4 Pauli Algebra and Electrodynamics

4.1 Lorentz transformation and Lorentz force

The main importance of the Pauli algebra is to provide us with a stepping stone for the theory of spinor spaces to which we turn in Section 5. Yet it is useful to stop at this point to show that the formalism already developed provides us with an efficient framework for limited, yet important aspects of classical electrodyanmics (CED).

We have seen on page 26 that the effect on electric field on a test charge, a "boost," can be considered as an active Lorentz transformation, whereby the field is proportional to the "hyperbolic angular velocity $\dot{\mu}$."

This is in close analogy with the well known relation between the magnetic field and the cyclotron frequency, i.e., a "circular angular velocity" $\dot{\phi}$. These results had been obtained under very special conditions. The Pauli algebra is well suited to state them in much greater generality.

The close connection between the algebra of the Lorentz group and that of the electromagnetic field, is well known [24]. However, instead of developing the two algebras separately and noting the isomorphism of the results, we utilize the mathematical properties of the Lorentz group developed in Section 3 and translate them into the language of electrodynamics. The definition of the electromagnetic field implied by this procedure is, of course, hypothetical, and we turn to experience to ascertain its scope and limits. The proper understanding of the limitation of this conception is particularly important, as it serves to identify the direction for the deepening of the theory. The standard operational definition of the electromagnetic field involves the use of a test charge. Accordingly we assume the existence of particles that can act in such a capacity. The particle is to carry a charge e, a constant rest mass m, and the effect of the field acting during the time dt is to manifest itself in a change of the 4-momentum only, without involving any change in internal structure [25].

This means that the field has a sufficiently low frequency in the rest frame of the particle so as not to affect its internal structure. This is in harmony with the temporary exclusion of radiative interaction stated already.

Let the test charge be exposed to an electromagnetic field during a small time dt. We propose to describe the resulting change of the four-momentum $P \rightarrow P' = P + dP$ as an infinitesimal Lorentz transformation. In this preliminary form the statement would seem to be trivial, since it

[24] See [Syn35, Syn65] pp 87 & 343.

[25] This assumption suggests the drawing of a line between two kinds of electromagnetic interactions: (i) the Lorentz force; (ii) radiative interaction. We start with case (i), continue in Section 4.2 with the free field, but postpone the discussion of radiative interaction which affects the intrinsic structure and the rest mass of the particle. This subdivision of the interaction problem is not in the spirit of the classical theory. But then the classical radiation theory "breaks down" on the atomic level. Conforming to our program stated in Section 3.1 we do not wish to stretch theories beyond their empirical validity, and postpone the discussion of radiative interaction until we shall be ready to deal with it in the framework of quantum theory.

is valid for any force that does not affect the intrinsic structure, say a combination of gravitational and frictional forces. In order to characterize specifically the Lorentz force, we have to add that the characterization of the field is independent of the fourmomentum of the test charge, moreover it is independent of the frame of reference of the observer. These conditions can be expressed formally in the following.

Postulate 1. The effect of the Lorentz force on a particle (test charge) is represented as the transformation of the four-momentum space of the particle unto itself, and the transformations are elements of the active Lorentz group. Moreover, matrix representations in different Lorentz frames are connected by similarity transformations. (See Section 3.4.4.)

We now proceed to show that this postulate implies the known properties of the Lorentz force.

First, we show that an infinitesimal Lorentz transformation indeed reduces to the Lorentz force provided we establish a "dictionary" between the parameters of the transformation and the electromagnetic field (see below Equation 4.1.12). Consider a pure Lorentz transformation along $\hat{h}$

$$p' = p\cosh\mu + p_0\sinh\mu \tag{4.1.1}$$

$$p_0' = p\sinh\mu + p_0\cosh\mu \tag{4.1.2}$$

where $\vec{p} = p\hat{h} + \vec{p}$ with $\vec{p}\cdot\vec{h} = 0$. For infinitesimal transformations $\mu \to d\mu$:

