

26

Feedback Example: The Inverted Pendulum

In this lecture, we analyze and demonstrate the use of feedback in a specific system, the inverted pendulum. The system consists of a cart that can be pulled forward or backward on a track. Mounted on the cart is an inverted pendulum, i.e., a pendulum pivoted at its base and with the weight at the top. Consequently, with the cart stationary, the pendulum is unstable; even if balanced in unstable equilibrium, any external disturbance will cause the pendulum to fall.

To balance the pendulum, the cart can be moved forward or backward under the weighted rod. Thus, the forces acting on the system are the cart acceleration and any external disturbances. In principle, if all of the external disturbances are exactly specified (an unreasonable assumption) and if the system dynamics are precisely understood, then the cart acceleration can be specified in such a way as to maintain the rod in a vertical position. A more reasonable approach to balancing the rod or, equivalently, stabilizing this unstable system, however, is to constantly measure the angle of the rod and choose the cart acceleration based on this measurement. This then corresponds to a feedback system in which the measured angle is fed back through an appropriate choice of feedback dynamics to control the cart acceleration.

We carry out an analysis of the open-loop system and explore several possible choices for the feedback dynamics. Under the assumption that the angular displacement of the rod from perpendicular is kept small, the behavior of the inverted pendulum can be described through a second-order linear constant-coefficient differential equation, or equivalently through a system function with two real-axis poles, one in the left half of the s -plane and the other in the right half of the s -plane, and therefore associated with the system instability. A first, more or less obvious, choice for the feedback dynamics is to simply choose the cart acceleration proportional to the measured angle. The resulting system function again has two poles, the positions of which are, of course, dependent on the feedback gain. Examination of the locus of these poles as a function of gain (often referred to as the *root locus*) shows that for the feedback gain negative, the right half-plane pole moves even further into the right half-plane so that the instability becomes even more severe. If the feedback gain is greater than zero, as the gain increases the two poles move

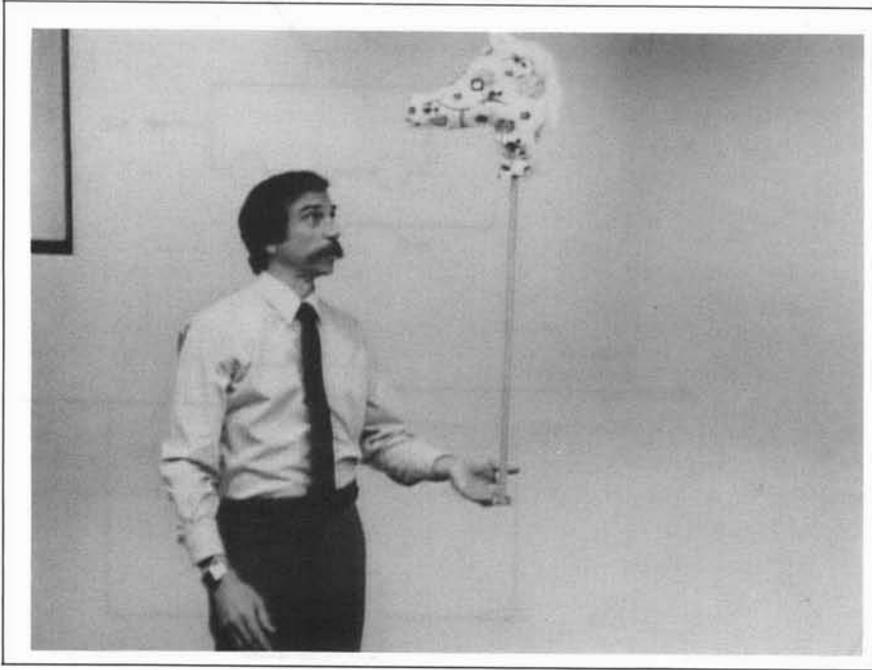
toward each other, meeting at the origin and then traveling along the imaginary axis. The presence of a pole in the right half-plane indicates that even with a very small input (such as a small displacement of the rod), the rod angle will increase exponentially. With the poles on the imaginary axis, any displacement will result in an oscillatory behavior. Consequently, the system remains unstable for all values of the feedback gain.

