PROFESSOR: Well, we're back. And today we will consider solutions to harmonic oscillators with damped harmonic oscillators. And what I thought would be best, in order to focus on the new part, the question of damping-- what effect it has on the motion of harmonic oscillators-- I decided I'll take an oscillator, which is one of those we considered last time, I could've taken any one of them, and add to it a drag force.

So let me discuss what we're going to consider. Suppose we have a rod, the uniform rod of mass m suspended from the pivot here. This is the vertical plane. So imagine a ruler, for example, with a nail through the end of it can oscillate in the plane of this board. So we at a gain considering rotations in two dimensions. We will consider rotations, which are anti-clockwise, like this, to have a positive sign. And if it's clockwise, that's a negative rotation.

This particular mass rod we assume has a mass m. It's uniform length l. And initially, it's just suspended like this. So at time equals 0. This is the position, and at that instant, we hit the bottom. There's an impulse acting on it. And that impulse instantaneously gives an angular velocity to this rod, which is of course the rate of change of the angle. And we assume it has some number.

And so that it's easy for you to remember what that number is, I'm calling it the angular velocity at t equals 0. But remember, this is simply the initial angular velocity that this rod has. At any instant of time later, this rod will be at some other angle. I've shown here, theta of t. This is at time t. And it may be moving at that time. And the angular velocity in this direction we call theta dot t. These are anti-clockwise rotations. And so these are positive. This is as that angle. And if this is a positive number, this is rotating in that direction.

So this is the same simple harmonic oscillator we considered last time. And what I now want to take a slightly more realistic situation. All harmonic oscillators in the universe have some drag, maybe very small, some mechanism by which they lose kinetic energy. So we were going to assume that this oscillator has some drag torque applied to it. It could be friction around the pivot, it could be the air, this could be inside the liquid, you name it.

Now, I am going to make the assumption that the presence of the damping mechanism exerts a drag torque on this rod. And I'm going to make the assumption that the drag force is proportional to the angular velocity. Since the drag force is in the opposite direction to the angular velocity obviously, there is a minus sign here and this is the constant of proportionality.

Now, you may well ask why did I assume this is the case? Is that because in nature it's normally like that? No. The reason why I chose this form is very simple. I know from experience that if I use this drag force, I can analytically solve my equations. If I take a more realistic version of the force, then the final solutions I'm going to get it are not analytically soluble. It doesn't mean you can't solve them. You can solve them by a computer. But for teaching you how to solve problems, it's not going to be very helpful if I then say go into a computer and get the computer to solve it for you.

So the choice of this resistive force which is proportional to the rate of change of position of the rod is chosen for that reason. The other thing I will do, once again-- because of what you saw last time that happened-- assume that the displacement of this rod is small. We're only going to consider small oscillations. In other words, we'll only assume it's sufficiently small so that we can approximate sine of the angle by the angle itself. Again, the reason why we do that is so that at the end, we'll be able to solve the equations of motion.

Now, if you ask yourself, hold on. We're doing all these approximations. Does this correspond to reality? And the answer is the following. In problems that we are doing on the board like this, we're taking idealized situations. For those idealized situations, the prediction of what will happen is exact. But since the description of the situations is it not exactly equal to what you'd normally have in the world, the answers that we get are only as accurate to the degree of the assumptions.

But certainly from solutions like this, we can get a sense in understanding what happens, because for example, the drag force slightly different will not qualitatively change what the actual motion is, et cetera. And this happens throughout so-called problem solving land in any physics that you consider. So this is the physical description of the situation. And what we want to do, as before, find out what this rod will do qualitatively as a function of time and also be able to predict exactly where it will be at the given instant of time. So that's what I mean by I'm going to solve this problem.

Now, I chose the same oscillator as I say so that you don't have to focus so much on the rising equation of motion, but more on what the frictional torque does, how that changes what we learned before. So here is the free body diagrams or the force diagram, which corresponds to this. Here is the pivot. This represents the rod. What forces act on this rod? Well, there's going to be the force of gravity.

And you know from the studies of rigid body motion that we can understand what torque this force exerts by considering that the force of gravity acts at the center of mass of this rod. So if you take the center of mass, this location, then there'll be a force of gravity on this mass. The mass of the rod will be m, so there'll be an mg force acting downwards there.