$$p' - p = p_0 d\mu \tag{4.1.3}$$

$$p_0' - p_0 = p d\mu \tag{4.1.4}$$

or

$$\dot{\vec{p}} = p_0\dot{\mu}\hat{h} \tag{4.1.5}$$

$$\dot{p}_0 = \vec{p}\cdot\hat{h}\dot{\mu} \tag{4.1.6}$$

By making use of

$$\vec{p} = mc\sinh\mu = \gamma m\vec{v} \tag{4.1.7}$$

$$p_0 = mc\cosh\mu = \gamma mc$$

we obtain

$$\dot{\vec{p}} = pmc\hat{h}\dot{\mu} \tag{4.1.8}$$

$$\dot{p}_0 = pm\frac{\vec{v}}{c}\cdot\hat{h}\dot{\mu}$$

Turning to rotation we have from Equation A.3.8 of Appendix A (See note on page 51).

$$\vec{p}_\perp' = \vec{p}_\perp\cos\phi + \hat{u}\times\vec{p}_\perp\sin\phi \tag{4.1.9}$$

For an infinitesimal rotation $\phi \simeq d\phi$, and by using Equation 4.1.7 we obtain, since $\vec{p}'_{\parallel} = \vec{p}_{\parallel}$ and $\vec{p}_{\parallel} \times \hat{u} = 0$,

$$\vec{p}' - \vec{p} = -\vec{p} \times \hat{u}\, d\phi = -\gamma m \vec{v} \times \hat{u}\, d\phi \tag{4.1.10}$$

or,

$$\dot{\vec{p}} = -\gamma m \vec{v} \times \hat{u}\, d\phi \tag{4.1.11}$$

With the definitions of 3.3.28 and 3.3.29 of page 26 written vectorially:

$$\begin{aligned} \vec{E} &= \frac{\gamma m c}{e} \dot{\mu} \hat{h} \\ \vec{B} &= \frac{-\gamma m c}{e} \dot{\phi} \hat{u} \end{aligned} \tag{4.1.12}$$

Equations 4.1.8 and 4.1.9 reduce to the Lorentz force equations.

Let us consider now an infinitesimal Lorentz transformation generated by

$$\begin{aligned} V &= 1 + \frac{\mu}{2} \hat{h} \cdot \vec{\sigma} - \frac{i\phi}{2} \hat{u} \cdot \vec{\sigma} && (4.1.13) \\ &= 1 + \frac{edt}{2\gamma m c} \left(\vec{E} + i \vec{B} \right) \cdot \vec{\sigma} && (4.1.14) \\ &= 1 + \frac{edt}{2\gamma m c} F && (4.1.15) \end{aligned}$$

with

$$\vec{f} = \vec{E} + i\vec{B}, \quad F = \vec{f} \cdot \vec{\sigma} \tag{4.1.16}$$

It is apparent from Equations 4.1.15 and 4.1.16 that the transformation properties of V and F are identical. Since the transformation of V has been obtained already in Section 3.4.4, we can write down at once that of the field $\vec{f}$.

Let us express the passive Lorentz transformation of the four-momentum P from the inertial frame Σ to Σ' as

$$P' = SPS^{\dagger} \tag{4.1.17}$$

where S is unimodular. The field matrix transforms by a similarity transformation:

$$F' = SFS^{-1} \tag{4.1.18}$$

with the complex reflections (contragradient entities) transforming as

$$\begin{aligned} \bar{P}' &= \bar{S} \bar{P} \bar{S}^{-1} && (4.1.19) \\ \bar{F}' &= \bar{S} \bar{F} \bar{S}^{\dagger} && (4.1.20) \end{aligned}$$

For

$$S = H = \exp(-\frac{\mu}{2} \hat{h} \cdot \vec{\sigma}) \tag{4.1.21}$$

we obtain the passive Lorentz transformation for a frame Σ_2 moving with respect to Σ_1, with-the velocity

$$\begin{aligned} \vec{v} &= c\hat{h}\tanh\mu && (4.1.22)\\ F' &= HFH^{-1} && (4.1.23) \end{aligned}$$