A second type of feedback to consider corresponds to choosing the cart acceleration proportional to the derivative of the angular displacement. This choice is motivated by the possibility that perhaps the pendulum can be stabilized by accelerating the cart faster if the angular displacement is increasing faster. Examination of the root locus with derivative feedback demonstrates again that for the feedback gain less than zero the system becomes increasingly unstable while with the feedback gain greater than zero the system becomes more stable but is never completely stabilized. Finally, we consider a combination of proportional plus derivative feedback, and in this case by appropriate choice of the two gain factors the system can be stabilized.

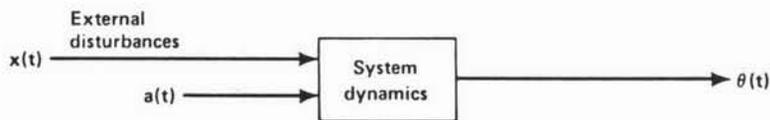
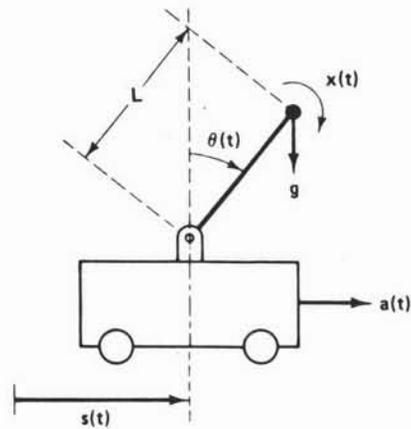
The demonstration accompanying this lecture, besides being entertaining, is hopefully very instructive. In addition to showing that the system can in fact be stabilized by appropriate choice of feedback, we are able to show that feedback is able to compensate for external disturbances and to changes in the system dynamics.

Suggested Reading

Section 11.3, Root-Locus Analysis of Linear Feedback Systems, pages 700–704



DEMONSTRATION 26.1
A simple inverted pendulum.

