PROBLEM 13.1

In static equilibrium, we have

$$-\nabla p - \rho g \overline{i}_1 = 0 \qquad \text{(a)}$$

Since $p = \rho RT$, (a) may be rewritten as

$$RT \frac{d\rho}{dx_1} + \rho g = 0 \qquad \text{(b)}$$

Solving, we obtain

$$\rho = \rho_o e^{-\frac{g}{RT} x_1} \qquad \text{(c)}$$

PROBLEM 13.2

Since the pressure is a constant, Eq. (13.2.25) reduces to

$$\rho v \frac{dv}{dz} = - J_y B \qquad \text{(a)}$$

where we use the coordinate system defined in Fig. 13P.4. Now, from Eq. (13.2.21) we obtain

$$J_y = \sigma(E_y + vB) \qquad \text{(b)}$$

If the loading factor K, defined by Eq. (13.2.32) is constant, then

$$- KvB = + E \qquad \text{(c)}$$

Thus, $J_y = \sigma v B(1-K)$ (d)

Then

$$\rho v \frac{dv}{dz} = - \sigma v B^2(1-K) \qquad \text{(e)}$$

or

$$\rho \frac{dv}{dz} = - \sigma B^2(1-K) = - \sigma(1-K) \frac{B_i^2 A_i}{A(z)} \qquad \text{(f)}$$

From conservation of mass, Eq. (13.2.24), we have

$$\rho_i v_i A_i = \rho A(z) v \qquad \text{(g)}$$

Thus

$$\frac{\rho_i v_i A_i}{v} \frac{dv}{dz} = -\sigma(1-K) B_i^2 A_i \qquad \text{(h)}$$

Integrating, we obtain

$$\ell n\, v = \frac{-\sigma(1-K)B_i^2}{\rho_i v_i} z + C \qquad \text{(i)}$$

or

$$v = v_i e^{-\frac{z}{\ell_d}} \qquad \text{(j)}$$

where $\ell_d = \dfrac{\rho_i v_i}{\sigma(1-K)B_i^2}$ and we evaluate the arbitrary constant by realizing that

$v = v_i$ at $z = 0$.

PROBLEM 13.3

Part a

We assume T, B_o, w, σ, c_p and c_v are constant. Since the electrodes are short-circuited, E = 0, and so

$$J_y = v\,B_o. \tag{a}$$

We use the coordinate system defined in Fig. 13P.4. Applying conservation of energy, Eq. (13.2.26), we have

$$\rho v \frac{d}{dz}\left(\frac{1}{2}v^2\right) = 0, \text{ where we have set } h = \text{constant}. \tag{b}$$

Thus, v is a constant, $v = v_i$. Conservation of momentum, Eq. (13.2.25), implies

$$\frac{dp}{dz} = -v_i B_o^2 \tag{c}$$

Thus, $p = -v_i B_o^2 z + p_i$ (d)

The mechanical equation of state, Eq. (13.1.10) then implies

$$\rho = \frac{p}{RT} = -\frac{v_i B_o^2 z + p_i}{R\,T} = \rho_i - \frac{v_i B_o^2 z}{RT} \tag{e}$$

From conservation of mass, we then obtain

$$\rho_i v_i w d_i = \left(-\frac{v_i B_o^2 z}{RT} + \rho_i\right) v_i\, w d(z) \tag{f}$$

Thus

$$d(z) = \frac{\rho_i d_i}{\left(\rho_i - \frac{v_i B_o^2 z}{RT}\right)} \tag{g}$$

Part b

Then

$$\rho(z) = \rho_i - \frac{v_i B_o^2\, z}{RT} \tag{h}$$

PROBLEM 13.4

Note:

There are errors in Eqs. (13.2.16) and (13.2.31). They should read:

$$\frac{1}{M^2}\,\frac{d(M^2)}{dx_1} = \frac{\{(\gamma-1)(1+\gamma M^2)E_3 + \gamma[2+(\gamma-1)M^2]v_1 B_2\}\,J_3}{(1-M^2)\gamma\, p v_1} \tag{13.2.16}$$

and

$$\frac{1}{M^2}\,\frac{d(M^2)}{dx_1} = \frac{1}{(1-M^2)}\left\{[(\gamma-1)(1+\gamma M^2)E_3 + \gamma\{2+(\gamma-1)M^2\}\,v_1 B_2]\,\frac{J_3}{\gamma p v_1} - \frac{[2+(\gamma-1)M^2]}{A}\frac{dA}{dx_1}\right\} \tag{13.2.31}$$

Part a

We assume that σ, γ, B_o, K and M are constant along the channel. Then, from the corrected form of Eq. (13.2.31), we must have

PROBLEM 13.4 (continued)

$$0 = \frac{1}{1-M^2}\left\{\left[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)\right]\frac{vB_o^2\sigma(1-K)}{\gamma p} - \frac{[2+(\gamma-1)M^2]}{A}\frac{dA}{dz}\right\} \quad \text{(a)}$$

Now, using the relations

$$v^2 = M^2\gamma RT$$

and

$$p = \rho RT$$

we write

$$\frac{v}{\gamma p} = \frac{M^2}{\rho v} \quad \text{(b)}$$

Thus, we obtain

$$\frac{1}{A^2}\frac{dA}{dz} = \frac{[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)]\dfrac{B_o^2\sigma(1-K)M^2}{\rho VA}}{2+(\gamma-1)M^2} \quad \text{(c)}$$

