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Integration of classical and quantum physics

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The perennial aspect of the Newtonian foundation of mathematical physics is that the concept of "motion," that is, "kinematics," is to serve as the interface between mathematics and physics. Kinematics subdivides into the theory of orbital translation and that of undulation and spinning. Newtonian mechanics is based on giving to translational kinematics a priority over the other modes, since planetary revolution can be represented as translation modified by gravitation. The so-called breakdown of classical physics stems from giving the translational priority a canonical status and extending it to the constituents of matter. We claim that in this case the priority is to be reversed. The main content of this paper is to establish the algebraic model for an indivisible, undulating entity that we call a "wave simplex." It is used as the point of departure for a non-Newtonian quantum dynamics in which physical and algebraic concepts are in close correspondence. The postulates of the classical phenomenological theories and those of the canonical theories based on translational priority are established as theorems under the proper limiting conditions, and forces are constructed rather than postulated. While the formal structure of two-level quantum mechanics is established as well, exception is taken to treating spin as a property of a point particle. It is considered self-evident that a spinning object is orientable, a property accounted for in terms of a unitary triplet. This is the point of departure for an intrinsic particle dynamics. A main result is the integration of classical and quantum physics, thus closing the gap created by the heuristic method of canonical quantization.

I. INTRODUCTION

The basic rules for connecting observation and mathematics in terms of proper concepts are implicit in the work of Newton. However, the formulation of these rules in the Newtonian tradition is flawed by the tacit assumption that the aim of mathematical physics is to establish a single deductive system that provides the "theory of everything."

This paper presents a method in which the unity of physics is developed by establishing the coherence of a plurality of theories.

By writing two major works Newton demonstrated the need for the parallel development of *phenomenological* and *fundamental* theories. The priorities of the former are empirical; mathematical postulates must be anchored in experience, and repealed or restricted in scope in the light of later experiments. Phenomenological theories are necessarily pluralistic, as demonstrated by the variety of theories making up macroscopic classical physics.

The integration of this plurality can be expected in terms of mathematically based fundamental theories. In the preface to the first edition of the *Principia,* Newton suggests that the concept of motion lends itself to a mediating role between mathematics and physics. We sharpen the term "motion" to mean "kinematics," with preference given to the even narrower definition of kinematics as a purely mathematical discipline, only one step removed from geometry.

Kinematics is a generic term that includes translation, rotation, and undulation. No simple mathematical theory can handle all of these simultaneously. In Newtonian mechanics the problem is solved by the exclusive use of translational kinematics. This theory is based on Newton's discovery that (i) "planetary revolution" can be seen as translation modified by gravitation, and (ii) that universal gravitation is sufficient for the synthesis of planetary orbits and terrestrial ballistics. The formalism is extremely simple since the translation of any object can be represented "as if" its mass were concentrated in its center of mass. The resulting permanent, structureless point particle is a theoretical construct for representing translational motion, the Newtonian model.

The formalism has been extended to many-body systems, as appropriate interactions have been postulated. The resulting canonical mechanics of point masses (CMP) is technically applicable to a wide variety of situations, although the restriction to translations and to point particles may not be justified, as for problems of microphysics.

In such cases of overextension, agreement with experiment can be achieved either by the traditional method of *correcting* the Newtonian model in terms of canonical quantization, or by *replacing* it, as demonstrated in this paper.

Canonical quantization was invaluable as a heuristic device, and canonical quantum mechanics (CQM) is satisfactory on the pragmatic grounds that it accounts for a wide range of experiments. However, it contradicts the canonical concepts which are used for its foundation. This defect is eliminated in the forthcoming rational reconstruction which involves the formation of a set of new concepts extracted from the experiments of microphysics. The canonical theory is recaptured under the proper limiting conditions.

The experimental discovery of the photon suggests that there exist undulatory entities of a fixed frequency which cannot be subdivided into smaller configurations of the same frequency. We shall see in the next section that this experimental finding is sufficient to establish an algebraic formalism, to become the point of departure for a non-Newtonian mechanics.

This is the essential step responsible for the divergence of the forthcoming developments from tradition, hence a preview of the argument is indicated.

It was evident from the outset that the photon cannot be described in terms of a linear wave equation, the solutions of which can be scaled down at will. It is, however, traditional to "save" the classical method by "quantization."

In the present context we bring the discrepancy to a head, and point out that the methods of classical mathematical physics are based on the continuous distribution of matter and radiation and hence their mathematical expression has the property of *self-similarity*, an important principle of inductive generalization through scaling, and a tool for problem solving.

Most of the methods of mathematical physics are self-similar. The scope of this property is apparent from the fact that it was stretched to such nonclassical situations as fractal geometry, the theory of chaos, and is invoked even in the deep inelastic scattering of particle physics.

It is, nevertheless, evident that self-similarity does not properly account for the theory of indivisible particles. This is closely connected with the existence of the elementary constant of action. However, the insertion of this constant into the self-similar theories has been rightly considered disruptive. The point of departure of the present program is the establishment of a special branch of mathematics in which self-similarity is replaced by "absolute" properties to be associated with the action $\hbar$. The elementary action entered physics in the context of electromagnetic radiation, and this is the problem we reconsider from the new point of view. Although we could not, without circularity, start our discussion with a complete theory of the photon, we can confine ourselves to the safe statement that there exist indivisible undulating entities. It is shown in the next section that this fragmentary information is sufficient to define an algebraic model, the wave simplex. A heuristic argument leads to its algebraic description where the infinitesimal formalism appropriate for wave fields is replaced by a specially chosen algebra of 2×2 matrices.

The non-Newtonian kinematics developed in terms of the wave simplex has a number of remarkable properties. It is developed as a purely mathematical discipline and the switch to dynamic concepts is achieved in terms of universal constants. Finally, we demonstrate a novel way to integrate the plurality of phenomenological theories. Whereas the canonical program aims to reduce all theories to a single formalism, the present development consists of a sequence of steps, where each deals with a precisely circumscribed problem and its completion is the point of departure for the next. The apparent simplicity of the developments depends critically on the chosen sequence, particularly on the first step.