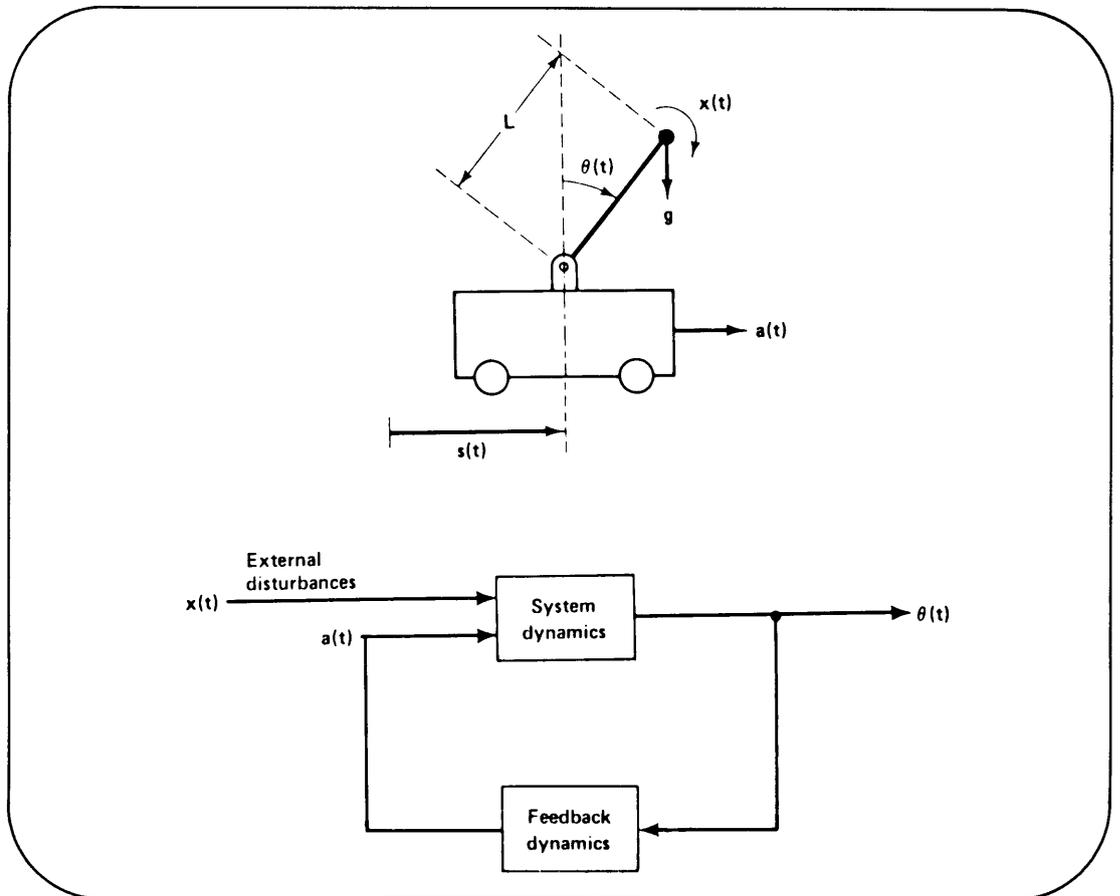


TRANSPARENCY 26.1
The inverted pendulum and a representation of the open-loop system.

TRANSPARENCY

26.2

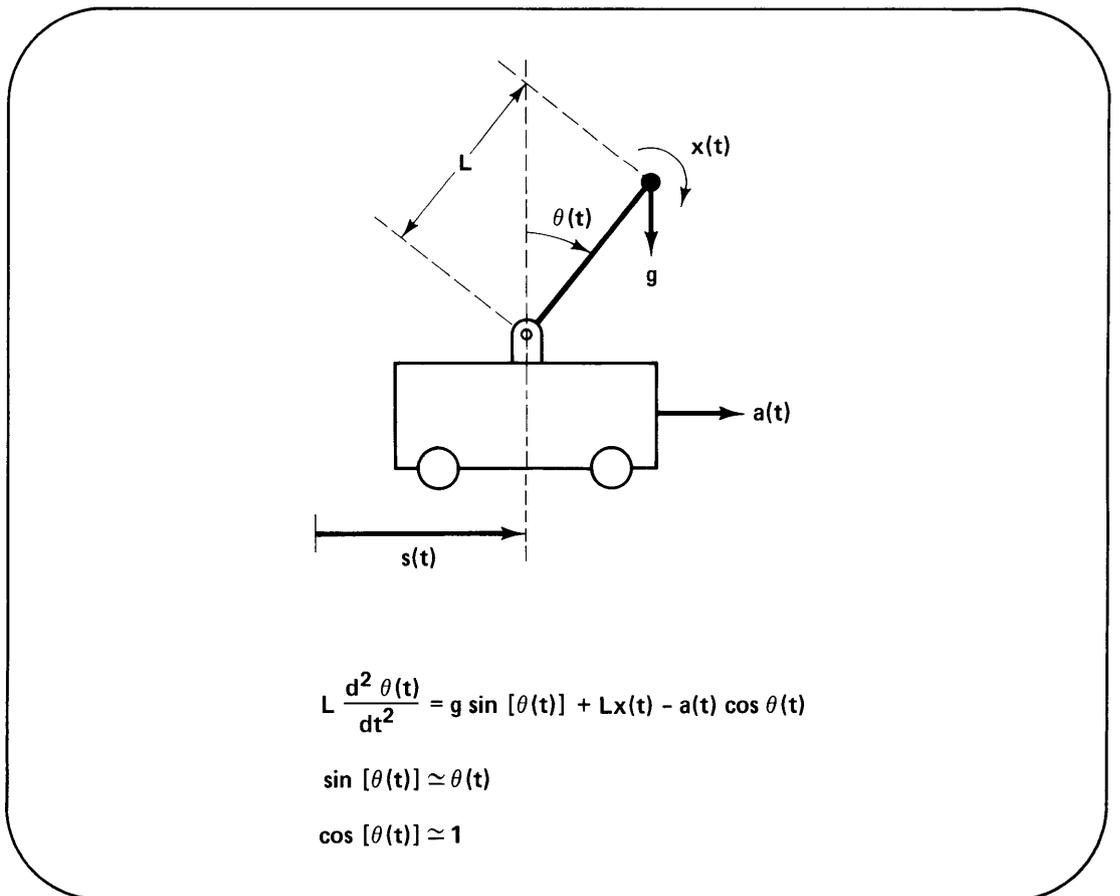
The inverted pendulum with a feedback loop from the pendulum angle to the cart acceleration.



