THE DISCRETE-TIME FOURIER TRANSFORM

Solution 4.1

The Fourier transform relation is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad \text{thus:}$$

(a) $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta(n-3)\, e^{-j\omega n} = e^{-j\omega 3}$

(b) $X(e^{j\omega}) = 1 + \frac{1}{2} e^{j\omega} + \frac{1}{2} e^{-j\omega} = 1 + \cos\omega$

(c) $X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$

(d) $X(e^{j\omega}) = \sum_{n=-3}^{+3} e^{-j\omega n} = e^{j3\omega} \sum_{n=0}^{6} e^{-j\omega n} = \frac{\sin(\frac{7\omega}{2})}{\sin(\frac{\omega}{2})}$

Solution 4.2

(a) In problem 2.4(c) we determined that the convolution of $\alpha^n\, u(n)$ and $\beta^n\, u(n)$ was given by

$$y(n) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right] u(n)$$

$$y(n) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right] u(n)$$

$$= \left[\alpha^n \left(\frac{\alpha}{\alpha-\beta}\right) + \beta^n \left(\frac{-\beta}{\alpha-\beta}\right)\right] u(n)$$

thus, $k_1 = \frac{\alpha}{\alpha-\beta}$ and $k_2 = \frac{\beta}{\beta-\alpha}$

(b) From problem 4.1 (c) it follows that

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

and

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

The Fourier transform of y(n) as obtained in (a)

$$Y(e^{j\omega}) = \sum_{n=0}^{\infty} \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} e^{-j\omega n}$$

$$= \frac{\beta}{\beta - \alpha} \sum_{n=0}^{\infty} \beta^n e^{-j\omega n} - \frac{\alpha}{\beta - \alpha} \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \frac{\beta}{\beta - \alpha} \frac{1}{1 - \beta e^{-j\omega}} - \frac{\alpha}{\beta - \alpha} \frac{1}{1 - \alpha e^{-j\omega}}$$

$$= \frac{1}{(1 - \beta e^{-j\omega})(1 - \alpha e^{-j\omega})}$$

Solution 4.3

(a) $$X_a(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} kx(n)e^{-j\omega n} = k\sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

$$= k\ X(e^{j\omega})$$

(b) $$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n - n_0)\ e^{-j\omega n}$$

Making the substitution of variables

$m = n - n_0$ or $n = m + n_0$

$$X_b\ (e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x(m)\ e^{-j\omega(m+n_0)} = \sum_{m=-\infty}^{+\infty} e^{-j\omega n_0}\ x(m)\ e^{-j\omega m}$$

$$= e^{-j\omega n_0}\ X(e^{j\omega})$$

(c) The transform of x(n) is given by

$$X\ (e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\ e^{-j\omega n}$$

thus $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} (-jn)\ x(n)\ e^{-j\omega n}$$

or

$$j\ \frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} n\ x(n)\ e^{-j\omega n}$$

$$= X_c\ (e^{j\omega})$$

Solution 4.4

(a) $$H_a(j\Omega) = \int_{-\infty}^{+\infty} h_a(t)\, e^{-j\Omega t}\, dt = \int_0^{\infty} a e^{-at}\, e^{-j\Omega t}\, dt = \frac{a}{j\Omega + a}$$

$$|H_a(j\Omega)|^2 = \frac{a^2}{\Omega^2 + a^2}$$. Thus $|H(j\Omega)|$ is as sketched below:

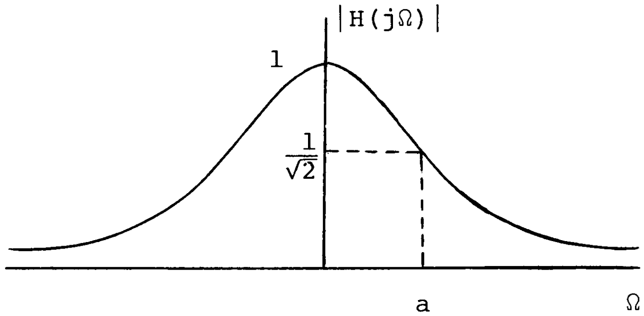


Figure S4.4-1

(b) $h_d(n) = cae^{-anT}\, u(n) = ca(e^{-aT})^n\, u(n)$

thus $$H_d(e^{j\omega}) = \frac{ca}{1 - e^{-aT}\, e^{-j\omega}}$$. For $H_d(e^{j0}) = 1$, $c = \frac{1 - e^{-aT}}{a}$