We call the totality of these systems *core theory* within which the postulates of the *pragmatic theories* are established as theorems, with their limits of validity spelled out.

The ground rules for carrying out this program are established in Sec. III. The postulates of the Maxwell-Lorentz-Einstein electrodynamics are derived from this basis in Sec. IV. This is achieved in a streamlined form, and an initial step is made toward the elimination of the latent discrepancy between the special relativity theory and quantum mechanics.

We deal in Sec. V with the problem of spin by postulating that spinning objects must be *orientable*. This property is represented in terms of a unitary triplet, a formalism which enables us to give a kinematic interpretation of the difference between the classical Einstein–de Haas and the quantum Stern-Gerlach experiments. This "kinematic quantization" provides us with a quantum system and a corresponding classical phenomenological limit. In such a fashion we are led to the beginning of two-level quantum mechanics, and a preview of further developments.

Our central move to replace the Newtonian model by that of the wave simplex is in contrast with the aim of the traditional critics of CQM who hoped to construct a theory which conforms more nearly to the Newtonian model. However, in a posthumously printed remark, Einstein finally conjectured that the representation of matter in terms of a field theory would have to yield to an algebraic one.[1]

II. FROM WAVE FIELD TO WAVE SIMPLEX

In order to construct a theory of wave configurations which is general enough to subsume both the self-similar and the absolute versions, we start from a minimal list of postulates.

(i) The experimental criterion for an entity to be considered undulatory is to exhibit diffraction which is accounted for in the case of a monochromatic beam in terms of a wave vector

$$\mathbf{k}=(2\pi/\lambda)\hat{\mathbf{k}} \ .$$

(ii) There exists a group of inertial frames in which light beams have a universal vacuum velocity c.

These are undoubtedly valid properties of light beams, and are sufficient to initiate a mathematical theory of undulation which naturally splits into two variants.

The content of the two postulates is expressed in terms of the special "dispersion relation," expressing in this case the absence of dispersion:

$$k_0^2 - \mathbf{k}^2 = 0 \ , \tag{2.1}$$

where $k_0 = \omega / c$.

This precise mathematical relation obtains because vacuum light velocity c has the character of a universal constant.

Equation (2.1) becomes significant only if it is placed into a wider mathematical context. Let us start with the standard method and map the wave vector components into differential operators:

$$(\mathbf{P}): \ (k_0, \mathbf{k}) \rightarrow i(\partial_0, -\nabla) \ .$$

Inserted into (2.1) this leads indeed to the linear wave equation, the standard tool for macroscopic wave fields. Here (P) refers to "point sets" and emphasizes the fact that this relation is part of a larger context of theories symbolically designated as $\{\mathcal{P}\}$ in which the points of space-time form the connecting link between physics and mathematics. We shall refer to entities which perform this mediating role briefly as an "interface." This Newtonian choice of an interface is in harmony with the unlimited divisibility of radiant energy in the classical theory, but is at variance with the quantal limitation of self-similarity.

Instead of taking care of the quantum "breakdown" of self-similarity by corrective prescriptions, we propose to find a consistent replacement for (P). The presence of the imaginary factor is not necessary for the derivation of the wave equation, but it enables us to state that (P) maps the wave vector components into Hermitian operators over the Hilbert space of plane waves.

This formulation suggests that the Hermitian operators over the infinite-dimensional Hilbert space $\mathcal{C}^\infty$, could be replaced by any of the finite-dimensional complex vector spaces. Actually, we have a situation which will become typical. The two-dimensional complex space $\mathcal{C}^2$ which is the furthest removed from the classical limit is not only the simplest, but provides also the point of departure for a wealth of formal constructions admitting physical interpretation. The two extreme cases undoubtedly play a privileged role. Thus we consider a mapping (V), a designation which refers to "vector space:"

$$(\mathbf{V}): \ (k_0, \mathbf{k}) \rightarrow k_0 I + \mathbf{k} \cdot \boldsymbol{\sigma} \equiv K \ .$$

Here the σ's stand for the Pauli matrices, defined as mathematical symbols: the invariant Hermitian basis of the algebra of 2×2 complex matrices. Among equivalent options we choose the conventional set of Pauli matrices. The vector $\mathbf{k}$ is on the left-hand side a physical wave vector, while on the right-hand side it is the mathematical generating vector of the matrix K. Thus (V) is a mapping of "wave phenomenology" into a matrix formalism, and optical manipulations of light beams immediately translate into matrix operations. We have a non-Newtonian "interface" between physics and mathematics.[2]

On multiplying (V) with the action constant we obtain on the left-hand side the definition of the four-momentum of a light quantum:

$$(p_0, \mathbf{p}) = \hbar (k_0, \mathbf{k}) \ . \tag{2.2}$$

By using this definition we obtain from (V)

$$(p_0, \mathbf{p}) \rightarrow p_0 I + \mathbf{p} \cdot \boldsymbol{\sigma} \equiv P \ . \tag{2.3}$$

The above relations describe an entity which we call a "wave simplex." It exhibits both the wave property (2.1) and the corpuscular property (2.2). The manifest, and almost trivial consistency of this description comes about because the corpuscular property is expressed in terms of an indivisible momentum, and no reference is made to localization. This is indeed what we obtain from the Compton effect, whereas localization on emission and absorption is intermittent rather than continuous.

The origin of the usual difficulties is the inconsistency of attributing a continuous orbit to an undulating entity. Since the Newtonian model is inseparable from the concept of a continuous orbit, there is no possibility for its consistent extension to include undulation. This difficulty is "covered up" by the principle of complementarity.

By contrast, the concept of the wave simplex provides us with the opportunity for the foundation of a non-Newtonian dynamics in which localization is recognized as a complicated problem not to be preempted by postulation.[3]

The new foundation starts with the wave simplex defined in terms of the relations (2.2) and (2.3). Since this concept straddles the "interface," it has an equally clear physical and mathematical meaning, but is strictly limited in content. Our program of development calls for the extension of this narrow base even while maintaining the close connection between experiment and formalism.