From conservation of mass,

$$\rho vA = \rho_i v_i A_i \quad \text{(d)}$$

Using (d), we integrate (c) and solve for $\frac{A(z)}{A_i}$

to obtain

$$\frac{A(z)}{A_i} = \frac{1}{1-\beta_1 z} \quad \text{(e)}$$

where

$$\beta_1 = \frac{[(\gamma-1)(1+\gamma M^2)(-K) + \gamma(2+(\gamma-1)M^2)]\sigma B_o^2 M^2(1-K)}{\rho_i v_i[2+(\gamma-1)M^2]}$$

We now substitute into Eq. (13.2.27) to obtain

$$\frac{1}{v}\frac{dv}{dz} = \frac{1}{(1-M^2)}\left[(\gamma-1)(-K)+\gamma\right]\frac{vB_o^2(1-K)\sigma}{\gamma p} - \frac{1}{A}\frac{dA}{dz} \quad \text{(f)}$$

Thus may be rewritten as

$$\frac{1}{v}\frac{dv}{dz} = \frac{1}{(1-M^2)}\left[\left[(\gamma-1)(-K)+\gamma\right]\frac{\sigma B_o^2(1-K)M^2}{\rho_i v_i A_i} - \frac{\beta_1}{A_i}\right] A \quad \text{(g)}$$

Solving, we obtain

$$\ln v = \frac{-\beta_2}{\beta_1}\ln(1-\beta_1 z) + \ln v_i \quad \text{(h)}$$

or

$$\frac{v(z)}{v_i} = (1-\beta_1 z)^{-\beta_2/\beta_1} \quad \text{(i)}$$

where

$$\beta_2 = \frac{1}{(1-M^2)}\quad\frac{[(\gamma-1)(-K)+\gamma]\sigma B_o^2\ (1-K)M^2 - \beta_1}{\rho_i\ v_i}$$

Now the temperature is related through Eq. (13.2.12), as

PROBLEM 13.4 (continued)

$$M^2\gamma RT = v^2 \tag{j}$$

Thus

$$\frac{T(z)}{T_i} = \left(\frac{v}{v_i}\right)^2 \tag{k}$$

From (d), we have

$$\frac{\rho(z)}{\rho_i} = \frac{v_i}{v}\frac{A_i}{A} \tag{ℓ}$$

Thus, from Eq. (13.1.10)

$$\frac{p(z)}{p_i} = \frac{v_i}{v}\frac{A_i}{A}\frac{T}{T_i} \tag{m}$$

Since the voltage across the electrodes is constant,

$$E = -\frac{V}{w(z)} = -Kv(z)B_o \tag{n}$$

or

$$w(z) = \frac{Kv_iB_ow_i}{Kv(z)B_o} = \frac{v_i}{v(z)}w_i \tag{o}$$

Thus,

$$\frac{w(z)}{w_i} = \frac{v_i}{v(z)} \tag{p}$$

Then

$$\frac{d(z)}{d_i} = \frac{A(z)}{A_i}\frac{w_i}{w(z)} \tag{q}$$

Part b

We now assume that σ, γ, B_o, K and v are constant along the channel. Then, from Eq. (13.2.27) we have

$$0 = \frac{1}{(1-M^2)}\left\{[(\gamma-1)(-K)+\gamma]v_iB_o^2\frac{(1-K)\sigma}{\gamma p} - \frac{1}{A}\frac{dA}{dz}\right\} \tag{r}$$

But, from Eq. (13.2.25) we know that

$$\frac{p}{p_i} = 1 - \frac{(1-K)\sigma v_iB_o^2z}{p_i} = 1-\beta_3 z \tag{s}$$

where $\beta_3 = (1-K)\dfrac{\sigma v_iB_o^2}{p_i}$

Substituting the results of (b), into (a) and solving for $\frac{A(z)}{A_i}$, we obtain

$$\frac{A(z)}{A_i} = \left(\frac{p}{p_i}\right)^{-\beta_4/\beta_3} \tag{t}$$

where $\beta_4 = [(\gamma-1)(-K)+\gamma]\dfrac{v_iB_o^2}{\gamma p_i}(1-K)\sigma$

From conservation of mass,

$$\frac{\rho(z)}{\rho_i} = \frac{A_i}{A(z)} \tag{u}$$

PROBLEM 13.4 (continued)

and so, from Eq. (13.1.10)

$$\frac{T(z)}{T_i} = \frac{p(z)}{p_i}\frac{\rho_i}{\rho(z)} \quad \text{(v)}$$

As in (p)

$$\frac{w(z)}{w_i} = \frac{v_i}{v(z)} = 1 \quad \text{(w)}$$

Thus

$$\frac{d(z)}{d_i} = \frac{A(z)}{A_i} \quad \text{(x)}$$

Part c

We wish to find the length ℓ such that

$$\frac{C_pT(\ell) + \frac{1}{2}[v(\ell)]^2}{C_pT(o) + \frac{1}{2}[v(o)]^2} = .9 \quad \text{(y)}$$

For the constant M generator of part (a), we obtain from (i) and (k)

$$\frac{C_p\left[\frac{v(\ell)}{v_i}\right]^2 T_i + \frac{1}{2}[v(\ell)]^2}{C_p\left[\frac{v(o)}{v_i}\right]^2 T_i + \frac{1}{2}[v(o)]^2} = \frac{C_p(1-\beta_1\ell)^{-2\beta_2/\beta_1}T_i + \frac{1}{2}[v_i(1-\beta_1\ell)]^{-2\beta_2/\beta_1}}{C_p T_i + \frac{1}{2}v_i^2} = .9 \quad \text{(z)}$$

