}(\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)}$$

$$+ N^2 \mu D j\omega \quad \text{(r)}$$

Part c

$Z(j\omega)$ has poles when

$$+ 2M\omega^2 (1 + \cos \alpha L \cosh \alpha L) = \frac{4}{3}(\alpha b)^3 ED(\cos \alpha L \sinh \alpha L + \sin \alpha L \cosh \alpha L)$$

PROBLEM 11.9

Part a

The flux above and below the beam must remain constant. Therefore, the $\overline{H}$ field above is

$$\overline{H}_a = \frac{H_o(a-b)}{(a-b-\xi)}\overline{i}_1 \tag{a}$$

and the $\overline{H}$ field below is

$$\overline{H}_b = \frac{H_o(a-b)}{(a-b+\xi)}\overline{i}_1 \tag{b}$$

Using the Maxwell stress tensor, the magnetic force on the beam is

$$T_2 = -\frac{\mu_o}{2}(H_a^2 - H_b^2) = -\frac{\mu_o}{2} H_o^2 (a-b)^2\left(+\frac{4\xi}{(a-b)^3}\right)$$

$$= -\frac{2\mu_o H_o^2 \xi}{(a-b)} \tag{c}$$

Thus, from Eq. 11.4.26, the equation of motion on the beam is

$$\frac{\partial^2\xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4\xi}{\partial x_1^4} = -\frac{\mu_o H_o^2\xi}{(a-b)b\rho} \tag{d}$$

Again, we let

$$\xi(x_1,t) = \text{Re}\,\hat{\xi}(x_1)e^{j\omega t} \tag{e}$$

with the boundary conditions

$$\xi(x_1=0) = 0 \qquad \xi(x_1=L) = 0$$
$$\delta_1(x_1=0) \qquad \delta_1(x_1=L) = 0 \tag{f}$$

Since $\delta_1 = -x_2\,\partial\xi/\partial x_1$ from Eq. 11.4.16, this implies that:

$$\frac{\partial\xi}{\partial x_1}(x_1=0) = 0 \text{ and } \frac{\partial\xi}{\partial x_1}(x_1=L) = 0 \tag{g}$$

Substituting our assumed solution into the equation of motion, we have

$$-\omega^2\hat{\xi} + \frac{Eb^2}{3\rho}\frac{\partial^4\hat{\xi}}{\partial x_1^4} + \frac{\mu_o H_o^2\hat{\xi}}{(a-b)b\rho} = 0 \tag{h}$$

Thus we see that our solutions are again of the form

$$\hat{\xi}(x) = A\sin\alpha x + B\cos\alpha x + C\sinh\alpha x + D\cosh\alpha x \tag{i}$$

PROBLEM 11.9 (continued)

where now

$$\alpha = \left[\left(\omega^2 - \frac{\mu_o H_o^2}{(a-b)b\rho}\right)\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4} \qquad (j)$$

Since the boundary conditions for this problem are identical to that of problem 11.7, we can take the solutions from that problem, substituting the new value of α. From problem 11.7, the solution must satisfy

$$\cos \alpha L \cosh \alpha L = 1 \qquad (k)$$

The first resonance occurs when

$$\alpha L \approx \frac{3\pi}{2}$$

or

$$\omega^2 = \frac{\left(\frac{3\pi}{2}\right)^4\left(\frac{Eb^2}{3\rho}\right)}{L^4} + \frac{\mu_o H_o^2}{(a-b)b\rho} \qquad (\ell)$$

Part c

The resonant frequencies are thus shifted upward due to the stiffening effect of the constant flux constraint.

Part d

We see that, no matter what the values of the system parameters $\omega^2 > 0$, so ω will always be real, and thus stable. This is expected as the constant flux constraint imposes a force which opposes the motion.