In order to achieve this goal, we have to establish two sets of ground rules, dealing with the algebraic and conceptual features, respectively. These are discussed in the next section.

III. THE POSTULATES OF WAVE SIMPLEX DYNAMICS

A. The algebraic foundation

The parallel between the maps (P) and (V) reflects the fact that "points" and "vectors" are closely related concepts, and in pure mathematics either one may be given the logical priority resulting in substantially different overall structures. The Bourbaki structure of modern mathematics is based on the point-set priority and may represent the ultimate formal response to the Newtonian heuristic use of infinitesimals over space and time. Switching to the vectorial preference means that we adjust the basic structure of mathematical physics to the experimental discovery of the quantum domain. As the limit of divisibility is reached, the description in terms of location in ordinary space yields to description in internal parameter spaces, which are mathematically vector spaces.

The basic space $\mathcal{C}^2$ is indeed the simplest of complex vector spaces, and the 2×2 matrix operators form a four-dimensional complex vector space. We make the choice of the formalism unique by the following specification.

Guideline: The algebraic objects obtained through construction on the right-hand side of the map (V) should admit translation into physical terms.

We call the formalism that satisfies this requirement $\{\mathcal{V}\}$. It will be built up in stages and starts with the algebra of 2×2 matrices supplemented by the group of conjugations

$$\mathcal{I}: A \to A = a_0 I + \mathbf{a}\cdot\boldsymbol{\sigma} \ , \quad (3.1a)$$

$$\mathcal{C}: A \to A^{\dagger} = a_0^* I + \mathbf{a}^*\cdot\boldsymbol{\sigma} \ , \quad (3.1b)$$

$$\mathcal{P}: A \to \tilde{A} = a_0 I - \mathbf{a}\cdot\boldsymbol{\sigma} \ , \quad (3.1c)$$

$$\mathcal{CP}: A \to \bar{A} = a_0^* I - \mathbf{a}^*\cdot\boldsymbol{\sigma} \ . \quad (3.1d)$$

Algebras admitting antiautomorphic conjugations are usually called *star algebras.* Since we have two independent conjugations (3.1b) and (3.1c) of this sort, we refer to the 2×2 matrix algebra enriched in this manner as the $\mathcal{H}_2^{**}$ algebra.[4]

We shall see that the conjugation relations are indispensable for connecting operational and matricial expressions.

At this point we note that by using (3.1c) we can write Eq. (2.1) as the determinant of K in the form

$$|K| = \tfrac{1}{2}\mathrm{Tr}(\tilde{K}K) = 0 \ . \quad (3.2)$$

The algebra $\mathcal{H}_2^{**}$ enables us not only to handle the incomplete concept of the wave simplex with precision, but leads also to the enlargement of the framework. An obvious step is to join invertible Hermitian matrices to the singular ones, and to generalize the concept of wave simplex by admitting also massive variants. The phenomenological mass concept enters the formalism through the determinant:

$$|P| = \tfrac{1}{2}\mathrm{Tr}(\tilde{P}P) = (mc)^2 \ . \quad (3.3)$$

Thus the wave simplex beams are represented as Hermitian matrices, where the generating vector is the linear momentum, and the determinant is the square of the mass.

Bringing the formal representation of massive and massless particles closer to each other corresponds to the experimental fact that all of them form beams subject to diffraction. The differences in charge, mass, and localization are left for later elaboration.

B. Events, experiments, and laws of nature

In the last section we specified the algebra appropriate for the development of the interface, and now we ask, how do we have to conceptualize experiments in order to bring them into harmony with the algebra? This is a prerequisite for explaining how is it possible to derive the laws of microphysics from actual experiments operating in terms of macroscopic instruments.

The Newtonian model was extracted from observational astronomy; it operates in terms of closed systems, and it is not surprising that, simple as the forthcoming discussion may be, it is not within the scope of the model.

We consider an experiment as an asymmetric coupling between a "device" and an "object." We know the phenomenological laws governing the former, and we find out something about the latter. An experiment can be translated into rigorous mathematics if it depends effectively only on a small number of parameters. In the case of the device a few phenomenological parameters suffice because we may ignore its atomic structure.

The case of the object is entirely different. Since the discovery of the particle beams at the turn of the century, we know that these play the role of "objects" in microphysical experiments. Identifying the relevant parameters involves the tacit use of the first two principles which we formulate explicitly.

(i) *The principle of class identity.* The particles arising from the decomposition of matter and radiation form classes of identical entities.

The parameters of a single particle account also for the homogeneous beam. Moreover, if either by observation or by method of preparation we know that we deal with say, electrons, then we know that the particles have all the properties of the electron class, regardless of where they are located in space-time, even on a cosmic scale.

The principle of class identity is based in chemistry, and is taken for granted by anyone involved with science or technology. It made astrophysics possible and is basic for the connection of particle physics and cosmology. This principle applies also to pure states; its importance has been recognized as a way to explain the intuitive meaning of quantum mechanics.[5]

By contrast, from the technical point of view, class identity is viewed in a negative role, as it interferes with the concept of orbital identity which is the mainstay of the Newtonian model, and is used for the foundation for canonical quantum mechanics.

The situation is different if the argument is centered around the wave simplex which can be directly identified with the beams of the above mentioned experiments. In this context, the principle of class identity is the basic physical principle from which the development of the interface originates. Our first task in the next section is to indicate how the classical macroscopic theories are deduced from the new point of departure.

It is apparent that the principle of class identity provides an important, but limited prediction with respect to future experiments. We claim that a limitation of predictability is unavoidable as it is stated in our second principle.

(ii) *The principle of contingency.* The observation of individual events reflects the interplay of law and contingency.

This principle is reminiscent of the "impossibility of perpetual motion," which was not the end, but the beginning of power engineering. Likewise in the present case, it helps to keep in mind what is impossible. Thus we cannot grant serious consideration to the Laplacian "principle of determinism" which invokes a divine intelligence in order not to acknowledge the contingent character of the initial and boundary conditions. Neither this "principle," nor the "law of induction" enables us to predict that the sun will rise in strictly periodical intervals. Actually, the intervals will fluctuate, and the time-keeping role of astronomical objects has been replaced by atomic clocks. We ought to establish the underlying quantum dynamic laws of nature.

