PROBLEM 14.1

Part a

We can specify the relevant variables as

$$\begin{aligned}\bar{v} &= \bar{i}_1\, v_1(x_2)\\ \bar{E} &= \bar{i}_2 E_2(x_2) + i_3 E_3(x_2)\\ \bar{J} &= \bar{i}_2\, J_o\\ \bar{B} &= \bar{i}_2 B_o + \bar{i}_1 B_1(x_3)\end{aligned} \tag{a}$$

The x_1 component of the momentum equation is

$$0 = \mu \frac{\partial^2 v_1}{\partial x_2^2} \tag{b}$$

with solution

$$v_1 = C_1 x_2 + C_2$$

Applying the boundary conditions

$$\begin{aligned}v_1 &= 0 \quad @ \quad x_2 = 0\\ v_1 &= v_o \quad @ \quad x_2 = d\end{aligned} \tag{c}$$

We obtain

$$v_1 = \frac{v_o x_2}{d} \tag{d}$$

We note that there is no magnetic force density since the imposed current and magnetic field are colinear. We apply Ohm's law for a moving fluid

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B}) \tag{e}$$

in the x_2 and x_3 directions to obtain

$$J_o = \sigma E_2 \tag{f}$$

and

$$0 = \sigma(E_3 + v_1 B_o) \tag{g}$$

since no current can flow in the x_3 direction.

Thus

$$E_2 = \frac{J_o}{\sigma} \tag{h}$$

and

$$E_3 = -\frac{v_o x_2 B_o}{d} \tag{i}$$

As from Eq. (14.2.5),

$$V = \int_0^d E_2 dx_2 = \frac{J_o}{\sigma} d \tag{j}$$

Thus, the electrical input p_e per unit area in an $x_1 - x_3$ plane is

$$p_e = J_o V = \frac{J_o^2 d}{\sigma} \tag{k}$$

PROBLEM 14.1 (continued)

We see that this power is dissipated as Ohmic loss. The moving fluid looks just like a resistor from the electrical terminals. The traction that must be applied to the upper plate to maintain the steady motion is

$$\tau = \mu \frac{\partial v_1}{\partial x_2}\bigg|_{x_2=d} = \frac{\mu v_o}{d} \qquad (\ell)$$

Again we note no contribution from the magnetic forces.

The mechanical input power per unit area is then

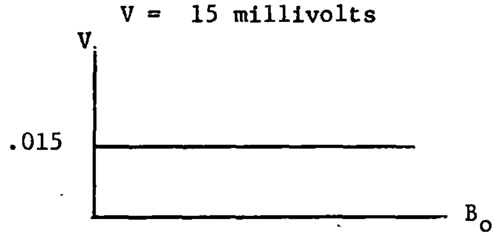
$$p_m = \tau_1 v_o = \frac{\mu v_o^2}{d} \qquad (m)$$

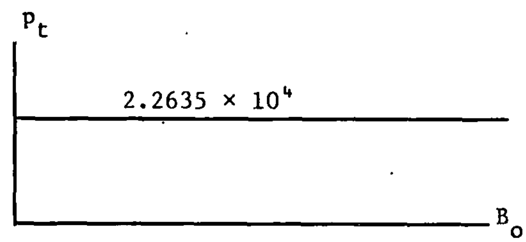
The total input power per unit area is thus

$$p_t = p_e + p_m = \frac{\mu v_o^2}{d} + \frac{J_o^2 d}{\sigma} \qquad (n)$$

The first term is due to viscous loss that results from simple shear flow, while the second term is simply the Joule loss associated with Ohmic heating. There is no electromechanical coupling. Using the parameters from Table 14.2.1, we obtain

V = 15 millivolts





and

$$p_t = 2.2635 \times 10^4 \text{ watts/m}^2, \text{ independent of } B_o.$$

These results correspond to the plots of Fig. 14.2.3 in the limit as

$$B_o \to 0$$

We see that the brush losses and brush voltage are much less for this configuration than for that analyzed in Sec. 14.2.1. This is because the electrical and mechanical equations were uncoupled when the applied flux density was in the x_2 direction. This configuration is better, because low voltages at the brush eliminate arcing, and because the net power input per unit area is less no matter the field strength B_o.