Reducing, we obtain

$$(1-\beta_1\ell)^{-2\beta_2/\beta_1} = .9 \quad \text{(aa)}$$

Substituting the given numerical values, we have

$$\beta_1 = .396 \quad \text{and} \quad \beta_2/\beta_1 = -7.3\times 10^{-2}$$

We then solve (aa) for ℓ, to obtain

$$\ell \approx 1.3 \text{ meters}$$

For the constant v generator of part (b), we obtain from (s), (t), (u) and (v)

$$\frac{C_pT_i\left[\frac{p(\ell)}{p_i}\frac{\rho_i}{\rho(\ell)}\right] + \frac{1}{2}v_i^2}{C_p T_i + \frac{1}{2}v_i^2} = .9 \quad \text{(bb)}$$

or

$$\frac{C_pT_i\,(1-\beta_3\ell)^{(1-\beta_4/\beta_3)} + \frac{1}{2}v_i^2}{C_p T_i + \frac{1}{2}v_i^2} = .9 \quad \text{(cc)}$$

Substituting the given numerical values, we have

PROBLEM 13.4 (continued)

$$\beta_3 = .45 \quad \text{and} \quad \beta_4/\beta_3 = .857$$

Solving for ℓ, we obtain

$$\ell \approx 1.3 \text{ meters.}$$

PROBLEM 13.5

We are given the following relations:

$$\frac{B(z)}{B_i} = \frac{E(z)}{E_i} = \frac{w_i}{w(z)} = \frac{d_i}{d(z)} = \left(\frac{A_i}{A(z)}\right)^{1/2}$$

and that v, σ, γ, and K are constant.

Part a

From Eq. (13.2.33),

$$J = (1-K)\sigma vB \tag{a}$$

For constant velocity, conservation of momentum yields

$$\frac{dp}{dz} = -(1-K)\sigma vB^2 \tag{b}$$

Conservation of energy yields

$$\rho vC_p\frac{dT}{dz} = -K(1-K)\sigma(vB)^2 \tag{c}$$

Using the equation of state,

$$p = \rho RT \tag{d}$$

we obtain

$$T\frac{d\rho}{dz} + \rho\frac{dT}{dz} = -\frac{(1-K)}{R}\sigma vB^2 \tag{e}$$

or

$$T\frac{d\rho}{dz} + \frac{(-K)(1-K)\sigma vB^2}{C_p} = -\frac{(1-K)\sigma vB^2}{R} \tag{f}$$

Thus,

$$T\frac{d\rho}{dz} = \sigma vB^2(1-K)\left(-\frac{1}{R} + \frac{K}{c_p}\right) \tag{g}$$

Also

$$B^2 = \frac{B_i^2(A_i)}{A(z)}$$

and

$$\rho_i A_i = \rho(z)A(z)$$

Therefore

$$T\frac{d\rho}{dz} = \frac{\sigma vB_i^2\ (1-K)(-\frac{1}{R} + \frac{K}{c_p})}{\rho_i}\rho(z) \tag{h}$$

and

$$\rho\, c_p\frac{dT}{dz} = -K(1-K)\sigma v\frac{B_i^2\rho}{\rho_i} \tag{i}$$

PROBLEM 13.5 (continued)

and so

$$\frac{dT}{dz} = -\frac{K(1-K)\sigma v B_i^2}{\rho_i c_p} \qquad \text{(j)}$$

Therefore

$$T = -K(1-K)\frac{\sigma v B_i^2}{\rho_i c_p} z + T_i \qquad \text{(k)}$$

Let

$$\alpha = \frac{-K(1-K)\sigma v B_i^2}{\rho_i c_p} \qquad (\ell)$$

Then

$$T = T_i\left(\frac{\alpha z}{T_i} + 1\right) \qquad \text{(m)}$$

$$\frac{d\rho}{\rho} = \frac{+\sigma v B_i^2(1-K)\left(\frac{K}{c_p} - \frac{1}{R}\right)}{\rho_i\,(\alpha z + T_i)} dz \qquad \text{(n)}$$

We let

$$\beta = \frac{+\sigma v B_i^2(1-K)\left(\frac{K}{c_p} - \frac{1}{R}\right)}{\rho_i\ \alpha}$$

$$= \frac{c_p}{KR} - 1$$

Integrating (n), we then obtain

$$\ell n\ \rho = \beta\ \ell n(\alpha z + T_i) + \text{constant}$$

or

$$\rho = \rho_i\left(\frac{\alpha z}{T_i} + 1\right)^\beta \qquad \text{(o)}$$

Therefore

$$A(z) = \frac{A_i}{\left(\frac{\alpha z}{T_i} + 1\right)^\beta} \qquad \text{(p)}$$

Part b

From (m),

$$\frac{T(\ell)}{T_i} = \frac{\alpha\ell + T_i}{T_i} = .8$$

or

$$\frac{\alpha\ell}{T_i} = -.2$$

Now

$$\frac{\alpha}{T_i} = -\frac{K(1-K)\sigma v_i B_i^2}{\rho_i c_p T_i}$$

But

$$c_p T_i = \frac{R\ T_i}{(1-\frac{1}{\gamma})} = \frac{p_i}{\rho_i(1-\frac{1}{\gamma})} = 2.5 \times 10^6$$

PROBLEM 13.5 (Continued)