Our two principles provide a logical basis for the dichotomy of theories tentatively suggested in the Introduction.

(A) The aim of *pragmatic theories* is to predict any kind of event which might be of practical interest. These theories are based on the best available postulates and handle contingent features statistically.

The weak point of this approach is that postulates may be useful for prediction, while not deserving to be considered "laws of nature." This situation is remedied in the following.

(B) In the *core theory* which arises from the development of the interface, the close relation between experiment and algebra extends the Newtonian standard of reliability to microphysics.

The procedure relies on the lining up of a sequence of seminal experiments that enable us to extract the laws relevant for the interpretation of an unlimited range of future experiments.

The two types of theories are connected as the pragmatic postulates are sorted out in terms of theorems within specified limits of validity in the core theory. Conversely, the development of the core theory is practical only because the derivation of the postulates of the existing pragmatic theories ensures us of empirical relevance.

While the principle of class identity is empirically reliable, it is very much in need of dynamical underpinning. One of the main goals of the core theory is to provide an explanation in terms of internal quantum dynamics. This can be achieved only in terms of carefully aligned steps, which we start by invoking the phenomenological implications of the same principle.

IV. FROM THE WAVE SIMPLEX TO THE MAXWELL THEORY

The formal developments start with a corollary of the principle of class identity: two inertial observers ought to agree whether they perceive the *same beam*. We may refer to this requirement as the *principle of objectivity*. While the generating vector varies relative to the frame, we expect the invariance of (i) the Hermitian property, and (ii) the determinant of the representing matrix.

In order to formalize the inertial transformation $P \rightarrow P'$ within the $\mathcal{K}_2^{**}$ algebra, we try the "ansatz"

$$P' = VPW \ . \tag{4.1}$$

The two conditions are satisfied if and only if

$$W = V^{\dagger} \ \text{ and } \ |V| = 1 \ . \tag{4.2}$$

In other words, we have for the inertial transformation

$$P' = VPV^{\dagger} \ , \tag{4.3}$$

where V is unimodular. In $\mathcal{C}^2$ unimodular matrices can be written in an exponential form that satisfies the generalized Euler relation

$$e^{\kappa\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}} = \cosh\kappa I + \sinh\kappa\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} \ , \tag{4.4}$$

where κ is a complex scalar, and $\hat{\mathbf{v}}$ is a complex unit vector.

It is a theorem of linear algebra that invertible matrices have a unique pair of polar forms:

$$V = UH = H'U \ , \tag{4.5}$$

where

$$U = U(\hat{\mathbf{u}}, \tfrac{1}{2}\phi) = \cos\tfrac{1}{2}\phi I - i\sin\tfrac{1}{2}\phi\hat{\mathbf{u}}\cdot\boldsymbol{\sigma} \tag{4.6}$$

is a unitary matrix generating a rotation by the angle ϕ around the axis $\hat{\mathbf{u}}$,

$$H = H(\hat{\mathbf{h}}, \tfrac{1}{2}\mu) = \cosh\tfrac{1}{2}\mu I + \sinh\tfrac{1}{2}\mu\hat{\mathbf{h}}\cdot\boldsymbol{\sigma} \tag{4.7}$$

is a Hermitian matrix generating a Lorentz transformation by the rapidity $\mu = \tanh^{-1}v/c$ along the direction $\hat{\mathbf{h}}$, and

$$H' = UHU^{-1} \ . \tag{4.7a}$$

All of these matrices are unimodular, and the set of all matrices of this type forms the special linear group conventionally denoted as SL(2,C). Inserted into (4.3) it generates the Lorentz transformations over the four-momentum space. We denote this subgroup as $[L;P]$. We shall consider other subgroups as we go along.

The conventional idea is that the physically significant parametrization of the Lorentz group is in terms of 4×4 parameters and SL(2,C) is known as the spinorial representation. However, I claim that this complex parametrization is not just one of infinitely many representations. The six parameters implicit in U and H constitute an intuitive and compact representation of the independent parameters left after the orthogonality constraints are applied to the sixteen real parameters of the Lorentz matrix. This is an analogy to the Lagrangian coordinates left after the removal of the mechanical constraints.

Specifically, the exponential form (4.5) implies the generalized de Moivre relation

$$[U(\hat{\mathbf{u}}, \phi)]^n = U(\hat{\mathbf{u}}, n\phi) \ , \tag{4.8}$$

which, with $\phi = \omega t$, is essential for representing stationary rotations.

We shall call the Lorentz transformation connecting two inertial frames a "passive transformation." The same transformation interpreted in an active sense represents a change of four-momentum which may be ascribed to interaction. This idea leads to a coherent representation of the varied aspects of electromagnetic interaction.

We introduce the concept of the electron in terms of a phenomenological charge and mass. In the Compton experiment the effect of the photon on the electron is described as an active Lorentz transformation, and so is the reaction of the electron on the photon. The two transformations are related by the conservation of four-momentum, but not determined by it.

Computing the probability amplitude of the process belongs in QED and not in the core theory. Our main line of argument at this point is to show that our approach subsumes macroscopic electrodynamics and mechanics.

We introduce a macroscopic unit charge q associated with a point mass to act as a test charge that is infinitesimal by macroscopic standards. It detects a local field which we describe as an active infinitesimal Lorentz transformation generated by a unimodular V matrix:

$$V = I + dH + dU$$
$$= I + \tfrac{1}{2}(d\mu\hat{\mathbf{h}} - id\phi\hat{\mathbf{u}})\cdot\boldsymbol{\sigma} \ . \tag{4.9}$$

Inserted into (4.3) this leads to the Lorentz force and power equations, provided we make the following identification:

$$\mathbf{E} + i\mathbf{B} = \frac{mc}{q}(\dot{\mu}\hat{\mathbf{h}} - i\dot{\phi}\hat{\mathbf{u}}) \ . \tag{4.10}$$

This is a typical interface relation between the electromagnetic field and non-Euclidean kinematics. The kinematic reduction of mass and charge is to come later. The connection of the magnetic field with an angular velocity, i.e., the cyclotron frequency is familiar. The hyperbolic angular velocity is associated with the electric field generating an "active" Lorentz transformation.[6]

By setting the magnetic field equal to zero and letting $1/c \to 0$, we arrive at the Newtonian equations in terms of central forces. Being *nonuniform*, this limiting process is not invertible, and therefore the canonical reduction program is not practical.

