A Supplementary material on the Pauli algebra

A.1 Useful formulas

$$A = a_0 1 + \vec{a}\cdot\vec{\sigma}\tilde{A} = a_0 1 - \vec{a}\cdot\vec{\sigma}A^{\dagger} = a_0^* 1 + \vec{a}^*\cdot\vec{\sigma}\bar{A} = \tilde{A}^{\dagger} = a_0^* 1 - \vec{a}^*\cdot\vec{\sigma}$$

$$\frac{1}{2}Tr(A) = a_0\,, \quad |A| = a_0^2 - \vec{a}^2 1\frac{1}{2}Tr(A\tilde{A}) \tag{A.1.1}$$

$$\frac{1}{2}Tr(A\tilde{B}) = a_0 b_0 - \vec{a}\cdot\vec{b} \tag{A.1.2}$$

$$A^{-1} = \frac{\tilde{A}}{|A|} \quad \text{for} \quad |A| = 1 : A^{-1} = \tilde{A} \tag{A.1.3}$$

$$(\vec{a}\cdot\vec{\sigma})\left(\vec{b}\cdot\vec{\sigma}\right) = \vec{a}\cdot\vec{b}1 + i\left(\vec{a}\times\vec{b}\right)\cdot\vec{\sigma} \tag{A.1.4}$$

$$\text{For} \quad \vec{a}\|\vec{b} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \quad \vec{a}\times\vec{b} = 0 \tag{A.1.5}$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) - (\vec{b}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) = [(\vec{a}\cdot\vec{\sigma}), (\vec{b}\cdot\vec{\sigma})] = 0 \tag{A.1.6}$$

$$\text{For} \quad A = a_0 1 + \vec{a}\cdot\vec{\sigma}, \qquad B = b_0 1 + \vec{b}\cdot\vec{\sigma} \tag{A.1.7}$$

$$[A, B] = 0 \qquad \text{iff} \quad \vec{a}\|\vec{b} \tag{A.1.8}$$

$$\text{For} \quad \vec{a}\perp\vec{b}, \quad \vec{a}\cdot\vec{b}$$

$$\left\{\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right\} \equiv (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) + (\vec{b}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma}) = 0 \tag{A.1.9}$$

$$A(\vec{b}\cdot\vec{\sigma}) = (\vec{b}\cdot\vec{\sigma})\tilde{A} \tag{A.1.10}$$

$$U = U\left(\hat{u}, \frac{\phi}{2}\right) = \cos\frac{\phi}{2}1 - i\sin\frac{\phi}{2}\hat{n}\cdot\vec{\sigma} = \exp\left(-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}\right) \tag{A.1.11}$$

$$H = H\left(\hat{h}, \frac{\mu}{2}\right) = \cosh\frac{\mu}{2}1 + \sinh\frac{\mu}{2}\hat{h}\cdot\vec{\sigma} = \exp\left(\frac{\mu}{2}\hat{h}\cdot\vec{\sigma}\right) \tag{A.1.12}$$

U unitary unimodular, H Hermitian and positive.

A.2 Lorentz invariance and bilateral multiplication

For Hermitian matrices: $K^\dagger = K, \bar{K} = \tilde{K}$ and the same for R. Why bilateral multiplication? To eliminate nonphysical factors indicated as $\underbrace{}$.

$$\begin{pmatrix} e^{(\mu-i\phi)/2} & 0 \\ 0 & e^{-(\mu-i\phi)/2} \end{pmatrix} \begin{pmatrix} k_0+k_3 & k_1-ik_2 \\ k_1+ik_2 & k_2-k_3 \end{pmatrix} = \tag{A.2.1}$$

$$\begin{pmatrix} e^{\mu/2}(k_0+u_3)\underbrace{e^{-i\phi/2}} & \underbrace{e^{\mu/2}}(k_1-ik_2)e^{-i\phi/2} \\ \underbrace{e^{-\mu/2}}(k_1+ik_2)e^{-i\phi/2} & e^{\mu/2}(k_0-k_3)\underbrace{e^{i\phi/2}} \end{pmatrix} \times \tag{A.2.2}$$

$$\begin{pmatrix} k_0+k_3 & k_1-ik_2 \\ k_1+ik_2 & k_2-k_3 \end{pmatrix} \begin{pmatrix} e^{(\mu+i\phi)/2} & 0 \\ 0 & e^{-(\mu+i\phi)/2} \end{pmatrix} = \tag{A.2.3}$$