This force applied to this rigid body will exert a torque. In this case, it'll be a clockwise and negative torque. In addition to that, there will be a torque acting on this due to the frictional force. And that, we said, we will call b theta dot. So what is the equation of motion for this rod? The acceleration of the angular acceleration of this rod from tau is equal to i alpha. The acceleration will be equal to the torque divided by the moment of inertia.

Since this is rotations of a rigid body in a plane in two dimensions, I don't have to write any vector quantities. On writing the equations, this is the component of the rotations about an axis perpendicular to the board here. So the angular acceleration will be equal to the total torque, which is the torque due to gravity plus the torque due to the drag divided by the moment of inertia. And by the way, the moment of inertia of a rod like this about one end is 1/3 ml squared.

So this is the torque due to gravity on this. It's minus because, as I mentioned here, it's a clockwise torque minus b theta dot. That's also minus that, because the drag force is always in opposition to the angular velocity divided by i. As we did last time for the harmonic oscillator, in order for me to write fewer numbers or algebraic symbols on the board, I define some quantities. I will define omega 0 squared to be equal to 3 g over l. I'll define gamma to be b over I-- that constant proportionality divided by the inertia of this system, the moment of inertia. And I will, as I said, assume that this angle's always sufficiently small for the accuracy with which one I want to predict what will happen, I can assume that sine delta t is equal to theta t.

Now let me be honest with you. I defined these quantities not only to make my life easier in writing things down, but as you'll see as we go on, it will serve a useful purpose for comparing how all sorts of harmonic oscillators behave with or without damping, et cetera. If I take these definitions and make this assumption and I take this and play around with the algebra, I'll end up with this equation.

I end up with an equation that theta double dot, the second derivative of theta and the angular acceleration times gamma times the angular velocity plus omega 0 squared times the angular position is equal to 0. This is the equation of motion for this system. And the other thing which we know that initially the theta at 0 is 0 and the initial angular velocity is this number, angular velocity at t equals 0.

These three equations are absolutely, from the point of view of physics, equivalent to the description we had. Here we had this physical description of somebody's-- this rod, et cetera. These mathematical equations describe this situation. In order to predict what will happen, what we now have to do is solve these equations. So that's what I will do now.

So I will solve this equation. And let me remind you, as I discussed in the past, I cannot emphasize it enough now. I am not trying to teach you mathematics. I am trying to help you learn how to solve physical systems, how to predict how a given physical system will behave as a function of time. So for me, mathematics is a tool. I need a solution of that equation. And this is a linear second order differential equation in time.

And I know from mathematics that if I find an equation which solves that and if it has 2 arbitrary constants-- in other words, the solution satisfies that equation for all values of those arbitrary constants. Then I have found the one and only solution of that equation. And so let me tell you. I will solve this by using this uniqueness theorem. I know that that equation, unfortunately, is a little harder to solve than the one we did last time for simple harmonic motion. One finds that the solution of that equation actually depends on these constants.

For the harmonic oscillator, the type of equation, which was the solution, did not depend on those constants. Here, it does. One finds that if you have gamma squared greater than omega 0 squared-- in other words, if you have this big constant bigger than a certain amount, this means you have heavy damping. You have one kind of a solution, while if it's that other way about, if this is bigger than that, you get a different kind of a solution. It's not obvious, but that's what mathematics tells you.

So let me consider first the case of strong damping. B is big. And so we're talking about situation, you have this oscillator. But there's lots of friction, so it doesn't move smoothly at all. This is greater than that. Under those conditions, you'll find that this equation satisfies our equation of motion. Now actually, Professor Walter Lewin, in his lectures, did show you how you can prove that this equation satisfies our equation of motion for these specific conditions. He used complex amplitude. And you can review his lectures and you can see that. Or you can look it up in books on the solution of differential equations.

For me, it is adequate that I found a solution which satisfies this equation. Now, note this is a solution of that equation where the alpha 1 and alpha 2 are defined here. It's all defined in known quantities. Gamma omega 0 are known quantities for any specific problem in particular for the one we're solving. So here, every term is known. The A and the B are the 2 arbitrary constants. In other words, this equation satisfies our equation of motion for all values of A and B. So again, by the uniqueness theorem, this is the most general solution to our problem provided the damping is very high.

