

CIRCULAR CONVOLUTION

1. Lecture 10 - 43 minutes

Circular Convolution $x_3(n) = x_1(n) \circledast x_2(n)$

$$= \left[\sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{x}_2(n-m) \right] \mathcal{R}_N(n)$$

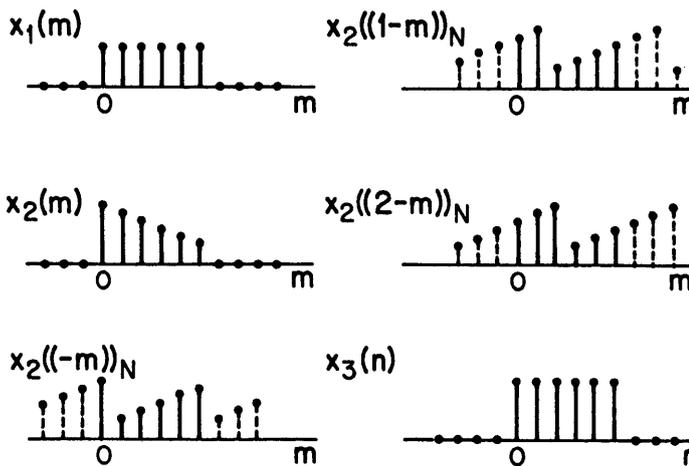
$$= \left[\sum_{m=0}^{N-1} x_1((m))_N x_2((n-m))_N \right] \mathcal{R}_N(n)$$

$$= \left[\sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N \right] \mathcal{R}_N(n)$$

$$= [x_1(n) * x_2((n))_N] \mathcal{R}_N(n)$$

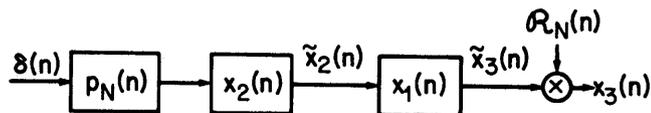
Circular convolution expressed in terms of periodic and linear convolution.

a.

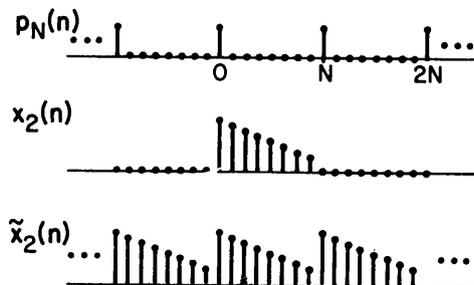


Example of circular convolution of two sequences.

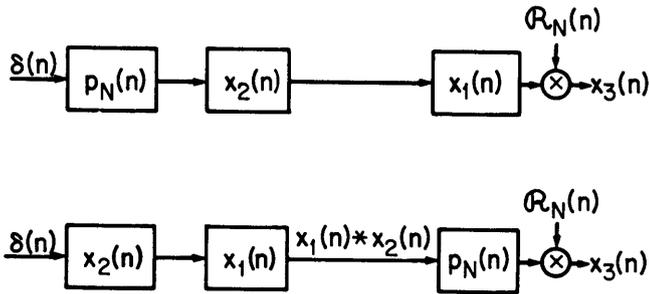
b.



An interpretation of circular convolution.



c.



Rearrangement of the operations in forming the circular convolution.

d.

"Circular Convolution =
Linear Convolution + Aliasing"

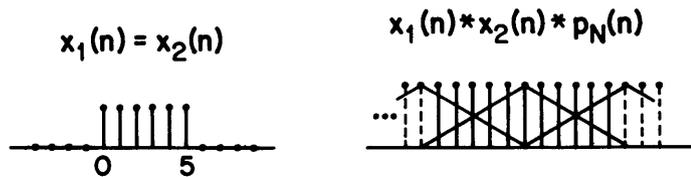
Interpretation of circular convolution as linear convolution followed by aliasing.

$$\hat{x}_3(n) = x_1(n) * x_2(n)$$

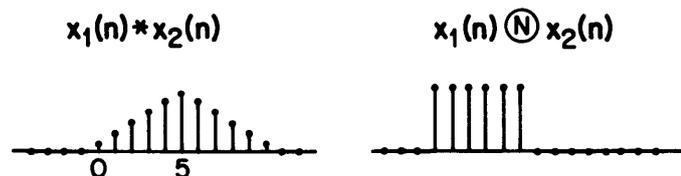
$$x_3(n) = x_1(n) \textcircled{N} x_2(n)$$

$$x_3(n) = \left[\sum_{r=-\infty}^{+\infty} \hat{x}_3(n+rN) \right] \mathcal{R}_N(n)$$

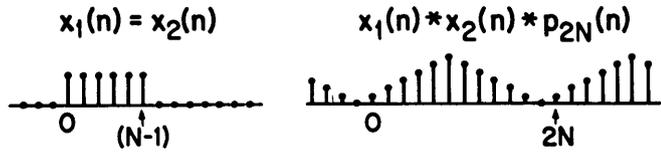
e.