$$\begin{pmatrix} e^{\mu/2}(k_0+k_3)\underbrace{e^{i\phi/2}} & \underbrace{e^{-\mu/2}}(k_1-ik_2)e^{-i\phi/2} \\ \underbrace{e^{-\mu/2}}(k_1+ik_2)e^{i\phi/2} & e^{\mu/2}(k_0-k_3)\underbrace{e^{-i\phi/2}} \end{pmatrix} \times \tag{A.2.4}$$

$$\begin{pmatrix} e^{(\mu-i\phi)/2} & 0 \\ 0 & e^{-(\mu-i\phi)/2} \end{pmatrix} \begin{pmatrix} k_0+k_3 & k_1-ik_2 \\ k_1+ik_2 & k_2-k_3 \end{pmatrix} \begin{pmatrix} e^{(\mu+i\phi)/2} & 0 \\ 0 & e^{-(\mu+i\phi)/2} \end{pmatrix} = \tag{A.2.5}$$

$$\begin{pmatrix} e^{\mu/2}(k_0+k_3) & e^{-i\phi/2}(k_1-ik_2) \\ e^{i\phi/2}(k_1+ik_2) & e^{\mu/2}(k_0-k_3) \end{pmatrix} \tag{A.2.6}$$

Or, in 4×4 matrix form:

$$\begin{pmatrix} k'_1 \\ k'_2 \\ k'_3 \\ k'_0 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cosh\mu & \sinh\mu \\ 0 & 0 & \sinh\mu & \cosh\mu \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_0 \end{pmatrix} \tag{A.2.7}$$

Circular rotation around the z-axis by ϕ and hyperbolic rotation along the same asix by th ehyperbolic angle μ: Lorentz four-screw: $\mathcal{L}(\phi, \hat{z}, \mu)$. These transformations form an Abelian group.

In the Pauli algebra the formal simplicity of these relations is maintained even for arbitrary axial directions. To be sure, obtaining explicit results from the bilateral products may become cumbersome. However, the standard vectorial results can be easily extracted.

A.3 Typical Examples

Example 1

$$K' = HKH, \quad H = \exp\left(\frac{\mu}{2}\hat{h}\cdot\vec{\sigma}\right)$$
$$\vec{k} = \vec{k}_{\parallel} + \vec{k}_{\perp} \quad \vec{k}_{\parallel} = (\vec{k}\cdot\hat{h})\hat{h} \tag{A.3.1}$$

By using (6a) and (7b):

$$\vec{k}_{\parallel}\cdot\vec{\sigma}H = H\vec{k}_{\parallel}\cdot\vec{\sigma}\,, \quad \vec{k}_{\perp}\cdot\vec{\sigma}H = H^{-1}\vec{k}_{\perp}\cdot\vec{\sigma}$$
$$\vec{k}'_{\parallel} = \vec{k}_{\parallel} = k\hat{h} \tag{A.3.2}$$

$$\begin{aligned} \left(k'_0 + \vec{k}'_{\parallel}\cdot\vec{\sigma}\right) &= H^2\left(k_0 + \vec{k}_{\parallel}\cdot\vec{\sigma}\right) \\ &= \left(\cosh\mu + \sinh\mu\hat{h}\cdot\vec{\sigma}\right)\left(k_0 + \vec{k}_{\parallel}\cdot\vec{\sigma}\right) \end{aligned} \tag{A.3.3}$$

$$\begin{aligned} k'_0 &= k_0\cosh\mu + k\sinh\mu \\ k' &= k_0\sinh\mu + k\cosh\mu \end{aligned} \tag{A.3.4}$$

Example 2

$$K' = UKU^{-1}, \quad U = \exp\left(-i\frac{\phi}{2}\hat{u}\cdot\vec{\sigma}\right)$$
$$\vec{k} = \vec{k}_{\parallel} + \vec{k}_{\perp} \quad \vec{k}_{\parallel} = (\vec{k}\cdot\hat{u})\hat{u} \tag{A.3.5}$$

$$\vec{k}_{\parallel}\cdot\vec{\sigma}U^{-1} = U^{-1}\vec{k}_{\parallel}\cdot\vec{\sigma}\,, \qquad \vec{k}_{\perp}\cdot\vec{\sigma}U^{-1} = U\vec{k}_{\perp}\cdot\vec{\sigma}$$
$$\vec{k}'_{\parallel} = \vec{k}_{\parallel} \tag{A.3.6}$$
$$\tag{A.3.7}$$

$$\begin{aligned} \vec{k}'_{\perp}\cdot\vec{\sigma} &= \left(\cos\frac{\phi}{2}1 - i\sin\frac{\phi}{2}\hat{u}\cdot\vec{\sigma}\right)^2\vec{k}_{\perp}\cdot\vec{\sigma} \\ &= \left(\cos\phi 1 - i\sin\phi\hat{u}\cdot\vec{\sigma}\right)\vec{k}_{\perp}\cdot\vec{\sigma} \\ \vec{k}'_{\perp} &= \cos\phi\vec{k}_{\perp} + \sin\phi\hat{u}\times\vec{k}_{\perp} \end{aligned} \tag{A.3.8}$$

A.4 On the use of Involutions

The existence of the three involutions (see Equations A.1.1 above), provides a great deal of flexilbity. However, the most efficient use of these concepts calls for some care.