PROBLEM 11.10

Part a

We choose a coordinate system as in Fig. 11.4.12, centered at the right end of the rod. Because $\frac{d}{D} = \frac{1}{10}$, we can neglect fringing and consider the right end of the rod as a capacitor plate. Also, since $\frac{D}{\ell} = \frac{1}{10}$, we can assume that the electrical force acts only at $x_1 = 0$. Thus, the boundary conditions at $x_1 = 0$ are

$$-\int_{-b}^{b} T_{21}\, D\, dx_2 + f^e = 0 \qquad (a)$$

where $T_{21} = \frac{(x_2^2 - b^2)}{2} E \frac{\partial^3 \xi}{\partial x_1^3}$ (Eq. 11.4.21)

since the electrical force, f^e, must balance the shear stress T_{21} to keep the rod in equilibrium,

PROBLEM 11.10 (continued)

and

$$T_{11}(0) = -x_2 E \frac{\partial^2 \xi}{\partial x_1^2}(0) = 0 \qquad \text{(b)}$$

since the end of the rod is free of normal stresses. At $x_1 = -\ell$, the rod is clamped so

$$\xi(-\ell) = 0 \qquad \text{(c)}$$

and

$$\delta_1(-\ell) = -x_2 \frac{\partial \xi}{\partial x_1}(-\ell) = 0 \qquad \text{(d)}$$

We use the Maxwell stress tensor to calculate the electrical force to be

$$f^e = \frac{\varepsilon A}{2}\left[\frac{(V_o + v_s)^2}{[d - \xi(0)]^2} - \frac{(v_s - V_o)^2}{[d + \xi(0)]^2}\right] \qquad \text{(e)}$$

$$\approx \frac{2\varepsilon A V_o}{d^2}\left[v_s + \frac{V_o\,\xi(0)}{d}\right]$$

The equation of motion of the beam is (example 11.4.4)

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{Eb^2}{3\rho}\frac{\partial^4 \xi}{\partial x_1^4} = 0 \qquad \text{(f)}$$

We write the solution to Eq. (f) in the form

$$\xi(x,t) = \text{Re}\,\hat{\xi}(x)e^{j\omega t} \qquad \text{(g)}$$

where

$$\hat{\xi}(x) = A_1 \sin \alpha x + A_2 \cos \alpha x + A_3 \sinh \alpha x + A_4 \cosh \alpha x$$

with

$$\alpha = \left[\omega^2\left(\frac{3\rho}{Eb^2}\right)\right]^{1/4}$$

Applying the four boundary conditions, Eqs. (a), (b), (c) and (d), we obtain the equations

$$-A_1 \sin \alpha\ell + A_2 \cos \alpha\ell - A_3 \sinh \alpha\ell + A_4 \cosh \alpha L = 0$$

$$A_1 \cos \alpha\ell + A_2 \sin \alpha\ell + A_3 \cosh \alpha\ell - A_4 \sinh \alpha\ell = 0 \qquad \text{(h)}$$

$$-A_2 + A_4 = 0$$

$$-\frac{2}{3}b^3 DE\alpha^3 A_1 + \frac{2\varepsilon_o A V_o^2}{d^3}A_2 + \frac{2}{3}b^3 DE\alpha^3 A_3 + \frac{2\varepsilon_o A V_o^2}{d^3}A_4 = -\frac{2\varepsilon_o A V_o \hat{v}_s}{d^2}$$

PROBLEM 11.10 (continued)

Now $i_s = \dfrac{dq_s}{dt}$ (i)

where
$$q_s = \frac{\varepsilon_o A}{d-\xi(0)}(V_o + v_s) + \frac{\varepsilon_o A(v_s - V_o)}{d + \xi(0)}$$

$$\approx \frac{2\varepsilon_o A v_s}{d} + \frac{2\varepsilon_o A V_o}{d^2}\,\xi(0) \qquad \text{(j)}$$