We introduce the complex six-vector

$$\mathbf{f} = \mathbf{E} + i\mathbf{B} \tag{4.11}$$

and the field matrix

$$F = \mathbf{f}\cdot\boldsymbol{\sigma} \ . \tag{4.12}$$

Thus a six-dimensional parameter space imposes itself which is closely associated with four-dimensional space-time.

In order to integrate the Maxwell equations into the present context, we consider a field in the self-similar limit, in which we are allowed to use the map (P). In order to harmonize it with our expression for the Lorentz force, we define an operator that is part of the $\mathcal{H}_2^{**}$ algebra, but its generating vector is a differential operator acting on plane waves. Thus the operator has a "hybrid" character enabling us to extend the parameter-space formalism to ordinary space-time:

$$D = \partial_0 - \nabla\cdot\boldsymbol{\sigma} \sim iK \ . \tag{4.13}$$

We write now the homogeneous Maxwell equations concisely as

$$D\bar{F} = 0 \ . \tag{4.14}$$

It is easy to establish the rules by which the Lorentz covariance of these equations becomes evident.[4] By iteration of (4.14) we have

$$D\bar{D} = \partial_0^2 - \nabla^2 = 0 \ . \tag{4.15}$$

These equations have spherical wave solutions which are also Lorentz invariant and we have justified the Lorentz group acting over space-time $[L; ct, \mathbf{r}]$, which is not valid within the strict assumptions of the beginning of this section. This rounds out our junction with the standard ($\mathcal{P}$) formalism. However, we know that this is based on approximations involving point masses and plane waves, and now we proceed to remove these restrictions stepwise.

By adding to (4.14) the four-current matrix J, we obtain the inhomogeneous equation

$$D\bar{F} = J \ . \tag{4.16}$$

We continue to have an equation which is formally Lorentz invariant. We can use charges and currents as sources of the electromagnetic field and obtain the usual macroscopic interactions under static and stationary conditions. However, the classical theory fails to account for radiative interactions and this is closely connected with the difficulties of harmonizing the principles of special relativity with those of quantum mechanics.[7]

Let us suppose that we have a classical theory of the emission of radio waves. The wave pulse can be transformed into an optical or an x-ray pulse, but the same transformation will not carry the radio antenna into an atomic source of radiation: the conjectured invariance of the laws of nature with respect to inertial transformations is inconsistent with the quantum limitation of scale invariance.

This does not affect the validity of the Lorentz group $[L; P]$, which we derived above on the basis of the invariance of the mapping (V) rather than (P). Conceptually it is based on the *principle of objectivity* rather than on *relativity*.

We conclude that radiative interaction should be considered in a consistently quantum context. In order to avoid chemical complications, we consider at first magnetic resonance transitions involving spin flip. Thus our next move is to consider spin and magnetic moment.

V. FROM RIGID ROTATION TO SPIN $\frac{1}{2}$

A. The algebra of rigid rotations

In the last section we have established the classical Lorentz and Maxwell-type electromagnetic interactions from the beginning of wave simplex dynamics in the context of the core theory. This was based on the simplifying assumption that there exist canonical pointlike test charges coupled to the field.

We proceed now to deepen the discussion by considering spin phenomena. In canonical quantum mechanics it is acceptable to endow point particles with spin because this provides the correct experimental predictions in atomic spectroscopy and quantum chemistry. However, this success notwithstanding, a spinning point is an intrinsic contradiction. We suggest instead that no object is spinning unless it is orientable.[8] This requirement is very strong. If orientation is specified in terms of Eulerian angles, formalization in terms of the $\mathcal{H}_2^{**}$ formalism admits only two rigorous elaborations which correspond to a macroscopic and a quantum concept of rigidity, respectively.

The problem of orientable objects was solved by Euler in terms of the relative orientation of two orthonormal triads. We combine the geometrical method of Eulerian angles with SU(2) representation of rotations, and write the "Eulerian triplet" of rotations as

$$U_{\mathrm{tr}}=U(\hat{\mathbf{z}},\tfrac{1}{2}\alpha)U(\hat{\mathbf{y}},\tfrac{1}{2}\beta)U(\hat{\mathbf{z}},\tfrac{1}{2}\gamma)\ , \quad (5.1)$$

where α,β,γ are the Eulerian angles. By using (4.6) we obtain explicitly

$$U_{\mathrm{tr}}=\begin{bmatrix}\xi_0 & -\xi_1^* \\ \xi_1 & \xi_0^*\end{bmatrix}\equiv U(\xi)\ , \quad (5.2)$$

where we introduced the Cayley-Klein parameters

$$\xi_0=e^{-i\alpha/2}\cos\frac{\beta}{2}e^{-i\gamma/2}\ ,$$
$$\xi_1=e^{i\alpha/2}\sin\frac{\beta}{2}e^{-i\gamma/2} \quad (5.3)$$

with

$$|\xi_0|^2+|\xi_1|^2=1\ . \quad (5.4)$$

Since every SU(2) matrix can be parametrized in either of the two alternative ways, we can develop triad kinematics by expressing an arbitrary rotational displacement as

$$U(\hat{\mathbf{u}},\tfrac{1}{2}\phi)U(\xi)=U(\xi')\ , \quad (5.5)$$

where the "operator" acts on an "object" to change its orientation state. This equation is the point of departure of what we call the compound SU(2) formalism, a part of our program for translating experiment into formalism.

The group property of SU(2) matrices leads to the simple inference that any two of the matrices in (5.5) uniquely determines the third. This is a proposition of great generality and we have to seek out the special conditions under which it is significant.