For any matrix of $\mathcal{A}_2$

$$A^{-1} = \frac{\tilde{A}}{|A|} \quad |A| = \frac{1}{2}Tr(A\tilde{A}) \tag{A.4.1}$$

In the case of Hermitian matrices we have two alternatives:

$$k_0 r_0 - \vec{k} \cdot \vec{r} = \frac{1}{2}Tr(K\tilde{R}) \tag{A.4.2}$$

or

$$k_0 r_0 - \vec{k} \cdot \vec{r} = \frac{1}{2}Tr(K\bar{R}) \tag{A.4.3}$$

It will appear, however from later discussions, that the complex reflection of Equation A.4.3 is more appropriate to describe the transition from contravariant to covariant entities.

A case in point is the formal representation of the mirroring of a four-vector in a plane with the normal along $\hat{x}_1$. We have

$$\begin{aligned} K' &= \sigma_1 \bar{K} \sigma_1 = \sigma_1 (k_0 1 - k_1\sigma_1 - k_2\sigma_2 - k_3\sigma_3)\,\sigma_1 \\ &= \sigma_1^2 (k_0 1 - k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3) \\ &= k_0 1 - k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3 \end{aligned} \tag{A.4.4}$$

More generally the mirroring in a plane with normal x is achieve by means of the operation

$$K' = \hat{a} \cdot \vec{\sigma} \bar{K} \hat{a} \cdot \vec{\sigma} \tag{A.4.5}$$

Again, we could have chosen $\tilde{K}$ instead of $\bar{K}$.

However, Eq (22) generalizes to the inversion of the electromagnetic six-vector $\vec{f} = \vec{E} + i\vec{B}$:

$$\left(\vec{E}' + i\vec{B}'\right) \cdot \vec{\sigma} = \overline{\left(\vec{E} + i\vec{B}\right) \cdot \vec{\sigma}} = \left(-\vec{E} + i\vec{B}\right) \cdot \vec{\sigma} \tag{A.4.6}$$

This relation takes into account the fact that $\vec{E}$ is a polar and $\vec{B}$ an axial vector.

A.5 On Parameterization and Integration

The explicit performance of the bilateral multiplication provides the connection between the parameters of rotation and the elements of the 4×4 matrices. We consider here only the pure rotation generated by

$$U = \exp(-i\frac{\phi}{2}\hat{u} \cdot \vec{\sigma}) \tag{A.5.1}$$

Let

$$l_0 = \cos\phi/2, \qquad l_1 = \sin\phi/2\hat{u}_1 \tag{A.5.2}$$
$$l_2 = \sin\phi/2\hat{u}_3, \qquad l_3 = \sin\phi/2\hat{u}_3 \tag{A.5.3}$$
$$u_1 = \cos(\hat{u}\cdot\hat{x}_1), \ldots, etc. \tag{A.5.4}$$
$$u_1^2 + u_2^2 + u_3^2 = 1 \tag{A.5.5}$$

$$\begin{pmatrix} k_1' \\ k_2' \\ k_3' \end{pmatrix} = \begin{pmatrix} l_0^2 + l_1^2 - l_2^2 - l_3^2 & 2(l_1 l_2 - l_0 l_3) & 2(l_1 l_3 + l_0 l_2) \\ 2(l_1 l_2 + l_0 l_3) & l_0^2 - l_1^2 + l_2^2 - l_3^2 & 2(l_2 l_3 - l_0 l_1) \\ 2(l_1 l_3 - l_0 l_3) & 2(l_2 l_3 + l_0 l_1) & l_0^2 - l_1^2 - l_2^2 + l_3^2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \tag{A.5.6}$$

Such expression are, of course not very practical. One usually considers infinitesimal relations with the parameters $d\phi\mu_k$. Integration of the infinitesimal operations into those of the finite group can be achieved within the general theory of Lie groups and Lie algebras [57].

In our approach the integration is achieved by explicit construction for the special case of the restricted Lorentz group. This is the first step in our program of using group theory to supplement or replace method of differential equations.

[57]These matters are extensively treated in the mathematical literature. A monograph aiming at physicists is [Gil06]