We extract from here the standard expressions by using the familiar decomposition

$$\vec{f} = \vec{f}_{\|} + \vec{f}_{\perp} = (\vec{f}\cdot\hat{h})\hat{h} \tag{4.1.24}$$

we get

$$\begin{aligned} \vec{f}'_{\|} &= \vec{f}_{\|} && (4.1.25)\\ \vec{f}_{\perp}\cdot\vec{\sigma} &= H\left(\vec{f}_{\perp}\cdot\vec{\sigma}\right)H^{-1} = H^2\left(\vec{f}_{\perp}\cdot\vec{\sigma}\right) && (4.1.26)\\ &= \left(\cosh\mu - \sinh\mu\hat{h}\cdot\vec{\sigma}\right)\vec{f}_{\perp}\cdot\vec{\sigma} && (4.1.27) \end{aligned}$$

Hence

$$\begin{aligned} \vec{f}'_{\perp} &= \vec{f}_{\perp}\cosh\mu + i\sinh\mu\,\hat{h}\times\vec{f}_{\perp} && (4.1.28)\\ &= \cosh\mu\left(\vec{f}_{\perp} + i\tanh\mu\,\hat{h}\times\vec{f}_{\perp}\right) && (4.1.29)\\ &= \gamma\left(\vec{f}_{\perp} + i\frac{\vec{v}}{c}\times\vec{f}_{\perp}\right) && (4.1.30) \end{aligned}$$

where we used Equation 4.1.22. Inserting from Equation 4.1.16 we get

$$\begin{aligned} \vec{E}'_{\perp} &= \gamma\left(\vec{E}_{\perp} + \frac{\vec{v}}{c}\times\vec{B}_{\perp}\right)\\ \vec{B}'_{\perp} &= \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}}{c}\times\vec{E}_{\perp}\right) && (4.1.31)\\ \vec{E}'_{\|} = \vec{E}_{\|}, &\qquad \vec{B}'_{\|} = \vec{B}_{\|} \end{aligned}$$

It is interesting to compare the two compact forms 4.1.18 and 4.1.31. Whereas the latter may be the most convenient for solving specific problems, the former will be the best stepping stone for the deepening of the theory. The only Lorentz invariant of the field is the determinant, which we write for convenience with the negative sign:

$$-|F| = \frac{1}{2}TrF^2 = \vec{f}^2 = \vec{E}^2 - \vec{B}^2 + 2i\vec{E}\cdot\vec{B} = g^2e^2\Psi \tag{4.1.32}$$

Hence we obtain the well know invariants

$$\begin{aligned} I_1 &= \vec{E}^2 - \vec{B}^2 = g^2\cos 2\psi && (4.1.33)\\ I_2 &= 2\vec{E}\cdot\vec{B} = g^2\sin 2\psi \end{aligned}$$

We distinguish two cases

1. $\vec{f}^2 \neq 0$
2. $\vec{f}^2 = 0$

These cases can be associated with the similarity classes of Table 3.2. In the case (i) F is unimodular axial, for (ii) it is nonaxial singular. (Since F is traceless, the two other entries in the table do not apply.) We first dispose of case (ii). A field having this Lorentz invariant property is called a null-field. The F matrix generates an exceptional Lorentz transformation (Section 3.4.4). In this field configuration $\vec{E}$ and $\vec{B}$ are perpendicular and are of equal size. This is a relativistically invariant property that is characteriestic of plane waves to be discussed in Section 4.2.

In the "normal" case (i) it is possible to find a canonical Lorentz frame, in which the electric and the magnetic fields are along the same line, they are parallel, or antiparallel. The Lorentz screw corresponds to a Maxwell wrench[26]. It is specified by a unit vector $\hat{s}n$ and the values of the fields in the canonical frame E_{can} and B_{can} . The wrench may degenerate can can with $E_{can} = 0$, or $B_{can} = 0$. The canonical frame is not unique, since a Lorentz transformation along $\hat{n}$ leaves the canonical fields invariant.