Therefore

$$\hat{i}_s = j\omega\frac{2\varepsilon_o A}{d}\left[\hat{v}_s + \frac{V_o}{d}\,\hat{\xi}(0)\right] \qquad \text{(k)}$$

where
$$\hat{\xi}(0) = A_2 + A_4$$

We use Cramer's rule to solve Eqs. (h) for A_2 and A_4 to obtain:

$$A_2 = A_4 = \frac{-\dfrac{\varepsilon_o A V_o \hat{v}_s}{d^2}\,[\cos\alpha\ell\sinh\alpha\ell - \sin\alpha\ell\cosh\alpha\ell]}{\dfrac{2}{3}b^3\alpha^3 DE(1 + \cos\alpha\ell\cosh\alpha\ell) + \dfrac{2\varepsilon_o A V_o^2}{d^3}(\cos\alpha\ell\sinh\alpha\ell - \sin\alpha\ell\cosh\alpha\ell)} \qquad (\ell)$$

Thus, from Eq. (k) we obtain

$$Z(j\omega) = \frac{d}{j\omega 2\varepsilon_o A}\left[1 + \frac{3\varepsilon_o A V_o^2}{d^3(\alpha b)^3 ED}\,\frac{(\cos\alpha\ell\sinh\alpha\ell - \sin\alpha\ell\cosh\alpha\ell)}{(1 + \cos\alpha\ell\cosh\alpha\ell)}\right] \qquad \text{(m)}$$

Part b

We define a function $g(\alpha\ell)$ such that Eq. (m) has a zero when

PROBLEM 11.10 (Continued)

$$(\alpha L)^3 g(\alpha L) = \frac{(1+\cosh\alpha\ell\,\cos\alpha\ell)(\alpha\ell)^3}{\sin\alpha\ell\,\cosh\alpha\ell - \cos\alpha\ell\,\sinh\alpha\ell} = \frac{3\ell^3 V_o^2 A\varepsilon_o}{DEb^3d^3} \tag{n}$$

Substituting numerical values, we obtain

$$\frac{3\ell^3 V_o^2 A\varepsilon_o}{DEb^3d^3} \approx \frac{3\times10^{-3}(10^6)10^{-4}(8.85\times10^{-12})}{10^{-2}(2.2\times10^{11})10^{-9}\;10^{-9}} \approx 1.2\times10^{-3} \tag{o}$$

In Figure 1, we plot $(\alpha\ell)^3 g(\alpha\ell)$ as a function of $\alpha\ell$. We see that the solution to Eq. (n) first occurs when $(\alpha\ell)^3 g(\alpha\ell) \approx 0$. Thus, the solution is approximately