We introduce the Hermitian conjugate of Eq. (5.5):

$$U^{\dagger}(\xi)U(\hat{\mathbf{u}},-\tfrac{1}{2}\phi)=U^{\dagger}(\xi')\ , \quad (5.6)$$

where

$$U^{\dagger}(\xi)=\begin{bmatrix}\xi_0^* & \xi_1^* \\ -\xi_1 & \xi_0\end{bmatrix}\ . \quad (5.7)$$

While it is quite compelling to introduce the Cayley-Klein parameters in the context of SU(2), the parameters appear here in a redundant fashion, since either of the two columns in (5.2), or the two rows in (5.7) contains the same information as the whole matrix. Considered as two-component column and row vectors, or briefly *spinors*, these entities are considerably easier to handle than the matrices they might replace.

We have at this point an ambiguity, as we can single out either one of the pair of columns and rows. If the matrix formalism is justified, this choice ought to be irrelevant. We explore the implications of the map from the matrix (5.2) to its first column, along with the Hermitian conjugate relation, and will establish the range of validity of the method on the basis of experimental evidence.

Here are the maps which contain also the definitions of the spinors:

$$U(\xi)\leftrightarrow\begin{bmatrix}\xi_0 \\ \xi_1\end{bmatrix}\equiv|\xi\rangle\ , \quad (5.8)$$

$$U^{\dagger}(\xi)\leftrightarrow(\xi_0^*\ \xi_1^*)\equiv\langle\xi|\ . \quad (5.9)$$

The bra and ket symbols are introduced here in a purely mathematical manner. They are in harmony with the standard Dirac symbols of nonrelativistic quantum mechanics, but there are differences in their range of application. First of all, they are defined in the space $\mathcal{C}^2$ rather than $\mathcal{C}^{\infty}$, and thus deal only with two-state systems. However, within this restriction the formalism is richer. Thus we introduce spinors of finite "length," rather than normalizing them to unity. This enables us to apply them also to classical, rather than only to quantum objects.

We feature alternative notations to be used as convenient:

$$|s\rangle\equiv s^{1/2}|\xi\rangle\equiv|s,\tfrac{1}{2}\alpha,\tfrac{1}{2}\beta,\tfrac{1}{2}\gamma\rangle \quad (5.10)$$

and introduce the vector $\mathbf{s}$ with the spherical coordinates s,α,β. Its role in the spinor formalism is apparent from the dyadic product, conventionally called the *density matrix*

$$|s\rangle\langle s|=\tfrac{1}{2}(sI+\mathbf{s}\cdot\boldsymbol{\sigma})\ . \quad (5.11)$$

By taking the trace we recover the vector from the density matrix and hence from the spinor:

$$\mathbf{s}=\mathrm{Tr}(|s\rangle\langle s|\boldsymbol{\sigma})=\langle s|\boldsymbol{\sigma}|s\rangle\ , \quad (5.12a)$$

$$s=\langle s|s\rangle\ . \quad (5.12b)$$

More specifically we obtain

$$s_x+is_y=e^{i\alpha}s\sin\beta\ , \quad (5.13a)$$

$$s_z=s\cos\beta\ , \quad (5.13b)$$

$$s_x^2+s_y^2+s_z^2=s^2\ . \quad (5.13c)$$

These relations parametrize a two-dimensional spherical surface of radius s in an abstract Euclidean three-space

$$\mathcal{S}^2\subset\mathbf{R}^3\ , \quad (5.13d)$$

the axes of which we interpret as the components of intrinsic angular momentum.

Since the compound SU(2) formalism involves operations on this space, rather than dealing with its metric properties, it should be properly called kinematics rather than geometry. Accordingly we introduce the concept of time by setting $\phi=\omega t$ to account for stationary rotation. As a rule we put the distinguished $\hat{\mathbf{z}}$ direction along the rotational axis and describe the process as

$$\alpha\rightarrow\alpha+\chi\omega t\ . \quad (5.13e)$$

Here $\chi=\pm 1$ is a "chirality factor" which enables us to deal with both senses of spinning even while considering the frequency ω and time t as positive and the rotational axis fixed.

Equations (5.13a)–(5.13e) describe a mode of motion often called *conical rotation.* It is understood that the conical angle β is constant.[9] It is often convenient to average over this rapid motion:

$$\langle s_x^2 \rangle_{\text{per}} = \langle s_y^2 \rangle_{\text{per}} = \tfrac{1}{2} s^2 \sin^2\beta \ . \tag{5.13f}$$

The symbol $\langle \ \rangle_{\text{per}}$ is to indicate that the smoothing is over a rapid periodical process, that is, "deterministic" if the frequency is constant. The "smoothed" orbits are specified in terms of the parameters s,β.

We arrived at this mode of motion by a simple global method. The same results follow also in terms of familiar differential equations.

We insert the time parameter into the kinematic equation (5.5), use the explicit operator from (5.1) and going from matrices to spinors according to (5.9), we obtain first for the kets

$$e^{-(1/2)i\chi\omega t \hat{\mathbf{u}}\cdot\boldsymbol{\sigma}} |s(0)\rangle = |s(t)\rangle \ . \tag{5.14}$$

Expanding the exponential for infinitesimal times, we obtain a first-order differential equation:

$$i\partial_0 |s\rangle = \mathcal{H}|s\rangle \tag{5.15}$$

with the "rotational Hamiltonian"

$$\mathcal{H} = \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma} \ , \tag{5.16}$$

where $\boldsymbol{\omega}=\chi\omega\hat{\mathbf{u}}$.

By taking the Hermitian conjugate of (5.15), we obtain for the bras

$$-i\langle s|\partial_0 = \langle s|\mathcal{H} \ , \tag{5.15$'$}$$

where the operators act toward the left on the bras.

