Z-TRANSFORM PROPERTIES

Solution 7.1

(a) - (iii)

(b) - (i)

(c) - (ii)

Solution 7.2

(a)

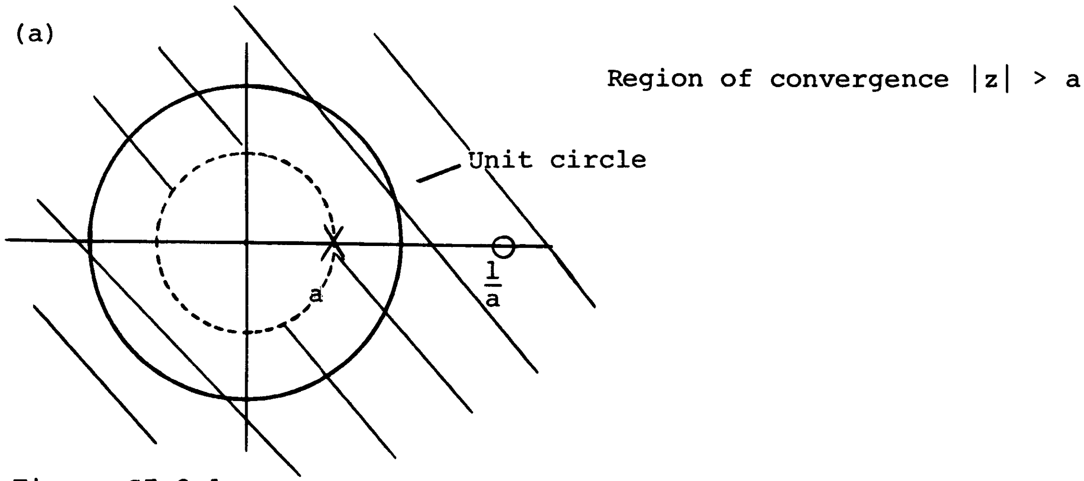

Figure S7.2-1

(b)

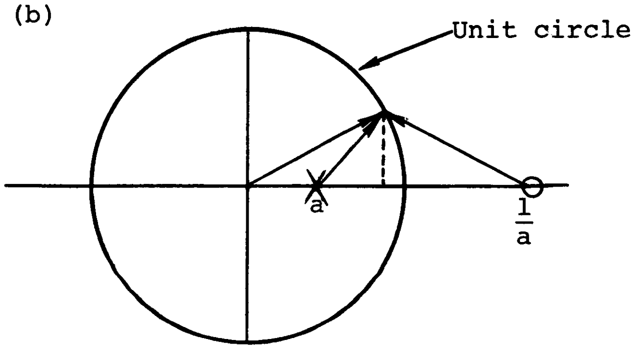

Figure S7.2-2

Shown above is the appropriate vector diagram, which we expand out below:

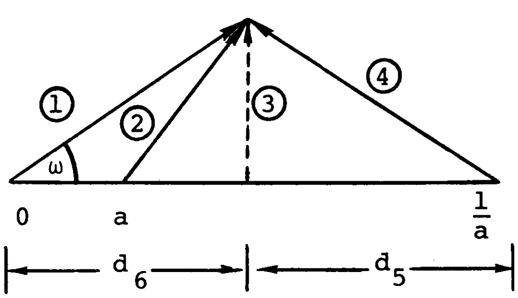

Figure S7.2-3

Let d_1, d_2, d_3 and d_4 denote the lengths of vectors ①, ②, ③ and ④ respectively. We wish to determine $\frac{d_4}{d_2}$. We know that vector ① is at an angle of ω to the horizontal axis and that $d_1 = 1$. Therefore:

$$d_6 = \cos\omega$$

$$d_3 = \sin\omega$$

$$d_5 = \frac{1}{a} - d_6 = \frac{1}{a} - \cos\omega$$

then

$$d_2^2 = (d_6 - a)^2 + d_3^2 = 1 + a^2 - 2a\cos\omega$$

Also,

$$d_4^2 = d_3^2 + d_5^2 = 1 + (\frac{1}{a})^2 - \frac{2}{a}\cos\omega = (\frac{1}{a})^2\left[1 + a^2 - 2a\cos\omega\right]$$

Thus

$$\frac{d_4}{d_2} = \frac{1}{a}$$

Solution 7.3

(i)

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n}$$

$$X(\frac{1}{z}) = \sum_{n=-\infty}^{+\infty} x(n)z^{n}$$

With the substitution of variables $m = -n$ in the above summation,

$$X(\frac{1}{z}) = \sum_{m=-\infty}^{+\infty} x(-m)z^{-m}$$

which we recognize as the z-transform of $x(-n)$.

(ii)

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} -nx(n)z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{+\infty} nx(n)z^{-n}$$

We recognize the right-hand side as the z-transform of n x(n).