Thus
$$\frac{\alpha}{T_i} = \frac{-.5(.5)50(700)16}{.7(2.5\times 10^6)} = -8.0\times 10^{-2}$$

Solving for ℓ, we obtain

$$\ell = \frac{.2}{8}\times 10^2 = 1.25 \text{ meters}$$

Part c

$$\rho = \rho_i\left(\frac{\alpha z}{T_i}+1\right)^{\beta}$$

Numerically

$$\beta = \frac{c_p}{KR} - 1 = \frac{1}{(1-\frac{1}{\gamma})K} - 1 \approx 6.$$

Thus

$$\rho(z) = .7(1-.08z)^{6}$$

Then it follows:

$$p(z) = \rho RT = p_i(1-.08z)^7 = 5\times 10^5(1-.08z)^7$$

$$T(z) = T_i(1-.08z)$$

From the given information, we cannot solve for T_i, only for

$$RT_i = \frac{p_i}{\rho_i} = \frac{v_i^{\,2}}{\gamma M_i^2} \approx 7\times 10^5$$

Now
$$M^2(z) = \frac{v_i^{\,2}}{\gamma RT(z)} = \frac{v_i^2}{\gamma p(z)}\rho(z) = \frac{v_i^2}{\gamma}\,\frac{\rho_i\left(\frac{\alpha z}{T_i}+1\right)^{\beta}}{p_i\left(\frac{\alpha z}{T_i}+1\right)^{(\beta+1)}}$$

$$= \frac{.5}{1-.08z}$$

Part d

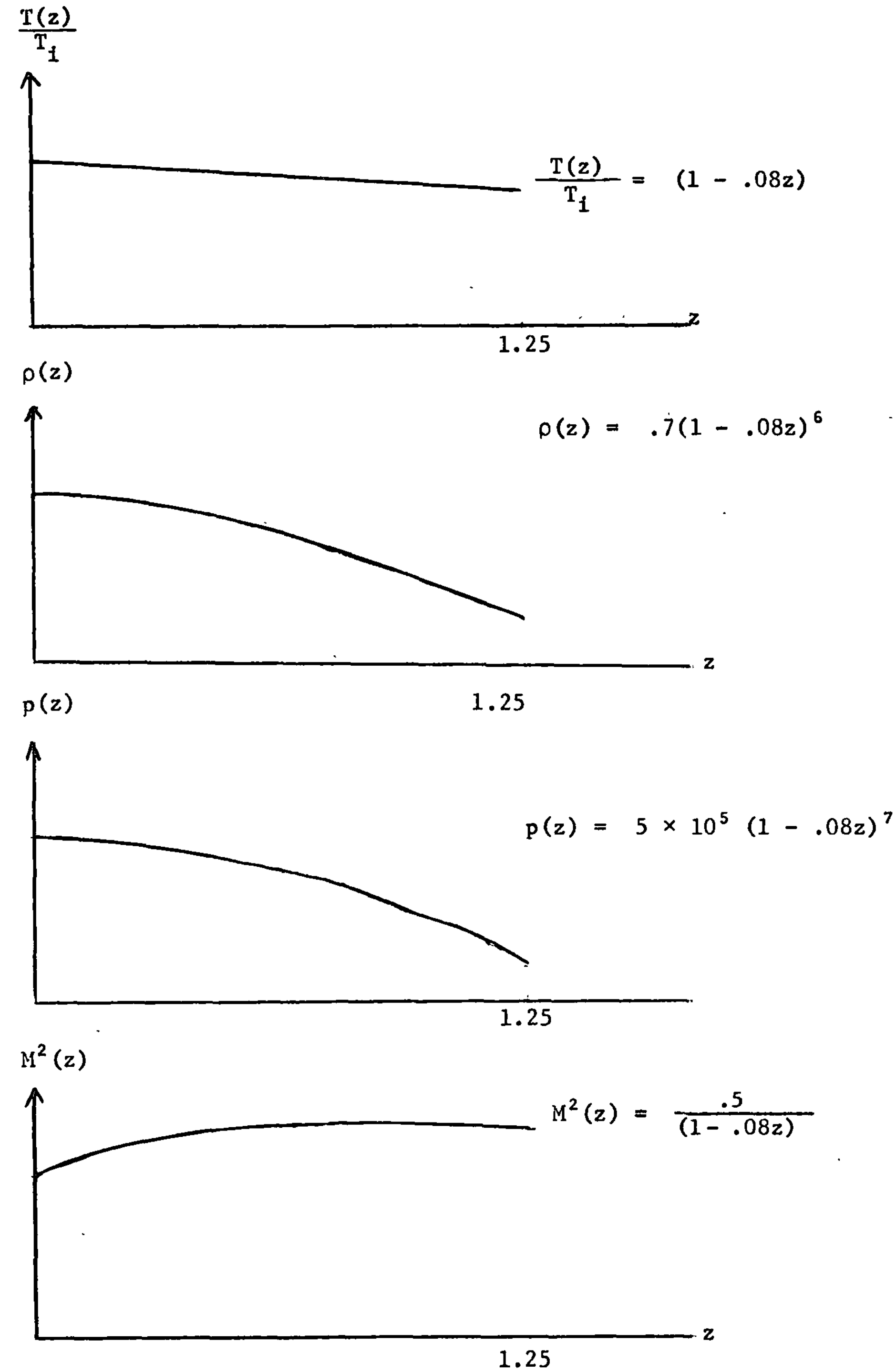
The total electric power drawn from this generator is

$$P^e = VI = -E(z)w(z)J(z)\ell d(z)$$
$$= -E(z)(1-K)\sigma vB(z)\ell d(z)w(z)$$
$$= -E_i w_i(1-K)\sigma vB_i d_i\ell$$