In spite of their similarity to the Schrödinger equation, the relations (5.15) are entirely classical. Since $\mathbf{s}$ is the angular momentum vector, its time derivative is the torque $\boldsymbol{\tau}$. We compute it by applying the operator in (5.14) to (5.11):

$$U|s\rangle\langle s|U^{-1} = U\tfrac{1}{2}(s+\mathbf{s}\cdot\boldsymbol{\sigma})U^{-1} \ . \tag{5.17}$$

Instead of expanding $U(\hat{\mathbf{u}},\tfrac{1}{2}\omega t)$, we reduce it in global terms according to (4.4) and obtain

$$\boldsymbol{\tau} = \partial_0 \mathbf{s} = \boldsymbol{\omega}\times\mathbf{s} \ . \tag{5.18}$$

This relation is indeed manifestly classical. Although the similarity to the Schrödinger equation is significant, the discussion of the subtle connection is deferred.

Our next step is to establish the transition from kinematics to dynamics. Within a mechanical context, the procedure would be to consider the constitutive relations between angular momentum and angular velocity. However, this method is impractical, since the moment of inertia is not among the attributes of elementary particles.

By contrast, particles do have magnetic moments which respond to macroscopic magnetic fields and in the next section we translate the foregoing results into a gyromagnetic language.

B. Macroscopic and microscopic gyromagnetism

We start from the phenomenological relations for the energy and torque of a macroscopic magnetic moment $\boldsymbol{\mu}$ in a homogeneous magnetic field $\mathbf{B}$:

$$E = -\mathbf{B}\cdot\boldsymbol{\mu} \ , \tag{5.19}$$

$$\boldsymbol{\tau} = -\mathbf{B}\times\boldsymbol{\mu} \ . \tag{5.20}$$

We introduce the standard gyromagnetic factor γ (not to be confused with the Eulerian angle). By using (5.18), we arrive at the "magnetomechanical dictionary," well known in the context of the Larmor precession:

$$\boldsymbol{\mu} = \gamma\mathbf{s} \ , \tag{5.21a}$$

$$\mathbf{B} = -\frac{1}{\gamma}\boldsymbol{\omega} \ . \tag{5.21b}$$

We obtain also

$$E = \mathbf{s}\cdot\boldsymbol{\omega} \ , \tag{5.22}$$

to complete the torque equation (5.18) into a neutral pair.

This "dictionary" enables us to analyze the types of gyromagnetic precession in terms of conical rotations.

The presence of the magnetic field injects a preferred direction into the gyromagnetic problem, and the system's response to this imposed anisotropy enables us to make a clean-cut distinction between classical and quantum systems.

(i) The coupling of a macroscopic magnetic moment to the external field in a dissipative environment leads to an alignment along the field to satisfy the energy minimum principle. The field imposes on the object a maximal anisotropy

$$s_z/s \to 1 \ . \tag{5.23}$$

It is understood that by convention s_z is positive if the precession is around the stable equilibrium of spin alignment. This is to be contrasted with the fictitious precession around the unstable equilibrium. We thus divide the angular momentum sphere into a "physical" and a "nonphysical" hemisphere.

(ii) Next we consider the Stern-Gerlach experiment involving a system with an arbitrary spin. The continuum range of the classical variables s,β is sharpened into a discrete set of states which is symmetrically located in the physical and the nonphysical hemispheres, respectively. Thus the Stern-Gerlach experiment reveals to us the operation of a nonclassical stability principle: whereas the classical energy minimum principle leads to the distinction between the physical and the nonphysical hemispheres, the new principle is based on hemispheric symmetry.

This qualitative criterion for quantum systems is very strong and we proceed to cast it into a form that serves as the basis for quantitative developments for which the case of spin-$\frac{1}{2}$ systems is particularly suited.

We may say in all generality that quantum systems cannot be aligned with the field to satisfy the anisotropy condition (5.23). However, empirically, there is something special about spin-$\frac{1}{2}$ particles, which I propose to ascribe to an underlying principle.

While the macroscopic system submits with some delay to the imposed anisotropy, the spin-$\frac{1}{2}$ particles line up only one component, but otherwise maintain an internal isotropy, or rather "quasi-isotropy," since isotropy obtains only in terms of the smoothed motion.

The principle of quasi-isotropy:

$$\langle s_x^2 \rangle_{\text{per}} = \langle s_y^2 \rangle_{\text{per}} = s_z^2 = \tfrac{1}{3} s^2 \ . \tag{5.24}$$

The connection with experiment and with existing theory is ensured by setting

$$s = s_0 = \hbar\sqrt{3}/2 \tag{5.25a}$$

which yields

$$s_z = \pm \tfrac{1}{2}\hbar \ . \tag{5.25b}$$

The two signs are expressions of the hemispheric symmetry of the sphere of Eq. (5.13d). This symmetry is "broken" by the energy principle and leads to the distinction between stable ground states and metastable excited states.

The measure of the symmetry breaking is the energy difference obtained by using (5.22):

$$\Delta E = \hbar\omega \ . \tag{5.25c}$$

This is the Bohr frequency condition, however, it is now an implication of the principle of quasi-isotropy within the context of the core theory. The connection between energy and frequency stems from classical gyromagnetic relations combined with the quantization of spin in terms of the action constant. Note that the transition frequency is identical with that of the conical rotation defined above.

In contrast to the present approach, the conventional procedure fails to recognize that quantization involves the emergence of the hemispheric symmetry. The origin of the discrepancy is not in the quantum system, for which the semiquantitative descriptions are so far identical, but in the selection of the classical baseline. It is characteristic of the Copenhagen program that "classical physics" is identified with "canonical mechanics." The latter operates in terms of closed systems and the conservation laws are based on Noether's theorem. In particular, the conservation of angular momentum is established for an isolated many-body system in isotropic space, a totally fictitious situation. The quasi-isotropy of the spin-$\frac{1}{2}$ particle is significant only because it obtains in an external field, which imposes its anisotropy on the classical system.[10]

We shall see that the handling of the transition from hemispheric asymmetry to symmetry belongs into non-Euclidean spherical geometry. Therefore, we have the parallel situation that linear and angular momenta belong under the purview of hyperbolic and of spherical non-Euclidean geometry, respectively.