$$\alpha\ell = 1.875$$

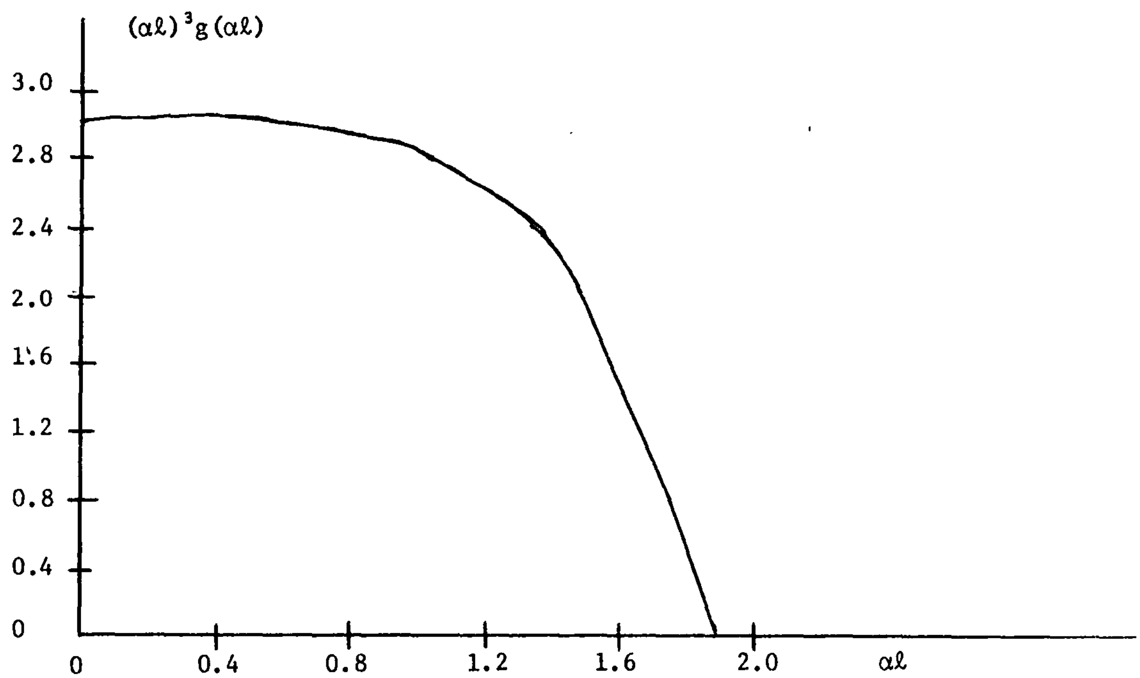


Figure 1

PROBLEM 11.10 (Continued)

From Eq. (g)

$$\alpha\ell = \left[\omega^2 \frac{3\rho}{Eb^2}\right]^{1/4} \ell = 1.875$$

Solving for ω, we obtain

$$\omega \approx 1080 \text{ rad/sec.} \qquad (p)$$

Part c

The input impedance of a series LC circuit is

$$Z(j\omega) = \frac{1 - LC\omega^2}{j\omega C} \qquad (q)$$

Thus the impedance has a zero when

$$\omega_o^2 = \frac{1}{LC} \qquad (r)$$

We let $\omega = \omega_o + \Delta\omega$, and expand (q) in a Taylor series around ω_o to obtain

$$Z(j\omega) \approx + j \frac{2\Delta\omega}{C\omega_o^2} = + 2j\, L\Delta\omega \qquad (s)$$

(m) can be written in the form

$$Z(j\omega) = \frac{1}{2j\omega C_o}[1 - f(\omega)] \quad \text{where } f(\omega_o) = 1 \quad \text{and } C_o = \frac{\varepsilon_o A}{d} \qquad (t)$$

For small deviations around ω_o

$$Z(j\omega) \approx \frac{j}{2\omega C_o} \left.\frac{\partial f}{\partial \omega}\right|_{\omega_o} \Delta\omega$$

Thus, from (q), (r) (s) and (t), we obtain the relations

$$2L = \frac{1}{2\omega C_o} \left.\frac{\partial f}{\partial \omega}\right|_{\omega_o} \qquad (u)$$

and

$$C = \frac{1}{\omega_o^2 L} \qquad (v)$$

now

$$f(\omega) = \frac{K}{(\alpha\ell)^3 g(\alpha\ell)} \qquad (w)$$

where

$$K = \frac{3\ell^3 \varepsilon_o A V_o^2}{d^3 (EDb^3)} = 1.2 \times 10^{-3}$$

PROBLEM 11.10 (Continued)

and $g(\alpha\ell) = \dfrac{1 + \cos\alpha\ell \cosh\alpha\ell}{\sin\alpha\ell \cosh\alpha\ell - \cos\alpha\ell \sinh\alpha\ell}$

Thus, we can write

$$\left.\frac{df(\omega)}{d\omega}\right|_{\omega_o} = \left\{\frac{d}{d(\alpha\ell)}\left[\frac{K}{(\alpha\ell)^3 g(\alpha\ell)}\right]\frac{d(\alpha\ell)}{d\omega}\right\}_{\omega_o} \tag{y}$$