TRANSPARENCY

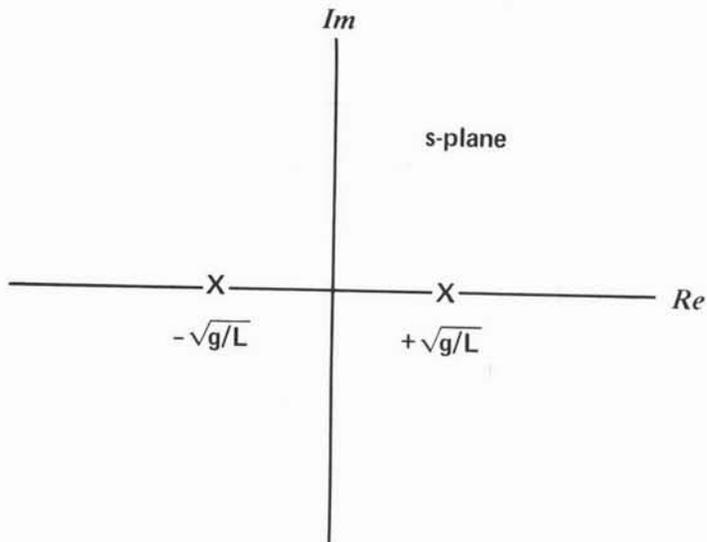
26.3

Equations describing the dynamics of the inverted pendulum.



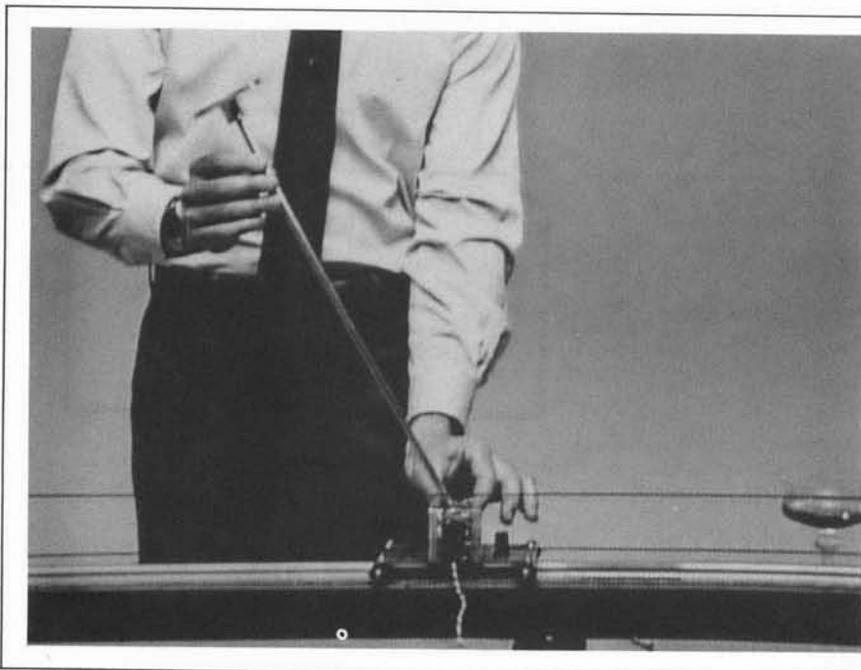
$$L \frac{d^2\theta(t)}{dt^2} - g \theta(t) = Lx(t) - a(t)$$

$$\Theta(s) = \frac{1}{Ls^2 - g} [LX(s) - A(s)]$$



TRANSPARENCY
26.4

Linearized dynamic equation when $\theta(t)$ is small and the corresponding Laplace transform and pole-zero pattern.



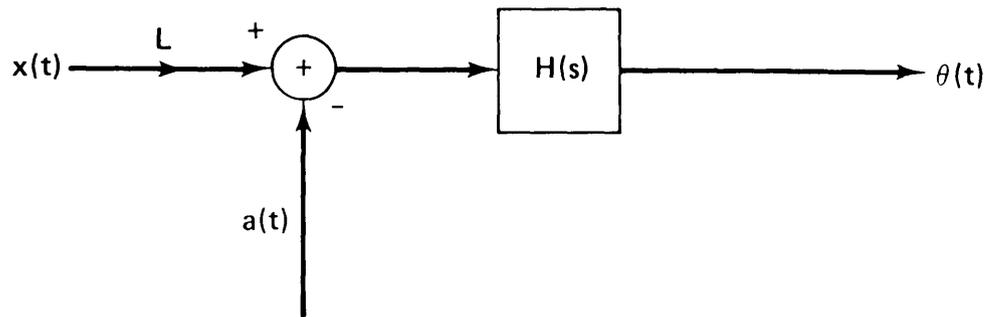
DEMONSTRATION
26.2

The cart and inverted pendulum represented by Transparency 26.2.