With this choice of c

$$|H_d(e^{j\omega})|^2 = \frac{(1 - e^{-aT})^2}{1 - 2e^{-aT}\cos\omega + e^{-2aT}}$$

thus $|H_d(e^{j\omega})|$ is:

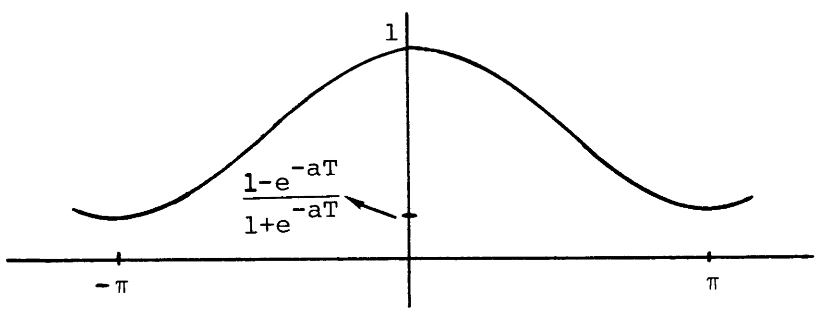


Figure S4.4-2

Note in particular that while the frequency response of the continuous-time filter asymptomatically approaches zero the frequency response of the digital filter doesn't. However as the sampling period T decreases, the value of $|H_d(e^{j\omega})|$ at $\omega = \pi$ decreases toward zero. The difference in the minimum values of $|H_a(j\Omega)|$ and $|H_d(e^{j\omega})|$ is of course due to aliasing.

Solution 4.5

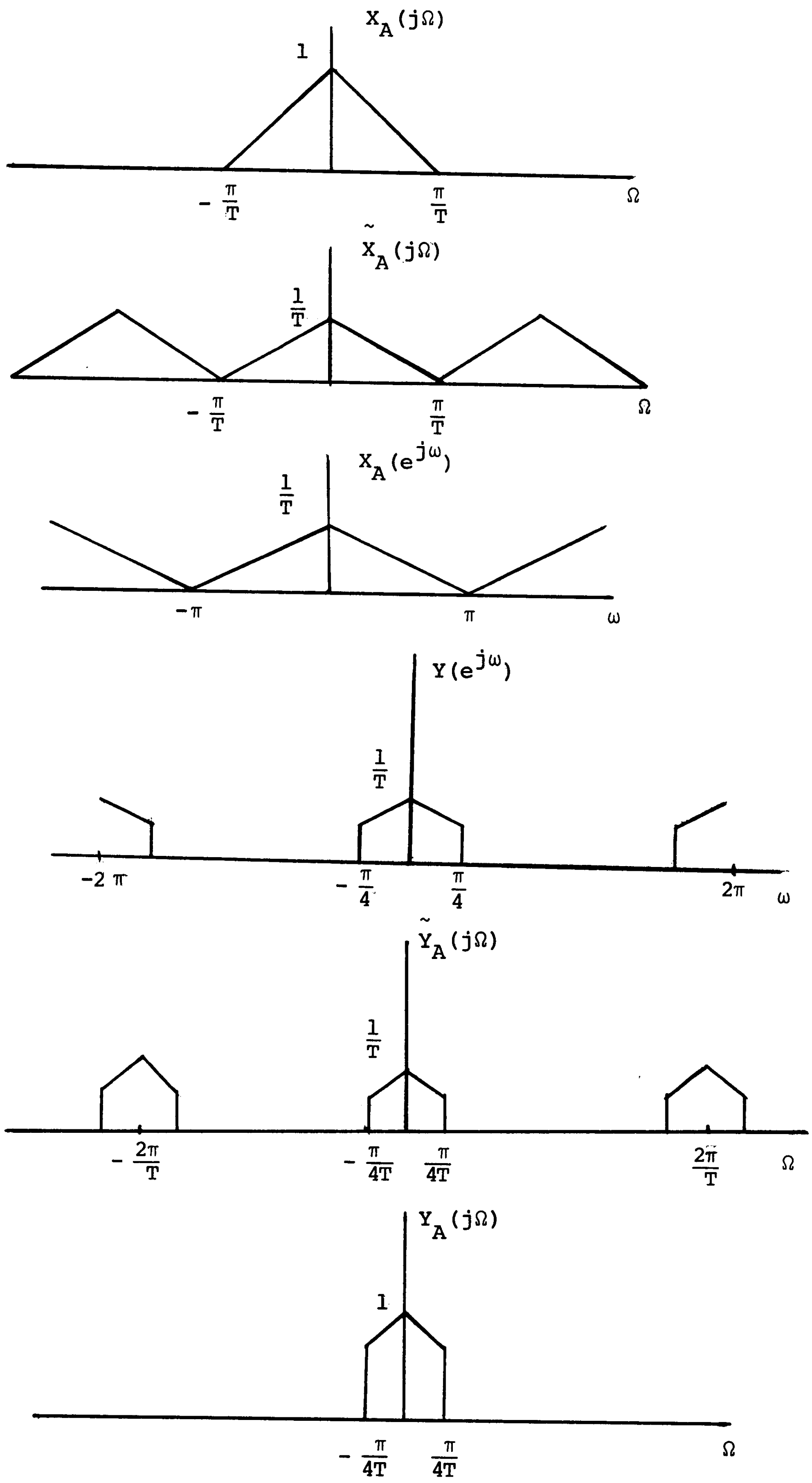
From the discussion in the lecture, we know that