But

$$E_i = -KvB_i$$

Thus
$$P^e = K(vB_i)^2\, w_i d_i\sigma(1-K)\ell$$
$$= .5(700)^2 16(.5)50(.5)1.25$$
$$= 61.3\times 10^6 \text{ watts} = 61.3 \text{ megawatts}$$


T(z)/T_i
T(z)/T_i = (1 - .08z)
z
1.25
ρ(z)
ρ(z) = .7(1 - .08z)^6
z
1.25
p(z)
p(z) = 5 × 10^5 (1 - .08z)^7
1.25
M^2(z)
M^2(z) = .5/(1 - .08z)
z
1.25

PROBLEM 13.6

Part a

We are given that

$$\bar{E} = \bar{i}_x \frac{4}{3} \frac{V_o}{L^{4/3}} x^{1/3} \tag{a}$$

and

$$\rho_e = \frac{4}{9} \frac{\varepsilon_o V_o}{L^{4/3} x^{2/3}} \tag{b}$$

The force equation in the steady state is

$$\rho_m v_x \frac{dv_x}{dx} \bar{i}_x = \rho_e \bar{E} \tag{c}$$

Since $\rho_e/\rho_m = q/m =$ constant, we can write

$$\frac{d}{dx}\left(\frac{1}{2} v_x^2\right) = \frac{q}{m} \frac{4}{3} \frac{V_o}{L^{4/3}} x^{1/3} \tag{d}$$

Solving for v_x we obtain

$$v_x = \sqrt{\frac{2q}{m} V_o \left(\frac{x}{L}\right)^{2/3}} \tag{e}$$

Part b

The total force per unit volume acting on the accelerator system is

$$\bar{F} = \rho_e \bar{E} \tag{f}$$

Thus, the total force which the fixed support must exert is

$$\bar{f}_{total} = -\int F dV \bar{i}_x$$

$$= -\int_0^{\ell} \frac{16}{27} \frac{\varepsilon_o V_o^2}{L^{8/3}} x^{-1/3} A dx\, \bar{i}_x$$

$$\bar{f}_{total} = -\frac{8}{9} \frac{\varepsilon_o V_o^2}{L^2} A\, \bar{i}_x$$

PROBLEM 13.7

Part a

We refer to the analysis performed in section 13.2.3a. The equation of motion for the velocity is, Eq. (13.2.76),

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x_1^2} \tag{a}$$

The boundary conditions are

$$v(-L) = V_o \cos \omega t$$

$$v(0) = 0$$

We write the solution in the form

PROBLEM 13.7 (continued)

$$v(x_1 t) = \text{Re}[A\, e^{j(\omega t - kx_1)} + B\, e^{j(\omega t + kx_1)}] \qquad \text{(b)}$$

where $k = \frac{\omega}{a}$

Using the boundary condition at $x_1 = 0$, we can alternately write the solution as

$$v = \text{Re}[A \sin kx_1 e^{j\omega t}]$$

Applying the other boundary condition at $x_1 = -L$, we finally obtain

$$v(x_1,t) = -\frac{V_o}{\sin kL} \sin kx_1 \cos \omega t. \qquad \text{(d)}$$

The perturbation pressure is related to the velocity through Eq. (13.2.74)

$$\rho_o \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial x_1} \qquad \text{(e)}$$

Solving, we obtain

$$\frac{\rho_o V_o \omega}{\sin kL} \sin kx_1 \sin \omega t = -\frac{\partial p'}{\partial x_1} \qquad \text{(f)}$$

or

$$p' = \frac{\rho_o V_o \omega}{k \sin kL} \cos kx_1 \sin \omega t \qquad \text{(g)}$$

where ρ_o is the equilibrium density, related to the speed of sound a, through Eq. (13.2.83).

Thus, the total pressure is

$$p = p_o + p' = p_o + \frac{\rho_o V_o \omega}{k \sin kL} \cos kx_1 \sin \omega t \qquad \text{(h)}$$

and the perturbation pressure at $x_1 = -L$ is

$$p'(-L, t) = \frac{\rho_o V_o a}{\sin kL} \cos kL \sin \omega t \qquad \text{(i)}$$

Part b

There will be resonances in the pressure if

$$\sin kL = 0 \qquad \text{(j)}$$

or

$$kL = n\pi \qquad n = 1,2,3.... \qquad \text{(k)}$$

Thus

$$\omega = \frac{n\pi}{L} a \qquad (\ell)$$

PROBLEM 13.8

Part a

We carry through an analysis similar to that performed in section 13.2.3b. We assume that

$$\bar{E} = \bar{i}_2 E_2(x_1,t)$$

$$\bar{J} = \bar{i}_2 J_2(x_1,t)$$

PROBLEM 13.8

$$\overline{B} = \overline{i}_3\,[\mu_o H_o + \mu_o H_3'(x_1,t)]$$

Conservation of momentum yields

$$\rho\frac{Dv_1}{Dt} = -\frac{\partial p}{\partial x_1} + J_2\mu_o(H_o + H_3') \qquad (a)$$

Conservation of energy gives us

$$\rho\frac{D}{Dt}\left(u + \frac{1}{2}v_1^2\right) = -\frac{\partial}{\partial x_1}(pv_1) + J_2E_2 \qquad (b)$$