Now from (g),

$$\left.\frac{d(\alpha\ell)}{d\omega}\right|_{\omega_o} = \left(\frac{3\rho}{Eb^2}\right)^{1/4} \frac{\ell}{2\omega_o^{1/2}} \tag{z}$$

and

$$\frac{d}{d(\alpha\ell)}\left[\frac{K}{(\alpha\ell)^3 g(\alpha\ell)}\right]_{\omega_o} = \left.\frac{-K}{[(\alpha\ell)^3 g(\alpha\ell)]^2}\frac{d}{d(\alpha\ell)}[(\alpha\ell)^3 g(\alpha\ell)]\right|_{\omega_o}$$

$$\approx -\left.\frac{1}{K}\frac{d}{d(\alpha\ell)}[(\alpha\ell)^3 g(\alpha\ell)]\right|_{\omega_o} \tag{aa}$$

since at $\omega = \omega_o$

$$(\alpha\ell)^3 g(\alpha\ell) = K\,. \tag{bb}$$

Continuing the differentiating in (aa), we finally obtain

$$\left.\frac{d}{d(\alpha\ell)}\left[\frac{(\alpha\ell)^3 g(\alpha\ell)}{-K}\right]\right|_{\omega_o} = -\frac{1}{K}\left[g(\alpha\ell)3(\alpha\ell)^2 + (\alpha\ell)^3\frac{d}{d(\alpha\ell)}g(\alpha\ell)\right]_{\omega_o}$$

$$= \left.\frac{-3}{\alpha\ell}\right|_{\omega_o} - \left.\frac{(\alpha\ell)^3}{K}\frac{d}{d(\alpha\ell)}g(\alpha\ell)\right|_{\omega_o} \tag{cc}$$

Now

$$\frac{d}{d(\alpha\ell)}g(\alpha\ell) = \frac{-\sin\alpha\ell\cosh\alpha\ell + \cos\alpha\ell\sinh\alpha\ell}{(\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell)}$$

$$-\frac{(1+\cos\alpha\ell\cosh\alpha\ell)(+\cos\alpha\ell\cosh\alpha\ell + \sin\alpha\ell\sinh\alpha\ell + \sin\alpha\ell\sinh\alpha\ell - \cos\alpha\ell\cosh\alpha\ell)}{(\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell)}$$

$$= -1 - \frac{2g(\alpha\ell)\,(\sin\alpha\ell\sinh\alpha\ell)}{(\sin\alpha\ell\cosh\alpha\ell - \cos\alpha\ell\sinh\alpha\ell)} \tag{dd}$$

PROBLEM 11.10 (Continued)

Substituting numerical values into the second term of (cc), we find it to have value much less than one at $\omega = \omega_o$.

Thus,

$$\frac{d}{d(\alpha\ell)} g(\alpha\ell) \approx -1 \qquad \text{(ee)}$$

Thus, using (y), (z), (aa), (bb) and (dd), we have

$$\left.\frac{df}{d\omega}\right|_{\omega_o} \approx \left(\frac{3\rho}{Eb^2}\right)^{1/4} \frac{\ell}{2\omega_o^{1/2}} \left[-\left.\frac{3}{\alpha\ell}\right|_{\omega_o} + \left.\frac{(\alpha\ell)^3}{K}\right|_{\omega_o} \right] \approx 4.8 \qquad \text{(ff)}$$

Thus, from (v) and (w)

$$L \approx \frac{4.8 \times 10^{-3}}{4(1080)(8.85 \times 10^{-12})(10^{-4})} = 1.25 \times 10^{9} \text{ henries}$$

and

$$C \approx \frac{1}{1.25 \times 10^{9}(1080)^2} = 6.8 \times 10^{-16} \text{ farads.}$$

PROBLEM 11.11

From Eq. (11.4.29), the equation of motion is

$$\rho \frac{\partial^2 \delta_3}{\partial t^2} = G\left(\frac{\partial^2 \delta_3}{\partial x_1^2} + \frac{\partial^2 \delta_3}{\partial x_2^2}\right) \quad \text{(a)}$$