If this is the theta of t, then of course the angular velocity theta dot of t, just differentiate here. And I get minus alpha 1 A e to the minus alpha 1 t and minus alpha 2 B to the minus alpha t. We want to find the solution to our particular problem. So we found it that satisfies the equation of motion for our system. We now want to make sure that we have a solution which satisfies the boundary conditions.

In other words, which is consistent with what we said this rod was doing at t equals 0. At t equals 0, we said the rod was vertically, so theta of 0 is 0. And we also said that at that instance, there was an impulse given to the rod. And so its angular velocity at t equals 0 was this number, the angular velocity at t equal 0.

So we must now make sure that the A and B, these arbitrary constants, are consistent with these initial conditions. And it's obvious how you do that. You take this equation and I take what is theta at t equals 0. Well, it's 0. So that's 0 equals to t equals 0. This is one, so it's A, plus when t equals 0, this is B. So it's A plus B. So we know that A must be equal to minus B. That's to satisfy the boundary conditions that the rod was vertical at t equals 0.

How about the other boundary condition? We know that t equals 0, the angular velocity theta dot is angular velocity at t equals 0. And so angular at t equals 0 must be equal to minus alpha 1 times A. And t equals 0, this is 1. So it's minus alpha 1 A. And here, minus alpha 2 B. OK, we now have 2 equations with 2 knowns, 2 algebraic equations. Therefore, we can find A and B. In fact, we found already that A is equal to minus B from this equation. And then replacing B by minus A, we find that A is angular velocity t equals 0 and alpha t over 1.

So what do we find? We have found that the specific solution to our equation of motion which satisfies these boundary conditions is theta of t is equal A e to the minus alpha 1 t. A is angular velocity at t equals 0 into e to the minus alpha 1 t. And B is minus A. So it's minus e to the minus alpha 2t. So we have solved that problem. This equation describes the motion of this compound pendulum, the rod, completely.

If you ask me for a time, I will tell you exactly the angle at which this will be hanging. There are no more unknowns in this. Furthermore, it gives me a sense of what that rod is doing. And actually, it's a bit of a surprise. We've considered here an oscillator, we'd expect an oscillator to oscillate. This equation doesn't oscillate. It is the sum of 2 exponential functions. If I just qualitatively plot this, if you compare alpha 2 to alpha 1, you find that alpha 2 is greater than alpha 1. So this exponential has a bigger exponent. It will be dropping faster than this one.

So what do we have? This is my angular velocity. Velocity at t equals 0. Let's consider one of them. This is 1. It's an exponential function. Here, I'm plotting the angle as a function of time. I'll get some exponential function like this, starting at that value. And the other function's also an exponential one, which has a higher value of the exponent. And therefore, it'll drop more quickly, starts with the same amplitude, so it'll go more quickly like this.

The net motion is the difference between those 2. Take this one minus that one. What does that look like? Well, qualitative you can see I start at 0. Far out here, it'll be 0, because the 2 exponentials will approach 0. So out here, and it'll be 0 or approaching 0. It'll be some magnitude in here. And so I will get something like this happening. It's the difference of the 2 exponential functions.

Isn't it amazing? Let's think for a second what we have shown. We've taken this harmonic oscillator, we've put it in a situation with a lot of friction, we gave it a kick, and we described it in terms of these mathematical equations. I then blindly applied mathematics, solved these equations, and predicted that this will be the motion. No oscillations.

Let's stop now for a second. By common sense, what would you expect? Imagine you had this rod and something which is very, very viscous-- very high drag. You give it a kick, it will move, because of inertia it will move. But it will get slowed down. It will finally come to a halt. And it'll slowly come back. But if the resistive torque is so big, it'll never overshoot. So this is exactly what you'd expect. And as I say, it is amazing. I always find it amazing, because this is a magnificent example that mathematics is the language that one can use to describe the nature, and that solving the mathematical equations predicts what will actually happen in a given situation.

Enough philosophizing. Let's now consider the other possibility. So I'll now do a second problem where I will have very weak damping. So imagine a more realistic situation. We take our pendulum, this compound pendulum, and we include the resistive torque due to the air, for example, that's there, et cetera. Again, I am taking the same pendulum so that we just focus on the effect of the damping. So I think I have it here.