$$\tilde{X}_A(j\Omega) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_A(j\Omega + \frac{2\pi r}{T})$$

and

$$X(e^{j\omega}) = \tilde{X}_A(j\Omega)\Big|_{\Omega T = \omega}$$

Let us assume that $X_A(j\Omega)$ has some arbitrary shape as indicated below. Since we are assuming that T is sufficiently small to prevent aliasing, $X_A(j\Omega)$ must be zero for $|\Omega| \geq \frac{\pi}{T}$. Then $\tilde{X}_A(j\Omega)$ and $X(e^{j\omega})$ are as shown in figure S4.5-1. $Y(e^{j\omega})$ corresponding to the output of the filter and $\tilde{Y}_A(j\Omega)$ and $Y_A(j\Omega)$ follow in a straightforward way and are as indicated in figure S4.5-1. Thus $y_A(n)$ could be obtained directly by passing x(n) through an ideal lowpass filter with unity gain in the passband and a cutoff frequency of $\frac{\pi}{4T}$ rad/sec. For the case in part (a) the cutoff frequency of the overall continuous-time filter is $\frac{\pi}{4} \times 10^4$ rad/sec and for the case in part (b) the cutoff frequency is $\frac{\pi}{8} \times 10^4$ rad/sec.



$X_A(j\Omega)$
1
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
Ω
$\tilde{X}_A(j\Omega)$
$\frac{1}{T}$
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
Ω
$X_A(e^{j\omega})$
$\frac{1}{T}$
$-\pi$
π
ω
$Y(e^{j\omega})$
$\frac{1}{T}$
-2π
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
2π
ω
$\tilde{Y}_A(j\Omega)$
$\frac{1}{T}$
$-\frac{2\pi}{T}$
$-\frac{\pi}{4T}$
$\frac{\pi}{4T}$
$\frac{2\pi}{T}$
Ω
$Y_A(j\Omega)$
1
$-\frac{\pi}{4T}$
$\frac{\pi}{4T}$
Ω

Figure S4.5-1

Solution 4.6*

(1) and (2) can be verified by direct substitution into the inverse Fourier transform relation. (3) and (4) follow from (1) since

$\text{Re}\,[x(n)] = \frac{1}{2}\,[x(n) + x^*(n)]$ and $j\text{Im}\,[x(n)] = \frac{1}{2}\,[x(n) - x^*(n)]$.

(5) and (6) follow from (2) since $\text{Re}[X(e^{j\omega})] = \frac{1}{2}\,[X(e^{j\omega}) + X^*(e^{j\omega})]$

and $j\text{Im}\,[X(e^{j\omega})] = \frac{1}{2}\,[X(e^{j\omega}) - X^*(e^{-j\omega})]$.