We let

$$\delta_3 = \text{Re}\ \hat{\delta}(x_2)e^{j(\omega t - kx_1)} \quad \text{(b)}$$

Substituting this assumed solution into the equation of motion, we obtain

$$-\rho\omega^2\hat{\delta} = G\left(-k^2\hat{\delta} + \frac{\partial^2 \hat{\delta}}{\partial x_2^2}\right) \quad \text{(c)}$$

or

$$\frac{\partial^2 \hat{\delta}}{\partial x_2^2} + \left(\frac{\rho\omega^2}{G} - k^2\right)\hat{\delta} = 0 \quad \text{(d)}$$

If we let $\beta^2 = \frac{\rho\omega^2}{G} - k^2$ (e)

the solutions for $\hat{\delta}$ are:

$$\hat{\delta}(x_2) = A \sin \beta x_2 + B \cos \beta x_2 \quad \text{(f)}$$

The boundary conditions are

$$\hat{\delta}(0) = 0 \quad \text{and} \quad \hat{\delta}(d) = 0 \quad \text{(g)}$$

This implies that $B = 0$

and that $\beta d = n\pi$.

Thus, the dispersion relation is

$$\omega^2 \frac{\rho}{G} - k^2 = \left(\frac{n\pi}{d}\right)^2 \quad \text{(h)}$$

Part b

The sketch of the dispersion relation is identical to that of Fig. 11.4.19. However, now the n=0 solution is trivial, as it implies that

$$\hat{\delta}(x_2) = 0$$

Thus, there is no principal mode of propagation.

PROBLEM 11.12

From Eq. (11.4.1), the equation of motion is

$$\rho \frac{\partial^2 \delta}{\partial t^2} = (2G + \lambda)\nabla(\nabla \cdot \delta) - G\nabla \times (\nabla \times \delta) \quad \text{(a)}$$

We consider motions

$$\delta = \delta_\theta(r,z,t)\bar{i}_\theta \quad \text{(b)}$$

Thus, the equation of motion reduces to

$$\rho \frac{\partial^2 \delta_\theta}{\partial t^2} - G\left[\frac{\partial^2 \delta_\theta}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial}{\partial r} r\delta_\theta\right)\right] = 0 \quad \text{(c)}$$

We assume solutions of the form

$$\delta_\theta(r,z,t) = \text{Re}\ \hat{\delta}(r)e^{j(\omega t - kz)} \quad \text{(d)}$$

which, when substituted into the equation of motion, yields

$$\frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial r} r\hat{\delta}(r)\right] + \left(\frac{\rho\omega^2}{G} - k^2\right) \hat{\delta}(r) = 0 \quad \text{(e)}$$

From page 207 of Ramo, Whinnery and Van Duzer, we recognize solutions to this equation as

$$\hat{\delta}(r) = A\ J_1\left[\left(\frac{\rho\omega^2}{G} - k^2\right)^{1/2} r\right] + BN_1\left[\left(\frac{\rho\omega^2}{G} - k^2\right)^{1/2} r\right] \quad \text{(f)}$$

On page 209 of this reference there are plots of the Bessel functions J_1 and N_1. We must have B = 0 as at r = 0, N_1 goes to $-\infty$. Now, at r = R

$$\hat{\delta}(R) = 0 \quad \text{(g)}$$

This implies that

$$J_1\left[\left(\frac{\rho\omega^2}{G} - k^2\right)^{1/2} R\right] = 0 \quad \text{(h)}$$

If we denote α_i as the zeroes of J_1 , i.e.

$$J_1\ (\alpha_i) = 0$$

we have the dispersion relation as

$$\frac{\rho}{G}\omega^2 - k^2 = \frac{\alpha_i^2}{R^2} \quad \text{(i)}$$