Everything up to here will be exactly the same. The only thing we've now changed, we've said that this little b is big such that the gamma squared is smaller, it doesn't have to be much smaller, so that gamma squared is less than 4 omega squared. In other words, B is less than this. So it's the weak damping case.

You may well ask, hold on. Why look for any other solutions? I have found a solution for this equation before. And up there was the two exponentials. Why can't we just take those and say look, that equation satisfies this equation, that value of theta, and just repeat this? Well, try it and you'll see you immediately get into difficulties. In this formulation, you see that alpha 1 and alpha 2 include a square root of gamma squared minus 4 omega squared.

We had no problem with that when gamma squared was greater than 4 omega squared and take the square root of it. But if gamma squared is less than 4 omega squared, that square root is the square root of a minus number. And how is that going to represent a physical situation? So the method breaks down. In other words, for this case, we had not found an equation which satisfies our equation of motion with these boundary conditions.

So we've got to try again. And there is an equation that does. And I've written it here. One finds that theta of t is equal to some arbitrary constant A e to the minus gamma over 2t times cosine, omega prime t plus phi where omega prime is well defined in terms of omega 0 and gamma is given here. And there's no problem. We have now gamma squared is less than that, so this is a real number. So it's no problem with that square root plus phi. So we have 2 arbitrary constants.

Try it. Take this equation, differentiate it twice, and calculate it. Then differentiate it once and multiply it by gamma. Don't differentiate. Take that equation and multiply the omega 0 squared. Add those 3 terms, you'll get 0. I tried it last night to make sure I didn't write the wrong thing here. So that really works. Again, if you want to know how did I find this equation, did I take it out of a hat, like a rabbit out of a hat? No. You can look it up in books, solutions of equations, or actually Professor Walter Lewin did derive this solution also using complex amplitudes. And you can find it in his lectures.

But from my point of view, I don't care where I got it from. That works. So this must describe the motion. And it has 2 arbitrary constants which I can find out, because I know the initial conditions for this pendulum. So once again, now I will try to find the 2 arbitrary constants.

How did we do it before? By taking our solution. Well, we know that theta of 0 is 0. Therefore, if 0 is equal to A e to the minus gamma over 2t, that's t 0. So that's 1 cosine t 0, so I get phi. So one boundary condition tells me that phi is pi over 2. Or there are others, minus pi over 2 works, et cetera. Doesn't matter. Take any one of them and providing you're consistent, the other one will make up for it to give you the same answer. So anything that satisfies this boundary condition is fine for one of those constants.

Now, the other one tells me something about the initial angular velocity of the rod. So I need to calculate the angular velocity of this rod. So here, I calculate theta dot t at time t by differentiating theta. And I get this lower equation. And now again, theta dot at 0 time is equal to-- we know it is-- this number, angular velocity at t equals 0.

So in this equation, I plug that in. I get that the angular velocity at t equals 0 will have to equal 2. A, e to the minus gamma over 2t. t is 0, so that's 1. And 2 minus omega prime sine of omega prime t, but t is 0. So that's 0. Phi, that's the sine of pi over 2, because we find phi. That's 1. So I don't do anything with that. Minus gamma over 2 cosine of omega prime t. t is 0, so it's cosine of phi. Phi is pi over 2, so that is 0. So that's all right. Therefore, A is equal to now angular velocity at t equals 0 divided by omega prime.

And therefore now, let me move over here. So now I can write the full solution. I have found everything, and I find that theta of t from the second line down is equal to A, which we found, angular velocity t equals 0 over omega prime minus-- there's a minus-- and now it's cosine omega prime t plus 90 degrees. So that's the same as minus sine of omega prime t. The 2 minuses cancel, so I get a plus.

And if earlier on you chose, for example here, minus pi over 2, then both of those signs would have been plus, and you still get the same answer. So it doesn't matter. The algebra looks after itself. So we have to found the solution of this equation of motion with these boundary conditions for this situation where the gamma and the omega 0 squared is such that this is angle to correspond to the low damping situation.

What kind of a motion is that? If I plot this, again as we do as a function of time, if I had a room full of students here who would correct me, then I would have caught the mistake. If you look at the second equation, it says take theta of t is A e to the minus gamma 2t cosine omega prime t. When I rewrote it plugging in the value of A, I forgot the e to the minus gamma over 2t. Sorry about that.