TRANSPARENCY
 26.5
 Block diagram and system function for the open-loop system.

$$\Theta(s) = H(s) [LX(s) - A(s)]$$

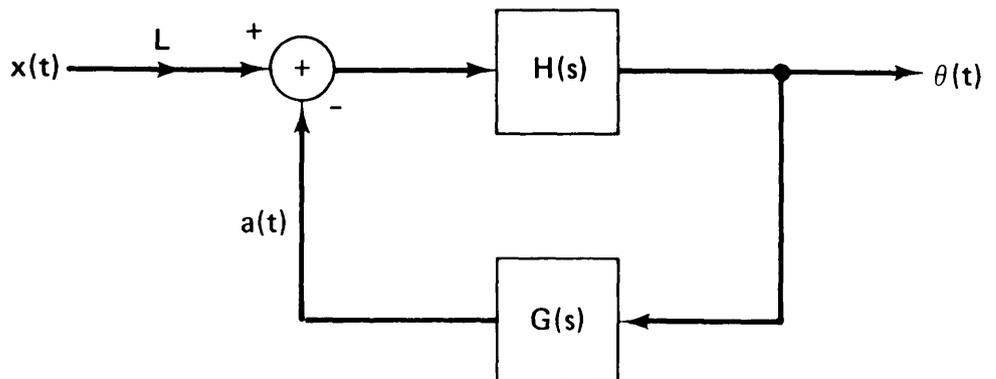
$$H(s) = \frac{1}{Ls^2 - g}$$



TRANSPARENCY
 26.6
 System function and block diagram with feedback.

$$\Theta(s) = H(s) [LX(s) - A(s)]$$

$$H(s) = \frac{1}{Ls^2 - g}$$



$$\Theta(s) = \frac{LH(s)}{1 + G(s)H(s)} X(s)$$

$$\Theta(s) = \frac{LH(s)}{1 + G(s)H(s)} X(s)$$

$$H(s) = \frac{1}{Ls^2 - g}$$

Proportional Feedback:

$$a(t) = K_1 \theta(t)$$

$$G(s) = K_1$$

$$\Theta(s) = \frac{1}{s^2 - \left(\frac{g - K_1}{L}\right)} X(s)$$

$$\text{Poles at } s = \pm \sqrt{\frac{g - K_1}{L}}$$

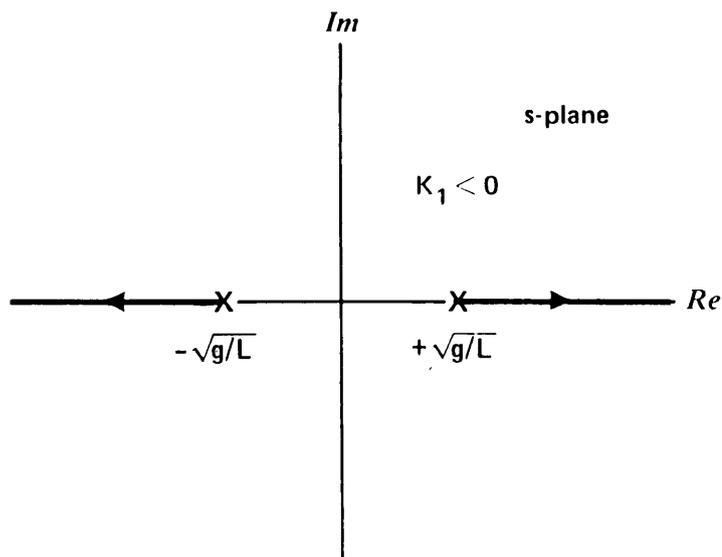
TRANSPARENCY

26.7

System function with
proportional feedback.

Proportional Feedback:

$$\Theta(s) = \frac{1}{s^2 - \left(\frac{g - K_1}{L}\right)} X(s)$$



TRANSPARENCY

26.8

Locus of system
function poles for
proportional feedback
with $k_1 < 0$.