Solution 7.4

Letting X(z) and Y(z) denote the z-transforms of x(n) and y(n), and using the properties of z-transforms, the z-transform of the difference equation results in

$$z^{-1} Y(z) - \frac{10}{3} Y(z) + z Y(z) = X(z)$$

thus the system function H(z) is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z^{-1}}{z^{-2} - \frac{10}{3}z^{-1} + 1}$$

$$= \frac{z^{-1}}{(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1})}$$

To determine the unit-sample response we can obtain the inverse transform of H(z) using any of the methods that we have discussed. For example, using contour integration,

$$x(n) = \frac{1}{2\pi j} \oint_{C} \frac{z^{-1}}{(1 - 3z^{-1})(1 - \frac{1}{3}z^{-1})} z^{n-1} dz$$

Since the system is stable, the region of convergence includes the unit circle. Thus for $n \geq 0$

$$x(n) = \text{Res}\left[\frac{z^{n}}{(z - 3)(z - \frac{1}{3})} \text{ at } z = \frac{1}{3}\right] = -\frac{3}{8} (\tfrac{1}{3})^{n}$$

For $n < 0$

$$x(n) = \frac{1}{2\pi j} \oint_{C} \frac{p}{(1 - 3p)(1 - \frac{1}{3}p)} p^{-n-1} dp$$

$$= \text{Res}\left[\frac{p^{-n}}{(1 - \frac{9}{3}p)(1 - \frac{1}{3}p)} \text{ at } p = \frac{1}{3}\right] = -\frac{3}{8} (\tfrac{1}{3})^{-n}$$

therefore,

$$x(n) = -\frac{3}{8}\left[(\tfrac{1}{3})^{n} u(n) + (3)^{n} u(-n-1)\right]$$

Solution 7.5

(a) According to the differentiation property, the z-transform of nx(n) is $-z \frac{dX(z)}{dz}$. For this problem,

$$-z\,\frac{dX(z)}{dz} = -\,\frac{az^{-1}}{(1 - az^{-1})}$$

The inverse z-transform of $-\,\frac{a}{(1 - az^{-1})}$ is $-a(a)^n\,u(n)$. From the shifting property, then, the inverse z-transform of $\frac{-az^{-1}}{(1 - az^{-1})}$ is $-a(a)^{n-1}\,u(n - 1)$.

Therefore

$$n\,x(n) = -a^n\,u(n - 1)$$

or

$$x(n) = -\,\frac{a^n}{n}\,u(n - 1) \qquad n \neq 0$$

Note that since we obtain $nx(n)$ from the differentiation property, this does not allow us to obtain $x(0)$. For this problem, however, we could obtain $x(0)$ from problem 5.7. Specifically, since $x(n)$ is causal,

$$x(0) = \lim_{z\to\infty} \log(1 - az^{-1}) = 0$$

(b) For $|\rho| < 1$, the power series expansion for $\log(1 - \rho)$ is

$$\log(1 - \rho) = -\sum_{n=1}^{\infty} \frac{\rho^n}{n}$$

thus,

$$\log(1 - az^{-1}) = -\sum_{n=1}^{\infty} \frac{a^n}{n} z^{-n}$$

thus we identify $x(n)$ as

$$x(n) = -\,\frac{a^n}{n}\,u(n - 1)$$

Note this problem is strongly related to the discussion in chapter 12 of the text. You may wish to look through some of the discussion in that chapter.

Solution 7.6

$$X_1(z) = \sum_{n=-\infty}^{+\infty} x_1(n) z^{-n} = \sum_{n=-\infty}^{+\infty} x(n) z^{-nM}$$

Therefore $X_1(z) = X(z^M)$

For $M = 2$

$$X_1(e^{j\omega}) = X(e^{j2\omega})$$

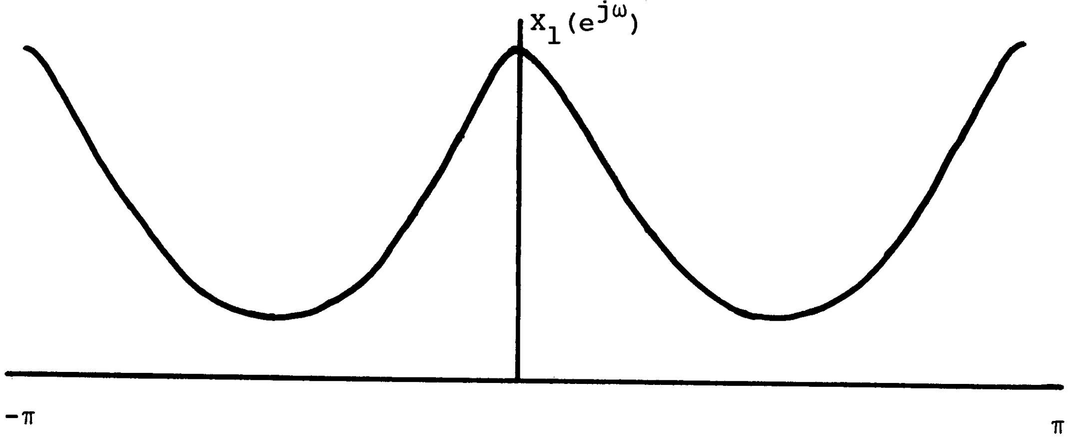

Figure S7.6-1