And we continue. So now if I plot this, what do I see? If it wasn't for the e to the minus gamma over 2t, it would be a sine. That's what it would look like. t, and this is the angle theta of t. This would be the simple harmonic motion with no drag, no damping. But this function is multiplied by an exponential, like this, which is dropping.

So the net result-- and excuse my sloppy drawing-- is this modulated by that. So it will be something like this. So what we get is damped harmonic oscillation. Now again, I can't resist pointing out. To me, it seems almost miraculous. You have the same equation in form with the same boundary conditions predicting completely different behavior if some of these constants are bigger than the others or vice versa.

So the mathematics has it built in to that kind of solution. Those two kinds of solutions exist. And that corresponds exactly what you would expect if you imagine an oscillator with damping. If you have this oscillator, if it's highly damped, as I've said before, you would expect this crazy motion reaching a maximum slowly coming down and coming to a grinding halt. Or if it's lightly damped, you would expect this to almost behave like a normal harmonic oscillator, but slowly losing energy due to damping and oscillating with smaller and smaller amplitude, which is exactly what the mathematics said would happen.

I always like to ask a crazy question, how does the oscillator know what the mathematical solution is? But somehow or other, it does. It's a crazy way to phrase it, but I always think of that. So let's put together actually what we have seen in combination of last time and this time, my problem solving, my attempts to teach you how to solve problems.

What we found last time was that we took three different harmonic oscillators. One was a mass on a spring. Another was this rod oscillating like this. The third was an electrical system, an LC circuit. In each case, we found that we the equation of motion looked exactly the same. The only difference was what was the name of the variable. For example, in the case of the spring, the displacement from the equilibrium was a distance. So this equation was of this form where the size was y.

When we came to this oscillating rod, the quantity which was displaced from equilibrium was an angle. So this sine was a theta. In the case of the LC circuit, the quantity was a charge. So the charge which was displaced from equilibrium. But I could have summarized the equations of motion for the three harmonic oscillators by this one equation and just tell you, well, this is the equation of motion for that and put in the appropriate quantity, which is displaced and also put in the appropriate value of this constant, which for the spring was k over n. For the rod, 3g over 2l. For the LC circuit is 1 over LC.

And believe me, we could have taken a million harmonic oscillator with no damping, anyone you can think of, and I could reduce it to this form. So here's something new that we observe, that you can take lots of physical situation but describe them in terms of the same mathematical equations. The same mathematical equation can describe millions and millions of different kinds of harmonic oscillators. So that is another feature of the scientific method that is very important. It means now we don't have to understand every little thing that happens in the universe. We can reduce it to a smaller subset of description.

Now, we've now add the new feature today. We added some loss mechanism. We took a specific kind where the resistive force or the resist torque is proportional to the velocity of the analysis. In our case, to the angular velocity of the rod. And we derived an equation of motion, which looks like this, where these psis for the oscillator we considered was the angular displacement.

I don't have to be an Einstein-- by analogies to what I said here-- to say look, I could have taken any one of the oscillators in the universe. And with damping, if the damping is proportional to the velocity of the analogous quantity of velocity, I'll end up with this equation of motion. So this is the equation of motion of simple harmonic motion. This is the equation of motion of damped harmonic oscillators. Every one of them.

But there is an interesting difference we saw today. There is only qualitatively one kind of solution here. In the case of the damped harmonic oscillator, there are two qualitatively different solutions to that. One is the sum of exponentials and the other is damped harmonic motion. Both the quality of the solution and the details, whether it's oscillating fast or slow, et cetera, depends on those quantities-- the gamma and the omega 0 squared, which in turn, for example, in the various cases it's k over n, 3g, et cetera.

So that will describe all oscillators. And in any particular oscillator, if you want to know what exactly happens at the given instant of time, you then need also to know those two constants, the boundary condition. If I have some mass on the spring oscillating, where it will be at some later time does depend where it is at this very instant.

Finally, there's just one more thing I would like to do today, and it's the following. Oscillators are not only nice in order to give you a lecture on oscillators or show you how to solve them, they are practical devices. So sometimes we want to know some property of a given oscillator. Well, there are two properties which describe the quality of a given oscillator and what its behavior is.