We use Ampere's and Faraday's laws to obtain

$$\frac{\partial H_3'}{\partial x_1} = -J_2 \qquad (c)$$

and

$$\frac{\partial E_2}{\partial x_1} = -\frac{\mu_o\partial H_3'}{\partial t} \qquad (d)$$

while

Ohm's law yields

$$J_2 = \sigma[E_2 - v_1B_3] \qquad (e)$$

Since $\sigma \to \infty$

$$E_2 = v_1B_3 \qquad (f)$$

We linearize, as in Eq. (13.2.91), so $E_2 \approx v_1\mu_oH_o$

Substituting into Faraday's law

$$\mu_oH_o\frac{\partial v_1}{\partial x_1} = -\mu_o\frac{\partial H_3'}{\partial t} \qquad (g)$$

Linearization of the conservation of mass yields

$$\frac{\partial\rho'}{\partial t} = -\rho_o\frac{\partial v_1}{\partial x_1} \qquad (h)$$

Thus, from (g)

$$\frac{\mu_oH_o}{\rho_o}\frac{\partial\rho'}{\partial t} = \mu_o\frac{\partial H_3'}{\partial t} \qquad (i)$$

Then

$$\frac{H_o}{H_3'} = \frac{\rho_o}{\rho'}$$

Linearizing Eq. (13.2.71), we obtain

$$\frac{Dp'}{Dt} = \frac{\gamma p_o}{\rho_o}\frac{D\rho'}{Dt} \qquad (k)$$

PROBLEM 13.8 (continued)

Defining the acoustic speed

$$a_s = \left(\frac{\gamma p_o}{\rho_o}\right)^{1/2} \text{ where } p_o \text{ is the equilibrium pressure,}$$

$$p_o = p_1 - \frac{\mu_o H_o^2}{2}$$

we have

$$p' = a_s^2 \rho' \tag{ℓ}$$

Linearization of convervation of momentum (a) yields

$$\rho_o \frac{\partial v_1}{\partial t} = -\frac{\partial p'}{\partial x_1} - \frac{\partial H_3'}{\partial x_1} \mu_o H_o \tag{m}$$

or, from (j) and (ℓ),

$$\rho_o \frac{\partial v_1}{\partial t} = \frac{\partial \rho'}{\partial x_1}\left(- a_s^2 - \frac{\mu_o H_o^2}{\rho_o}\right) \tag{n}$$

Differentiating (n) with respect to time, and using conservation of mass (h), we finally obtain

$$\frac{\partial^2 v_1}{\partial t^2} = \left(a_s^2 + \frac{\mu_o H_o^2}{\rho_o}\right)\frac{\partial^2 v_1}{\partial x_1^2} \tag{o}$$

Defining

$$a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \tag{p}$$

we have

$$\frac{\partial^2 v_1}{\partial t^2} = a^2 \frac{\partial^2 v_1}{\partial x_1^2} \tag{q}$$

Part b

We assume solutions of the form

$$V_1 = \text{Re}\ [A_1 e^{j(\omega t - kx_1)} + A_2 e^{j(\omega t + kx_1)}] \tag{r}$$

where $k = \frac{\omega}{a}$

The boundary condition at $x_1 = -L$ is

$$V(-L,t) = V_s \cos \omega t = V_s \text{Re}\ e^{j\omega t} \tag{s}$$

and at $x_1 = 0$

$$M \frac{dv_1(0,t)}{dt} = p'A\Big|_{x_1=0} + \mu_o H_o H_3' A\Big|_{x_1=0} \tag{t}$$

From (h), (j) and (ℓ),

$$\frac{1}{a_s^2}\frac{\partial p'}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \tag{u}$$

PROBLEM 13.8 (continued)

$$\frac{H_3'}{H_o} = \frac{p'}{a_s^2\rho_o} \tag{v}$$

Thus

$$M\frac{dv_1(0,t)}{dt} = A\left(\frac{\mu_o H_o^2}{a_s^2\rho_o} + 1\right)p' = A\frac{a^2}{a_s^2}p' \tag{w}$$

From (u), we solve for $p'\big|_{x_1=0}$ to obtain:

$$p'\Big|_{x_1=0} = -\frac{\rho_o a_s^2 k}{w}(A_2 - A_1)e^{j\omega t} \tag{x}$$

Substituting into (s) and (t), we have

$$Mjw(A_1 + A_2) = A\left(\frac{a}{a_s}\right)^2\left(\frac{\rho_o a_s^2 k}{w}\right)(A_1 - A_2)$$

and

$$A_1 e^{+jk\ell} + A_2 e^{-jk\ell} = V_s \tag{y}$$

Solving for A_1 and A_2, we obtain

$$A_1 = \frac{(Mjw + Aa\rho_o)V_s}{2(-Mw\sin k\ell + Aa\rho_o\cos k\ell)}$$

$$A_2 = \frac{(Aa\rho_o - Mjw)V_s}{2(-Mw\sin k\ell + Aa\rho_o\cos k\ell)} \tag{z}$$

Thus, the velocity of the piston is

$$v_1(0,t) = \text{Re}\ [A_1 + A_2]e^{j\omega t}$$

$$v_1(0,t) = \frac{Aa\rho_o V_s}{-Mw\sin k\ell + Aa\rho_o\cos k\ell}\cos\omega t \tag{aa}$$