The only effect of applying a flux density in the x_2 direction was to cause an electric field in the x_3 direction. However, since there was no current flow in the x_3 direction, there was no additional dissipated power. However, if E_3 became too large, the fluid might experience electrical breakdown, resulting in corona arcs.

PROBLEM 14.2

The momentum equation for the fluid is

$$\rho \frac{\partial \overline{v}}{\partial t} + \rho(\overline{v}\cdot\nabla)\overline{v} = -\nabla p + \mu\nabla^2\overline{v} \qquad \text{(a)}$$

We consider solutions of the form

$$\overline{v} = \overline{i}_z v_z(r)$$

and

$$p = p(z).$$

Then in the steady state, we write the z component of (a) in cylindrical coordinates as

$$\frac{\partial p}{\partial z} = \mu \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} \qquad \text{(b)}$$

Now, the left side of (b) is only a function of z, while the right side is only a function of r. Thus, from the given information

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L} \qquad \text{(c)}$$

Using the results of (c) in (b), we solve for $v_z(r)$ in the form

$$v_z(r) = \frac{p_2 - p_1}{4L\mu} r^2 + A \ln r + B \qquad \text{(d)}$$

where A and B are arbitrary constants to be evaluated by the boundary conditions

$$v_z(r = 0) \text{ is finite}$$

and

$$v_z(r = R) = 0$$

Thus the solution is

$$v_z(r) = \frac{(p_2 - p_1)}{4\mu L}(r^2 - R^2) \qquad \text{(e)}$$

We can also find relations between the flow rate and the pressure difference, since

$$\int_0^R v_z\, 2\pi r dr = Q$$

PROBLEM 14.3

Part a

We are given the pressure drop Δp, the magnetic field B_o, the conductivity σ, and the dimensions of the system.

Now

$$i = \int_{-d}^{+d} J\ell dx_2 = \sigma\ell \int_{-d}^{+d} (E_3 + v_1 B_o)dx_2 = \frac{V}{R} \qquad \text{(a)}$$

where

$V = -\frac{E}{w}$ is defined as the voltage across the resistor.

From Eq. (14.2.29), we have the solution for the velocity v_1 . We then perform the integrations of (a) and solve for the voltage V to obtain

PROBLEM 14.3(continued)

$$V = \frac{\frac{(\Delta p)2d}{B_o}\left(1 - \frac{\tanh M}{M}\right)}{\frac{1}{R} + \frac{2\sigma \ell d}{w}\frac{\tanh M}{M}} \qquad \text{(b)}$$

where

$$M = B_o d\left(\frac{\sigma}{\mu}\right)^{1/2}$$

Then, the power p^e dissipated in the resistor is

$$p^e = \frac{V^2}{R} = \frac{\left(\frac{\Delta p\ 2d}{B_o}\right)^2\left(1 - \frac{\tanh M}{M}\right)^2}{\left(\frac{1}{R} + \frac{1}{R_i}\frac{\tanh M}{M}\right)^2 R} \qquad \text{(c)}$$

where we have defined the internal resistance R_i as

$$R_i = \frac{w}{2\sigma \ell d}$$

Part b

To maximize p^e, we differentiate (c) with respect to R, solve for that value of R which makes this quantity zero, and then check that this value does indeed maximize p^e. Performing these operations, we obtain

$$R_{max} = \frac{M R_i}{\tanh M} \qquad \text{(d)}$$

Part c

We must convert the given numerical values to MKS units, using the conversions

$$10{,}000 \text{ gauss} = 1 \text{ Weber/meter}^2$$

and $\quad 100 \text{ cm} = 1 \text{ meter}$

For mercury

$$\sigma = 10^6 \text{ mhos/m}$$

and $\mu = 1.5 \times 10^{-3}$ kg/m-sec.

Thus

$$M = B_o d\left(\frac{\sigma}{\mu}\right)^{1/2} = 2 \times 10^{-2}\left(\frac{1}{1.5} \times 10^9\right)^{1/2}$$

$$M = 520$$

Then $\tanh M \approx 1$

and so

$$R_{max} = 520\left(\frac{10^{-1}}{2 \times 10^6 \times 10^{-2}}\right) \approx 2.60 \times 10^{-3} \text{ ohms.}$$