One you've seen already is to do the period or the frequency of the oscillator. In other words, you could have one oscillator which goes very slowly. You're going to have one which goes very fast. I have an oscillator and you want to tell somebody what it's like, a crucial number is what its period is, the 1 over the period which is the frequency.

Imagine, for example, a bell. That's an oscillator. You could have a bell which has a low tone or a very small one with a very high pitched tone. So that's one number that's important. But there is another number that's important. Whether it's a good or a bad oscillator. And it's obvious what is it good oscillator. A good oscillator is one that oscillates for a long time. A bad one is that doesn't continue oscillating for a very long time.

But again, you'd I'm sure agree with me that it isn't just how many oscillations it does, say, in a given time, like 1 second. If I take an oscillator that makes one oscillation per second, then if it survives 1 second, you wouldn't say it's a very good oscillator. It barely made one turn, while if the oscillator is making a million oscillations per second and that survives for a second, you would say that's a pretty good oscillator. It made a million oscillations before it ran out of steam.

So to quantify the quality of an oscillator, we use a term, and being very imaginative we call it the quality of the oscillator, or Q value. So I will finally want to derive for you for this particular oscillator what is its Q or Q value. And that's the last thing I will do today.

Now, it only makes sense to talk about the Q of an oscillator if it's at least a reasonably good oscillator. This one, this overdamped one who didn't even make one complete cycle, it doesn't make sense to talk about its Q. So the only case where it makes sense to talk about the Q of an oscillator is if it makes at least a few oscillations. It's got to be the underdamped case.

So I will now calculate for you what is the Q of this compound pendulum, this rod which oscillates on the conditions where the drag force is small so that it has this undamped motion. So let me do that. The Q value of an oscillator is defined by no good reason of pi multiply by the number of oscillations the oscillator makes before its amplitude drops by 1 over e. It's a formal definition.

So take some oscillator, which is a good oscillator. So it oscillates like this. It's decaying. This has an exponential form, as we saw for a damped harmonic oscillator. And we ask the question how many oscillations it makes before the amplitude drops by 1 over e. And for historic reason, 1 takes pi multiply by the number of oscillations for amplitude to go down by amplitude over e. 1 over e.

Because the energy stored in an oscillator is proportional to the amplitude squared, this is equivalent to 2 pi times the number of oscillations it takes for the energy to go to the energy over e. These are equivalent because 1 is the square of the other. And in fact, just let me tell you. I'm digressing here.

Originally this was the definition. The 2 pi is there because in the definition, one said what will be the change of phase of the oscillator for the energy stored in the oscillator to go down by 1 over e? If you're talking about the change of phase rather than number of oscillations, you have 2 pi there, because the phase in one oscillation, the phase changes by 2 pi. So historically, this came in. That's equivalent to this. And so we have to calculate that.

Now, for this particular oscillator, we actually calculated the motion, how the angle changes with time, and we see that the amplitude drops like e to the minus gamma over 2. So we saw that the amplitude drops like e to the minus gamma over 2t. So in what time will the amplitude drop by 1 over e? That time will be when gamma t over 2 is equal to 1. So you get e to the minus 1. So it will be in a time t equal to 2 over gamma.

Now, what we are interested in is how many oscillations will, in this time, the oscillator make. But you know the period T is equal to 2 pi over omega prime. I'm using here omega prime because that is the angular frequency of the damped oscillator. It was not omega 0, like the undamped one. And omega prime I defined for you earlier. But you know that if the damping is weak, omega prime is essentially equal to omega 0.

Q is a quality thing. We don't need it exactly. So it's just to give you a feel whether something is good or bad oscillator. So since omega prime is essentially equal to omega 0, I'll take the period to be approximately equal to this over omega 0. So in time t, how many periods will I have? I have to divide t by the period. That'll be the number of oscillations I'll make. So it's 2 over gamma divided by 2 pi over omega 0. And so this is the number of oscillations.

So the Q will be pi times the number of oscillations for the amplitude to drop by this. So the Q will be equal to pi times this quantity, which is 2 over gamma divided by 2 pi over omega 0, which is equal to omega 0 over gamma. So the quality of this particular oscillator will be equal to omega 0 over gamma. And I hope I still have it. Do I have that? Well, by going back through this video, omega 0 we defined earlier on and gamma also. And so you calculated this. That's as much as I was going to do about damped oscillations for you. Thank you.