PROBLEM 13.9

Part a

The differential equation for the velocity as derived in problem 13.8 is

$$\frac{\partial^2 v_1}{\partial t^2} = a^2\frac{\partial^2 v_1}{\partial x_1^2} \tag{a}$$

where

$$a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o}$$

with

$$a_s^2 = \left(\frac{\gamma p_o}{\rho_o}\right)^{1/2} \quad \text{where } p_o = p_1 - \frac{\mu_o H_o^2}{2}$$

Part b

We assume a solution of the form

PROBLEM 13.9 (continued)

$$V(x_1,t) = \text{Re }[De^{j(\omega t - kx_1)}] \quad \text{where } k = \frac{w}{a}$$

We do not consider the negatively traveling wave, as we want to use this system as a delay line without distortion. The boundary condition at $x_1 = -L$ is

$$V(-L,t) = \text{Re } V_s e^{j\omega t}$$

and at $x_1 = 0$ is

$$\cancel{M \frac{dV_1(0,t)}{dt}} \to 0 = p'(0,t)A - BV_1(0,t) + \mu_o H_o H_3' A \tag{b}$$

From problem 13.8, (h), (j) and (ℓ)

$$p' = a_s^2 \rho' \quad , \quad \frac{\partial \rho'}{\partial t} = -\rho_o \frac{\partial v_1}{\partial x_1} \quad \text{and} \quad \frac{H_3'}{H_o} = \frac{p'}{a_s^2 \rho_o}$$

Thus, (b) becomes

$$-BDe^{j\omega t} + \left(\frac{a}{a_s}\right)^2 p'A = 0 \tag{c}$$

where

$$p'\Big|_{x_1=0} = -\frac{\rho_o D(-jk)}{jw} a_s^2 e^{j\omega t} \tag{d}$$

Thus, for no reflections

$$-B + \left(\frac{a}{a_s}\right)^2 \frac{A\rho_o a_s^2}{a} = 0 \tag{e}$$

or

$$B = Aa\rho_o \tag{f}$$

PROBLEM 13.10

The equilibrium boundary conditions are

$$T[-(L_1 + L_2 + \Delta),t] = T_o$$

$$T[-(L_1 + \Delta),t]A_s = -p_o A_c$$

Boundary conditions for incremental motions are

1) $T[-(L_1 + L_2 + \Delta),t] = T_s(t)$
2) $-T[-(L_1 + \Delta),t]A_s - p(-L_1,t)A_c = M \frac{d}{dt} v_\ell(-L_1,t)$
3) $v_\ell(-L_1,t) = v_e[-(L_1 + \Delta),t]$ since the mass is rigid

and 4) $v_\ell(0,t) = 0$ since the wall at x=0 is fixed.

PROBLEM 13.11

Part a

We can immediately write down the equation for perturbation velocity, using equations (13.2.76) and (13.2.77) and the results of chapters 6 and 10.

PROBLEM 13.11 (continued)

We replace $\partial/\partial t$ by $\partial/\partial t + v\cdot\nabla$ to obtain

$$\left(\frac{\partial}{\partial t} + V_o \frac{\partial}{\partial x}\right)^2 v' = a_s^2 \frac{\partial^2 v'}{\partial x^2}$$

Letting $v' = \text{Re}\,\hat{V}\,e^{j(\omega t - kx)}$

we have

$$(\omega - kV_o)^2 = a_s^2 k^2$$

Solving for w, we obtain

$$\omega = k(V_o \pm a_s)$$

Part b

Solving for k, we have

$$k = \frac{\omega}{V_o \pm a_s}$$

For $V_o > a_s$, both waves propagate in the positive x- direction.

PROBLEM 13.12

Part a

We assume that

$$\bar{E} = \bar{i}_z E_z(x,t)$$
$$\bar{J} = \bar{i}_z J_z(x,t)$$
$$\bar{B} = \bar{i}_y \mu_o[H_o + H_y'(x,t)]$$

We also assume that all quantities can be written in the form of Eq. (13.2.91) .

$$\rho_o \frac{\partial v_x}{\partial t} = -\frac{\partial p'}{\partial x} - J_z \mu_o H_o \quad \text{(conservation of momentum linearized)} \tag{a}$$

The relevant electromagnetic equations are

$$\frac{\partial H_y'}{\partial x} = J_z \tag{b}$$

and

$$\frac{\partial E_z}{\partial x} = \mu_o \frac{\partial H_y'}{\partial t} \tag{c}$$

and the constitutive law is

$$J_z = \sigma(E_z + v_x \mu_o H_o) \tag{d}$$

We recognize that Eqs. (13.2.94), (13.2.96) and (13.2.97) are still valid, so

$$\frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} = -\frac{\partial v_x}{\partial x} \tag{e}$$

PROBLEM 13.12 (continued)

and

$$p' = a_s^2\rho' \qquad (f)$$

Part b

We assume all perturbation quantities are of the form

$$v_x = \mathrm{Re}[\hat{v}\, e^{j(\omega t - kx)}]$$

Using (b), (a) may be rewritten as

$$\rho_o j\omega\hat{v} = +jk\hat{p} + \mu_o H_o jk\hat{H} \qquad (g)$$

and (c) may now be written as

$$-jk\hat{E} = \mu_o j\omega\hat{H} \qquad (h)$$

Then, from (b) and (d)

$$-jk\hat{H} = \sigma(\hat{E} + \hat{v}\mu_o H_o) \qquad (i)$$

Solving (g) and (h) for $\hat{H}$ in terms of $\hat{v}$, we have

$$\hat{H} = \frac{\hat{v}\sigma\mu_o H_o}{\left(-jk + \sigma\mu_o \frac{\omega}{k}\right)} \qquad (j)$$