We can evaluate the invariant eqn:iii-8-18ab in the canonical frame and obtain

$$E_{can}^2 - B_{can}^2 = I_1 = g^2 \cos 2\psi \tag{4.1.34}$$
$$2E_{can}B_{can} = I_2 = g^2 \sin 2\psi \tag{4.1.35}$$

One obtains from here

$$E_{can} = g \cos \psi \tag{4.1.36}$$
$$B_{can} = g \sin \psi \tag{4.1.37}$$

The invariant character of the field is determined by the ratio

$$\frac{B_{can}}{E_{can}} = \tan \psi \tag{4.1.38}$$

that has been called its *pitch* by Synge (op. cit. p. 333) who discussed the problem of canonical frames of the electromagnetic field with the standard tensorial method.

The definition of pitch in problem #8 is the reciprocal to the one here given and should be changed to agree with Eq. (20)

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NOTE: Eq. (18) of Appendix A is incorrect. Eq. (6) is identical
to the unnumbered equation preceding (18).
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[26]See Synge, as quoted on page 47.

4.2 The Free Maxwell Field

Our apporach to CED thus far is unusual inasmuch as we have effectively defined, classifed and transformed the electromagnetic field at a small region of space-time without having used the Maxwell equations. This is, of course, an indication of the effectiveness of our definition of the field in terms of active Lorentz transformations.

In order to arrive at the Maxwell equations we invoke the standard principle of relativistic invariance, involving the passive interpretation of the Lorentz group.

Postulate 2. The electromagnetic field satisfies a first order differential equation in the space-time coordinates that is covariant under Lorentz transformations.

We consider the four-dimensional del operator $\{\partial_0, \nabla\}$ with $\partial_0 = \partial/\partial(ct) = \partial/\partial r_0$ as a four-vector, with its matrix equivalent

$$D = \partial_0 - \nabla \cdot \vec{\sigma} \tag{4.2.1}$$

The rationale for the minus sign is as follows. Let D operate on a function representing a plane wave:

$$\psi = \exp -i(\omega t - \vec{k} \cdot \vec{r}) = \exp -i(k_0 r_0 - \vec{k} \cdot \vec{r}) \tag{4.2.2}$$

we have

$$iD\psi = \left(k_0 1 + \vec{k} \cdot \vec{\sigma}\right) \psi = K\psi \tag{4.2.3}$$

Thus D has the same transformation properties as K:

$$D' = SDS^{\dagger} \tag{4.2.4}$$

while the complex reflection

$$\bar{D} = \partial_0 1 + \nabla \cdot \vec{\sigma} \tag{4.2.5}$$

transforms as $\bar{K}$, i.e.

$$\bar{D}' = \bar{S}\bar{D}S^{-1} \tag{4.2.6}$$

By using the transformation rules 4.1.18, 4.1.20 of the last section we see that $\bar{D}F$ transforms as a four-vector J:

$$\left(\bar{S}\bar{D}S^{-1}\right)\left(SFS^{-1}\right) = \bar{S}\bar{J}S^{-1} \tag{4.2.7}$$

Thus

$$\bar{D}F = \bar{J} \tag{4.2.8}$$

is a differential equation satisfying the conditions of Postulate 2. Setting tentatively

$$J = \rho 1 + \frac{\vec{j}}{c} \cdot \vec{\sigma} \tag{4.2.9}$$

with $\rho, \vec{j}$ the densities of charge and current, 4.2.8 is indeed a compact form of the Maxwell equations.

This is easily verified by sorting out the terms with the factors $(1, \sigma_k)$ and by separating the real and imaginary parts.

By operating on the Equation 4.2.9 with D and taking the trace we obtain

$$D\bar{D}F = (\partial_0^2 - \nabla^2)F = D\bar{J} \tag{4.2.10}$$

and

$$\frac{1}{2}TrD\bar{J} = \partial_0\rho + \frac{1}{c}\vec{\nabla}\cdot\vec{j} = 0 \tag{4.2.11}$$

These are standard results which are easily provided by the formalism. However, we do not have an explicit expression for J that would be satisfactory for a theory of radiative interaction.