Solution 4.7*

If $X(e^{j\omega})$ denotes the Fourier transform of $x(n)$, then

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,d\omega$$

Thus, with $y(n)$ denoting the convolution of $f(n)$ and $g(n)$ and since $Y(e^{j\omega}) = F(e^{j\omega})\,G(e^{j\omega})$, we wish to show that

$$y(0) = f(0)\,g(0) \quad .$$

But $$y(n) = \sum_{k=-\infty}^{+\infty} f(k)\,g(n-k)$$

so $$y(0) = \sum_{k=-\infty}^{+\infty} f(k)\,g(-k)$$

Since $f(k)$ is zero for $k < 0$ and $g(-k)$ is zero for $k > 0$,

$$\sum_{k=-\infty}^{+\infty} f(k)\,g(-k) = f(0)\,g(0)$$

Solution 4.8*

(a) Method A:

Consider $x(n)$ as a unit-sample $\delta(n)$. Then

$g(n)=h(n)$ and $r(n) = h(n) * g(-n) = \sum_{k=-\infty}^{+\infty} h(k)\,h(-n+k)$

Finally, $s(n) = r(-n) = \sum_{k=-\infty}^{+\infty} h(k)\,h(k+n)$

Consequently, $h_1(n) = \sum_{k=-\infty}^{+\infty} h(k)\,h(k+n)$

To show that this corresponds to zero phase, we wish to show that $h_1(n) = h_1(-n)$ since from Table 2.1 of the text with $h_1(n)$, if $h_1(n) = h_1(-n)$ then

$H_1(e^{j\omega}) = H_1^*(e^{j\omega})$ and hence the frequency response is real.

$$h_1(-n) = \sum_{k=-\infty}^{+\infty} h(k)\ h(k-n)$$

letting $k - n = r$,

$$h_1(-n) = \sum_{k=-\infty}^{+\infty} h(n+r)\ h(r)$$

which is identical to $h_1(n)$.

Alternatively we can show that $h_1(n)$ corresponds to a zero-phase filter by arguing in the frequency domain.
Specifically,

Let $\hat{g}(n) = g(-n)$. Then $\hat{G}_1(e^{j\omega}) = G_1^*(e^{j\omega}) = X^*(e^{j\omega})\ H^*(e^{j\omega})$

Also, $R(e^{j\omega}) = X^*(e^{j\omega})\ H\ (e^{j\omega})\ H^*(e^{j\omega}) = X^*(e^{j\omega})\ |H(e^{j\omega})|^2$

and $S(e^{j\omega}) = R^*(e^{j\omega})$

$= X(e^{j\omega})\ |H(e^{j\omega})|^2$

Thus, $H_1(e^{j\omega}) = |H(e^{j\omega})|^2$. Since $H_1(e^{j\omega})$ is real, it has a zero phase characteristic.

(b) <u>Method B</u>:

$G\ (e^{j\omega}) = X(e^{j\omega})\ H(e^{j\omega})$

$R(e^{j\omega}) = X^*(e^{j\omega})\ H(e^{j\omega})$

$Y(e^{j\omega}) = G\ (e^{j\omega}) + R^*(e^{j\omega})$

$= X(e^{j\omega})\ [H(e^{j\omega}) + H^*(e^{j\omega})]$

$= X(e^{j\omega})\ [2\ \mathrm{Re}\ H(e^{j\omega})]$

Therefore $H_2(e^{j\omega}) = 2\ \mathrm{Re}\ H(e^{j\omega}) = 2\ |H(e^{j\omega})|\ \cos[(\arg H(e^{j\omega})])$ and consequently is also zero phase.

(c) $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are sketched below. Clearly method A is the preferable method.

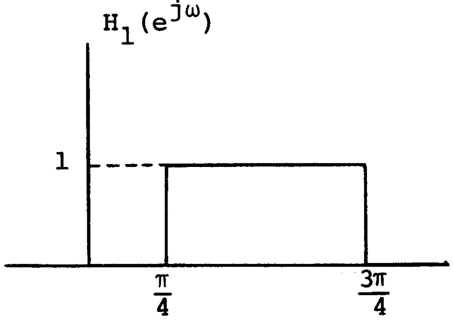


Figure S4.8-1

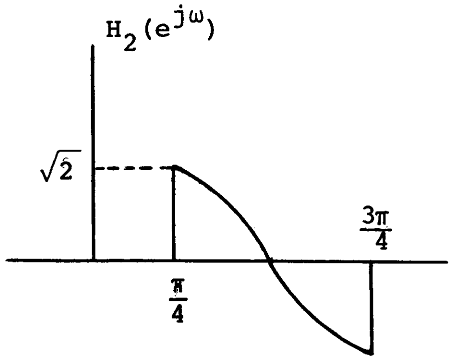


Figure S4.8-2