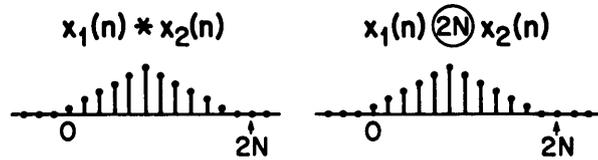
Example of a circular convolution formed by linear convolution followed by aliasing.



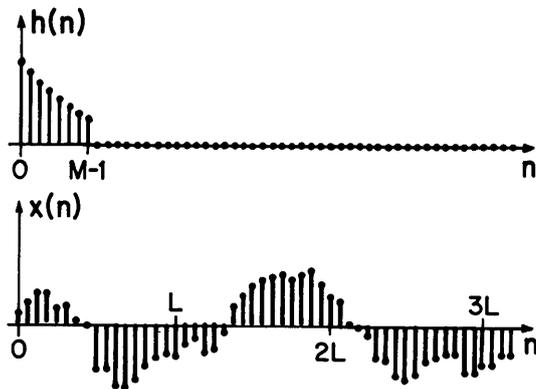
f.



Obtaining a linear convolution through the use of circular convolution.

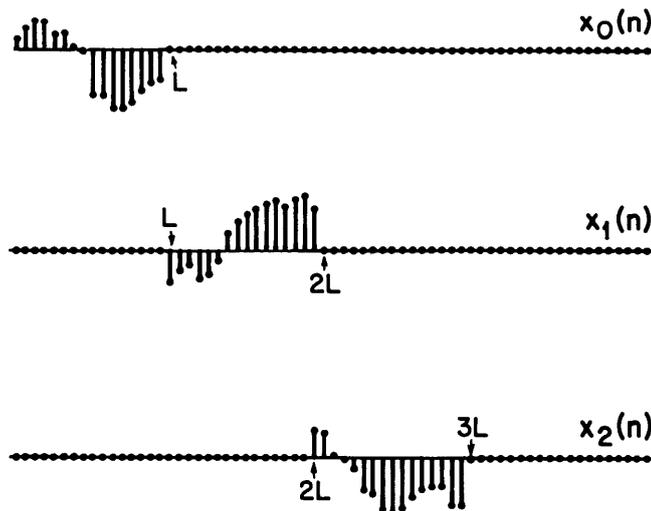


g.



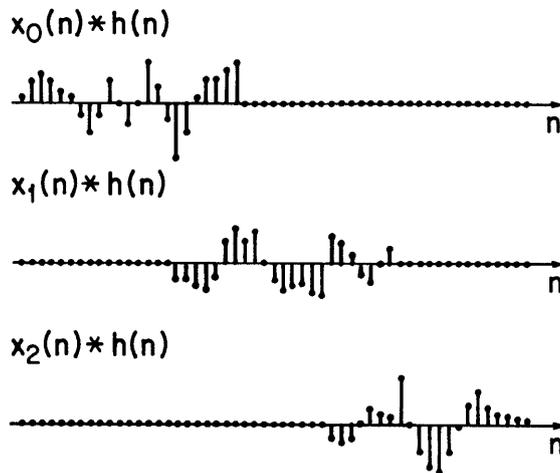
A finite length unit sample response and a sequence of indefinite length.

h.



Sectioning of the sequence $x(n)$.

i.



Linear convolution of $h(n)$ with the sections of $x(n)$. Note the overlap in the resulting output sections.

j.

2. Comments

In the last lecture we introduced the property of circular convolution for the Discrete Fourier Transform. The fact that multiplication of DFT's corresponds to a circular convolution rather than a linear convolution of the original sequences stems essentially from the implied periodicity in the use of the DFT, i.e. the fact that it essentially corresponds to the Discrete Fourier series of a periodic sequence. In this lecture we focus entirely on the properties of circular convolution and its relation to linear convolution. An interpretation of circular convolution as linear convolution followed by aliasing is developed.

As we will see in a later lecture, there is a highly efficient algorithm for the computation of the DFT and consequently it is often useful in practice to implement a convolution (for implementing a filter, for example) by computing the Discrete Fourier Transforms, multiplying, and then inverse transforming. This leads, of course, to a circular convolution, whereas it is generally a linear convolution that is desired. Consequently an important consideration is the use of circular convolution to implement a linear convolution. This can be accomplished by extending the lengths of the sequences to be convolved by "padding" with zeros. An additional consideration is the fact that a signal to be filtered may be arbitrarily long. Filtering can still be accomplished in this case using the DFT by first sectioning the input into finite length segments and implementing the convolution for each of these. The filtered segments can then be added with the appropriate synchronization

to produce the desired output. There are two basic procedures which are commonly used, referred to as the "overlap-add" and "overlap-save" methods. Although only the first of these is presented in the lecture, both are discussed in the text.