Yet, although the sphere is among the most investigated mathematical objects, there is no satisfactory spherical geometry and kinematics in the literature. The critical issue is that great circles diverging from a point on the sphere converge in the antipodal point. Felix Klein[11] found this situation inconsistent with projective geometry to which he wished to subsume the non-Euclidean geometries. Accordingly, he identified the antipodal points and called the resulting structure an "elliptic space." This convention left its mark on most if not all developments of the topology of the sphere.

Elliptic kinematics corresponds to the compound SU(2) formalism and the Einstein–de Haas experiment developed above. The undoing of the identification of the antipodal points is closely connected with the transition from the Einstein–de Haas to the Stern-Gerlach experiment and gives us an insight into the relation between classical rigid rotation and the quantum-mechanical spin-$\frac{1}{2}$ case. In the former a magnetic moment precesses around the stable orientation in the magnetic field, while for spin-$\frac{1}{2}$ systems there is also a mirrored "antistate" in which the precession is around the unstable orientation. Physically, quantum dynamics centers around the phenomenon of metastablility related by symmetry to a stable state. This point is not readily apparent in CQM in which the rigorous discussion is limited to closed systems.

The development of these ideas has many ramifications which remain to be discussed. The wealth of physically interpreted algebraic structure bears out the expectation as to the role of the interface, and in the next section we indicate in general terms where the argument is heading.

VI. DISCUSSION

To describe the spin doublet algebraically, one breaks up the $U(\xi)$ matrix into its two columns (and $U^\dagger$ into its two rows) to arrive at a bispinoral space in which the whole formalism of the quantum mechanics of two-level systems is derived from the SU(2) formalism. The geometrical Eulerian angles are *reinterpreted as phase and mixing angles.* This has the intuitive meaning of a transition from mechanical rigidity to the "quasirigidity" of wave configurations: rigid enough for class identity, yet also flexible to account for quantum transitions.

The concept of "pure state" extends the scope of the principle of class identity and "mixtures" enable us to deal with imperfectly identical classes.

An additional feature of the interface is that the SU(2) matrices can be mapped on a three-dimensional sphere in an abstract four-dimensional space:

$$\mathcal{S}^3 \subset \mathbf{R}^4 \ .$$

This bears out our above remark on the role of non-Euclidean spaces.

Four-dimensional Euclidean spaces have occasionally appeared in the literature, with the fourth dimension interpreted as imaginary time. Within the context of the interface, the interpretation of the fourth dimension emerges without ambiguity. Thus the four-dimensional spin space enables us to construct a continuous path for the spin-flip process. Moreover, the "collapse of the wave function" appears as a spherical wave that diverges from a point of the sphere $\mathscr{S}^3$ to converge toward the antipodal point.

Since an actual particle has both linear and angular momentum, particle dynamics emerges from the combination of spherical and hyperbolic kinematics, written in terms of appropriate spinors. The Dirac equation can be indeed seen as such a combination. When originally suggested, two of its features were rightly considered puzzling: what is the meaning of antiparticles and why does the equation not apply to protons? These forgotten questions are answered in the present program. Charge conjugation is a symmetry of spherical kinematics, and there are four different ways of setting up the combination between spherical and hyperbolic kinematics which are identified with photons, neutrinos, electrons, and protons, respectively. The different variants are specified in terms of their mass, charge and their measure of localization. These phenomenological properties are thus reduced to kinematics.

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[1]A. Einstein, *The Meaning of Relativity,* 5th ed. (Princeton University Press, Princeton, 1956), pp. 165–66.

[2]In producing this interface the σ matrices are essential. However, in order to satisfy this neutral mediating role, the σ's are to have no physical meaning of their own. This suggested usage is a departure from tradition, since Pauli discovered this formalism in the nonrelativistic quantum mechanics of the electron spin. Currently, the σ matrices are used phenomenologically for almost any dichotomic situation. The condition for giving a dynamic underpinning to the phenomenological theories is that these uses should be temporarily "forgotten," until they re-emerge from the orderly development of the new foundation.

[3]"Non-Newtonian" is an allusion to "non-Euclidean" geometry, which has been obtained from the Euclidean by abandoning self-similarity. The terminology has even more relevance, since the dynamics to be developed is expressed in terms of kinematic relations over the non-Euclidean hyperbolic and spherical spaces, just as Newtonian kinematics is over Euclidean space. Note that the non-Euclidean geometries refer to the parameter spaces associated with internal structure, and not a revision of the geometry of ordinary space.

[4]The idea of enriching the Pauli algebra by an additional conjugation operation was advanced by Cynthia Whitney (thesis, MIT, 1968) and used to clarify a subtle structural property of the Lorentz group [E. P. Wigner, Math. Am. **40**, 149 (1939)].

[5]L. Tisza, Rev. Mod. Phys. **25**, 151 (1963); V. F. Weisskopf, *Knowledge and Wonder,* 2nd. ed. (MIT Press, Cambridge, MA, 1979), Chap. 5. Also, *The Privilege of Being a Physicist* (Freeman, New York, 1988), Sec. 7.

[6]This analogy can be given a geometrical background, but only if we replace the conventional definition of angle in terms of "arc" with one in terms of "sectors." Thus ϕ and μ are defined as circular and hyperbolic sectors, respectively.

[7]The existence of difficulties is apparent from the infinities of field theory. However, the present analysis reveals the origin of the problem in a precise and intuitive form.

[8]The idea that deviation from pointlike behavior might be expressed as "orientability" was suggested to me by Robert Mills.

[9]So is the angle γ. However, this cancels out in the quadratic density matrix, thus the vector $\mathbf{s}$ does not fully represent the information contained in the spinor. This defect is corrected by using the third spherical angle γ in a space $\mathbf{R}^4$ which is indeed needed to map the manifold of U matrices.

[10]The partiality of the Copenhagen program for canonical as against phenomenological physics is purely subjective and its origin is in historical trends. However, the implications of this preference are very real and limit the scope of canonical quantum mechanics and particle physics.

[11]Felix Klein, *Vorlesungen über nicht-Euklidische Geometrie* (Springer, Berlin, 1928), Chap. VII.