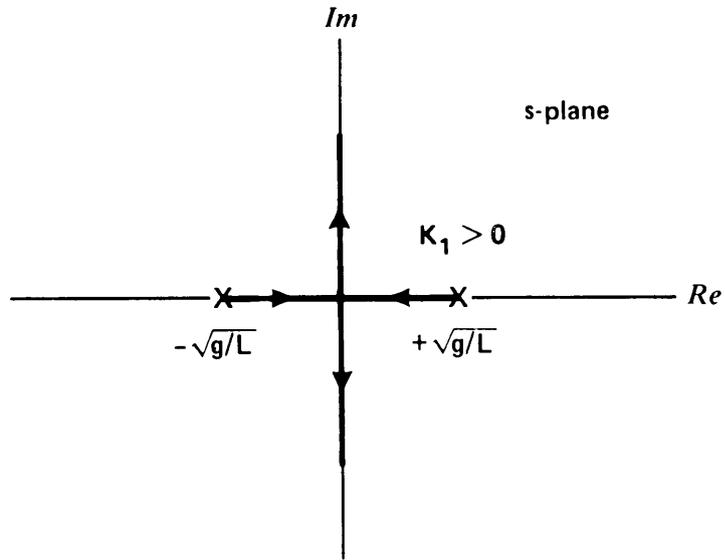
TRANSPARENCY

26.9

Locus of system function poles for proportional feedback with $k_1 > 0$.

Proportional Feedback:

$$\Theta(s) = \frac{1}{s^2 - \left(\frac{g - K_1}{L}\right)} X(s)$$



TRANSPARENCY

26.10

Overall system function with derivative feedback.

$$\Theta(s) = \frac{LH(s)}{1 + G(s)H(s)} X(s)$$

$$H(s) = \frac{1}{Ls^2 - g}$$

Derivative feedback:

$$a(t) = K_2 \frac{d\theta(t)}{dt}$$

$$G(s) = K_2 s$$

$$\Theta(s) = \frac{1}{s^2 + s(K_2/L) - g/L} X(s)$$

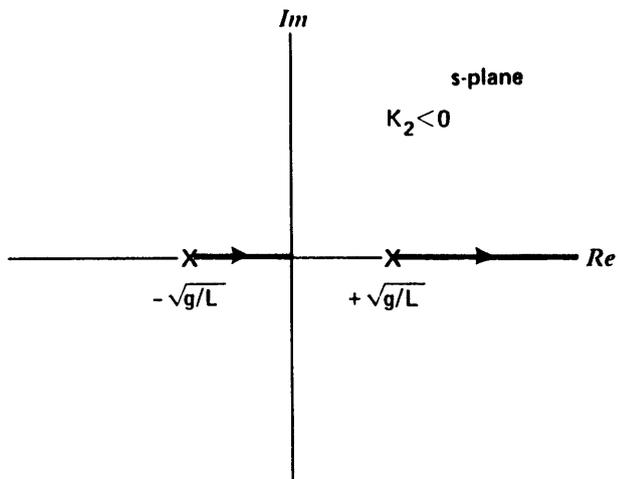
Poles at

$$s = -\frac{K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 + \left(\frac{g}{L}\right)}$$

Derivative feedback:

$$\Theta(s) = \frac{1}{s^2 + s(K_2/L) - g/L} X(s)$$

$$s = -\frac{K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 + \left(\frac{g}{L}\right)}$$

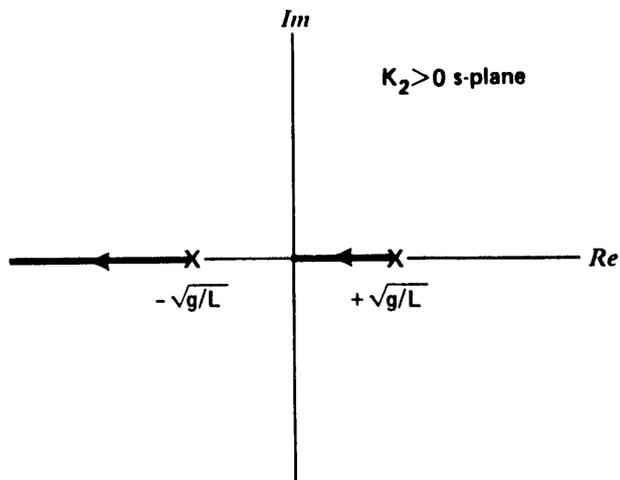


TRANSPARENCY 26.11
Locus of system function poles for derivative feedback with $k_2 < 0$.