3. Reading

Text: Sections 8.7.5. (page 542) and 8.9.

4. Problems

Problem 10.1

Let $x(n)$ and $h(n)$ denote two finite length sequences, both of length N .

- What is the maximum possible length of the linear convolution of $x(n)$ with $h(n)$?
- What is the maximum possible length of the N -point circular convolution of $x(n)$ with $h(n)$?

Problem 10.2

In the figure below is shown a four-point sequence $x(n)$.

- Sketch the linear convolution of $x(n)$ with $x(n)$.
- Sketch the four point circular convolution of $x(n)$ with $x(n)$.
- Sketch the ten-point circular convolution of $x(n)$ with $x(n)$.
- What is the smallest value of N for which an N -point circular convolution of $x(n)$ with $x(n)$ will be identical to the linear convolution?

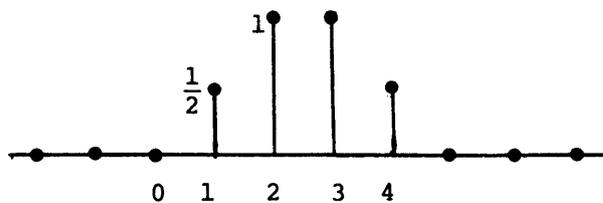


Figure P10.2-1

Problem 10.3

Consider two finite-duration sequences $x(n)$ and $y(n)$ where both are zero for $n < 0$ and with

$$x(n) = 0, \quad n \geq 8$$

$$y(n) = 0, \quad n \geq 20$$

The 20-point DFTs of each of the sequences are multiplied and the inverse DFT computed. Let $r(n)$ denote the inverse DFT. Specify which points in $r(n)$ correspond to points that would be obtained in a linear convolution of $x(n)$ and $y(n)$.

Problem 10.4*

We want to filter a very long string of data with a FIR filter whose unit-sample response is 50 samples long. We wish to implement this filter with an FFT using the overlap-save technique. To do this: (1) The input sections must be overlapped by V samples; and (2) from the output due to each section we must extract M samples such that when these samples from each section are butted together, the resulting sequence is the desired filtered output. Assume that the input segments are 100 samples long and that the size of the DFT is 128 ($=2^7$) points. Further assume that the output sequence from the circular convolution is indexed from point 0 to point 127.

- (a) Determine V .
- (b) Determine M .
- (c) Determine the index of the beginning and the end of the M points extracted; i.e., determine which of the 128 points from the circular convolution is extracted to be abutted with the result from the previous section.

Problem 10.5*

The problem often arises in which a signal $x(n)$ has been filtered by a linear time-invariant system that results in a distorted signal $y(n)$ and we wish to recover the original signal. This can often be done by processing $y(n)$ with a linear time-invariant system whose impulse response is such that the overall impulse response of the two systems in cascade is a unit sample. This is generally referred to as inverse filtering.

We have discussed the procedure for implementing an FIR filter using the DFT. The procedure involves, in part, multiplying the DFT of the input, $X(k)$ (or input sections), by $H(k)$, the DFT of the system unit sample response to produce $Y(k)$, the DFT of the output. It is often suggested, incorrectly, that the sequence $h_1(n)$ whose DFT is $1/H(k)$, is the impulse response of the inverse filter. The purpose of this problem is to indicate why that suggestion is incorrect.

Consider a linear time-invariant system with impulse response $h(n)$ given by

$$h(n) = \delta(n) - \frac{1}{2}\delta(n - n_0)$$

This system is an idealized example of a system that would introduce reverberation. Assume that $N = 4n_0$.

- (a) Determine the N-point DFT $H(k)$ of $h(n)$.
- (b) Now consider the N-point DFT $H_1(k)$ of a sequence $h_1(n)$ specified by

$$H_1(k) = \frac{1}{H(k)} \quad k = 0, 1, \dots, N - 1$$

Determine $h_1(n)$. Hint: If you have trouble evaluating the IDFT summation directly, express $H_1(k)$ as a polynomial in $W_N^{n_0 k}$ and observe that the $h_1(n)$ are coefficients of W_N^{nk} .

- (c) By evaluating the linear convolution of $h(n)$ and $h_1(n)$, show that $h(n) * h_1(n)$ is not a unit sample $\delta(n)$ and, consequently, $h_1(n)$ is not the unit sample response for the inverse system.
- (d) Compute the N-point circular convolution of $h(n)$ and $h_1(n)$.
- (e) Determine the unit-sample response $h_i(n)$ of the inverse system for $h(n)$. This can be done in a number of ways. One is to note that with $H(z)$ and $H_i(z)$ denoting the z-transforms of $h(n)$ and $h_i(n)$, $H_i(z) = 1/H(z)$. The inverse z-transforms of $H_i(z)$ can then be evaluated by long division.
- (f) Determine and verify numerically the relationship between $h_1(n)$ and $h_i(n)$.

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Resource: Digital Signal Processing
Prof. Alan V. Oppenheim

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