Therefore, in accordance with our program stated in Section 4.1 we set $J = 0$ and examine only the free field that obeys the homogeneous equations

$$\begin{aligned} \bar{D}F &= 0 \\ \left(\partial_0^2 - \nabla^2\right)F &= 0 \end{aligned} \tag{4.2.12}$$

The simplest elementary solution of 4.2.12 are monochromatic plane waves from which more complicated solutions can be built up. Hence we consider

$$F(\vec{r}, t) = F_+(\vec{k}, \omega)\exp\left\{i(\omega t - \vec{k}\cdot\vec{r})\right\} + F_-(\vec{k}, \omega)\exp\left\{i(\omega t - \vec{k}\cdot\vec{r})\right\} \tag{4.2.13}$$

where $F_\pm$ are matrices independent of space-time. Inserting into Equation 4.2.12 yields the condition

$$\omega^2 - c^2k^2 = 0 \tag{4.2.14}$$

Introducing the notation

$$\theta = k_0r_0 - \vec{k}\cdot\vec{r} \tag{4.2.15}$$

we write Equation 4.2.13 as

$$F(\vec{r}, t) = F_+\exp(-i\theta) + F_-\exp(i\theta) \tag{4.2.16}$$

Inserting into 4.2.12 we have

$$KF_\pm = K\left(\vec{E}_\pm + i\vec{B}_\pm\right)\cdot\vec{\sigma} = 0 \tag{4.2.17}$$

From here we get explicitly

$$\vec{k} \cdot \left(\vec{E}_{\pm} + i\vec{B}_{\pm}\right) = 0 \tag{4.2.18}$$

$$\vec{E}_{\pm} + i\vec{B}_{\pm} = i\hat{k}x \left(\vec{E}_{\pm} + i\vec{B}_{\pm}\right) \tag{4.2.19}$$

and we infer the well known properties of plane waves: $\vec{E}$ and $\vec{B}$ are of equal magnitude, and $\vec{E}, \vec{B}, \vec{k}$ from a right-handed Cartesian triad. We note that this constellation corresponds to the field of the type (ii) with $\vec{f}^2 = 0$ mentioned on page 51.

Since the classical $\vec{E}$, $\vec{B}$ are real, we have also the relations

$$\vec{E}_{-} = \vec{E}_{+}^{*}, \quad \vec{B}_{-} = \vec{B}_{+}^{*} \tag{4.2.20}$$

Consider now the case in which

$$\vec{f}_{-} = \vec{E}_{-} + i\vec{B}_{-} = 0 \tag{4.2.21}$$

which, in view of 4.2.20 implies

$$\vec{E}_{+} = i\vec{B}_{+} \tag{4.2.22}$$

Thus at a fixed point and direction in space the electric field lags the magnetic field by a phase $\pi/2$, and $\vec{f}_{+} = (\vec{E}_{+} + i\vec{B}_{+})$ is the amplitude of a circularly polarized wave of positive helicity, i.e., the rotation of the electric and magnetic vectors and the wave vector $\vec{k}$ form a right screw or the linear and angular momentum point in the same direction $+\vec{k}$. In the traditional optical terminology this is called a left circularly polarized wave. However, following current practice, we shall refer to positive helicity as right polarization R. The negative helicity state is represented by $\vec{f}_{-} = (\vec{E}_{-} + i\vec{B}_{-})$.

Actually, we have the alternative of associating $\vec{f}_{-}^{*} = (\vec{E}_{-} - i\vec{B}_{-})$ with R and $\vec{f}_{+}^{*} = (\vec{E}_{+} - i\vec{B}_{+})$ with L. However, we prefer the first choice and will use the added freedom in the formalism to describe the absorption and the emission process at a later stage [27]

Meanwhile, we turn to the discussion of polarization which has a number of interesting aspects, particularly if carried out in the context of spinor algebra.

[27] In the literature the Fourier decomposition of the field is usually written in terms of the vector potential rather than in terms of $\vec{E} + i\vec{B}$. The latter method, which alone leads naturally to a decomposition into helicity states, is considered by [Kra58], see page 86.