From (e) and (f), we solve for $\hat{p}$ in terms of $\hat{v}$ to be

$$\hat{p} = \frac{k}{\omega}\rho_o a_s^2 \hat{v} \qquad (k)$$

Substituting (j) and (k) back into (g), we find

$$\hat{v}\left[\rho_o j\omega - \frac{jk^2}{\omega}\rho_o a_s^2 - \frac{jk(\mu_o H_o)^2\sigma}{\left[-jk + \frac{\sigma\mu_o\omega}{k}\right]}\right] = 0 \qquad (\ell)$$

Thus, the dispersion relation is

$$(\omega^2 - k^2 a_s^2) - \frac{j(\mu_o H_o)^2 \omega\, k^2}{(+\frac{k^2}{\sigma} + j\mu_o\omega)\rho_o} = 0 \qquad (m)$$

We see that in the limit as $\sigma \to \infty$, this dispersion relation reduces to the lossless dispersion relation

$$\omega^2 - k^2\left(a_s^2 + \frac{\mu_o H_o^2}{\rho_o}\right) = 0 \qquad (n)$$

Part c

If σ is very small, we can approximate (m) as

$$\omega^2 - k^2 a_s^2 - j(\mu_o H_o)^2 \frac{\omega\sigma}{\rho_o}\left(1 - \frac{j\omega\mu_o\sigma}{k^2}\right) = 0 \qquad (o)$$

for which we can rewrite (o) as

PROBLEM 13.12 (continued)

$$k^4 a_s^2 - k^2\left[\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}\right] + \left(\frac{\mu_o H_o}{\rho_o}\right)^2 \omega^2\sigma^2\mu_o = 0 \qquad \text{(p)}$$

Solving for k^2, we obtain

$$k^2 = \frac{\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}}{2a_s^2} \pm \sqrt{\left[\frac{\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}}{2a_s^2}\right]^2 - \frac{\left(\frac{\mu_o\omega^2\sigma^2}{\rho_o}\right)(\mu_o H_o)^2}{a_s^2}} \qquad \text{(q)}$$

Since σ is very small, we expand the radical in (q) to obtain

$$k^2 = \frac{\left[\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}\right]}{2a_s^2} \pm \left[\frac{\omega^2 - \frac{j\omega\sigma}{\rho_o}(\mu_o H_o)^2}{2a_s^2} - \frac{\left(\frac{\mu_o\omega^2\sigma^2}{\rho_o}\right)(\mu_o H_o)^2}{\left[\omega^2 - \frac{j\omega\sigma}{\rho_o}(\mu_o H_o)^2\right]}\right] \qquad \text{(r)}$$

Thus, our approximate solutions for k^2 are

$$k^2 \approx \frac{\left[\omega^2 - j\omega\sigma\frac{(\mu_o H_o)^2}{\rho_o}\right]}{a_s^2} \qquad \text{(s)}$$

and

$$k^2 \approx \frac{\left(\frac{\mu_o\omega^2\sigma^2}{\rho_o}\right)(\mu_o H_o)^2}{\left[\omega^2 - \frac{j\omega\sigma}{\rho_o}(\mu_o H_o)^2\right]} \approx \left(\frac{\mu_o\sigma^2}{\rho_o}\right)(\mu_o H_o)^2 \qquad \text{(t)}$$

The wavenumbers for the first pair of waves are approximately:

$$k \approx \pm\left(\frac{\omega - j\frac{\sigma}{2\rho_o}(\mu_o H_o)^2}{a_s}\right) \qquad \text{(u)}$$

while for the second pair, we obtain

$$k \approx \pm\ \sigma(\mu_o H_o)\sqrt{\frac{\mu_o}{\rho_o}} \qquad \text{(v)}$$

The wavenumbers from (u) represent a forward and backward traveling wave, both with amplitudes exponentially decreasing. Such waves are called 'diffusion waves'. The wavenumbers from (v) represent pure propagating waves in the forward and reverse directions.

PROBLEM 13.12 (continued)

Part d

If σ is very large, then (m) reduces to

$$\omega^2 - k^2 a^2 - j\,\frac{H_o^2}{\rho_o}\,\frac{k^4}{\sigma\omega} = 0 \;;\quad a^2 = a_s^2 + \frac{\mu_o H_o^2}{\rho_o} \tag{w}$$

This can be put in the form

$$k^2 = \frac{\omega^2}{a^2} - j\,\frac{f(\omega,k)}{\sigma} \tag{x}$$

where

$$f(\omega,k) = \frac{H_o^2\;k^4}{\rho_o\;\omega a^2}$$

As σ becomes very large, the second term in (x) becomes negligible, and so

$$k^2 \approx \frac{\omega^2}{a^2} \tag{y}$$

However, it is this second term which represents the damping in space; that is,

$$k \approx \pm\left[\frac{\omega}{a} - j\,\frac{f(\omega,k)}{2\sigma\;\omega}\,a\right] \tag{z}$$

Thus, the approximate decay rate, k_i, is

$$k_i \approx \frac{f(\omega,k)\,a}{2\sigma\;\omega} = \frac{H_o^2\;k^4}{2\rho_o\,\omega a^2\sigma}\,\frac{a}{\omega} \tag{aa}$$

or

$$k_i \approx \frac{H_o^2}{2\rho_o\;a\sigma}\;\frac{k^4}{\omega^2} = \frac{H_o^2}{2\rho_o a^5\sigma}\;\omega^2$$