Derivative feedback:

$$\Theta(s) = \frac{1}{s^2 + s(K_2/L) - g/L} X(s)$$

$$s = -\frac{K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 + \left(\frac{g}{L}\right)}$$



TRANSPARENCY 26.12
Locus of system function poles for proportional feedback with $k_2 > 0$.

TRANSPARENCY

26.13

Overall system function with proportional plus derivative feedback.

Proportional plus derivative feedback:

$$a(t) = K_1 \theta(t) + K_2 \frac{d\theta(t)}{dt}$$

$$G(s) = K_1 + K_2 s$$

$$H(s) = \frac{1}{s^2 + s (K_2/L) - g/L + K_1/L}$$

Poles at

$$s = \frac{-K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 - \left(\frac{K_1 - g}{L}\right)}$$

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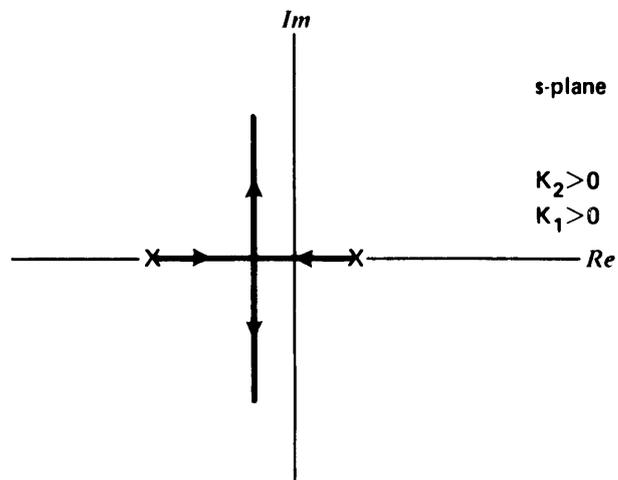
26.14

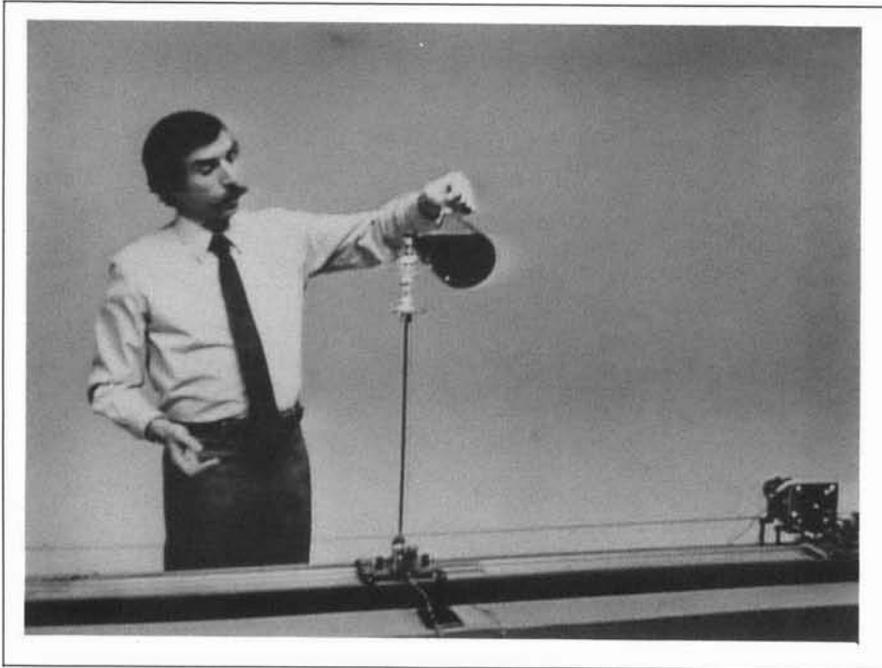
Locus of system function poles for proportional plus derivative feedback with $k_2 > 0$ and $k_1 > 0$.

Proportional plus derivative feedback:

$$H(s) = \frac{1}{s^2 + s (K_2/L) - g/L + K_1/L}$$

$$s = \frac{-K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 - \left(\frac{K_1 - g}{L}\right)}$$





**DEMONSTRATION
26.3**

The inverted pendulum with proportional plus derivative feedback operating as a stable system.

MIT OpenCourseWare
<http://ocw.mit.edu>

Resource: Signals and Systems
Professor Alan V. Oppenheim

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